

# STATICS



دانشگاه کردستان  
University of Kurdistan  
زانکۆی کوردستان

- Vector Mechanics for Engineers: Statics, 9th edition. Ferdinand Beer- E. Russell Johnston Jr. - Phillip Cornwell.
- Engineering Mechanics-Statics, 5th Edition. J. L. Meriam, L. G. Kraige.
- Other Reference: Brain P.Self“Lectures notes on Statics”

## Distributed Forces: Centroids and Centers of Gravity

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# Distributed Forces: Centroids and Centers of Gravity

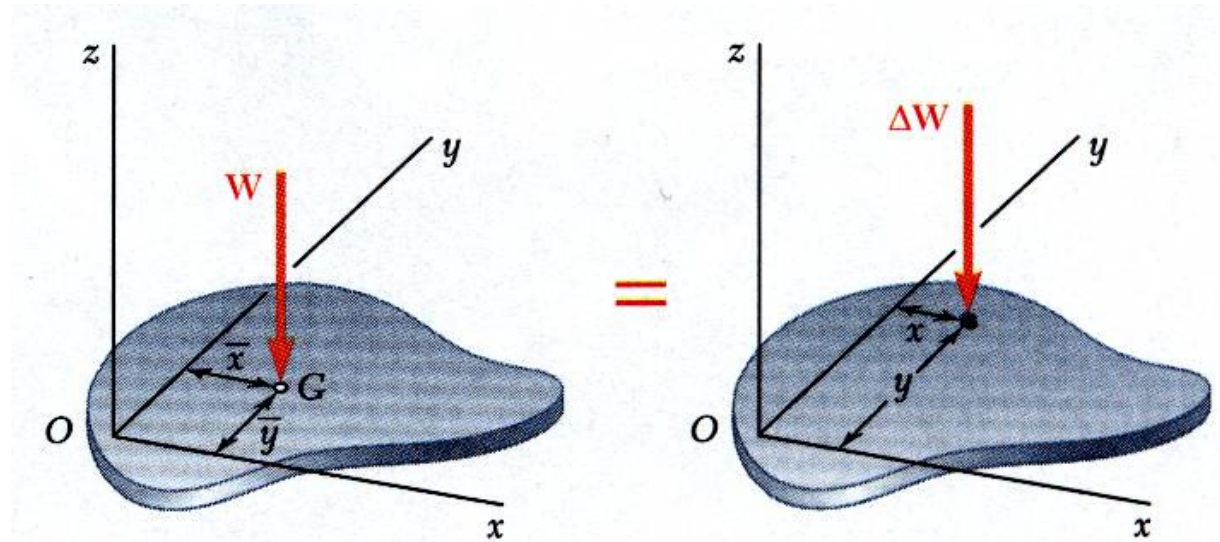
## □ Introduction

- The earth exerts a gravitational force on each of the particles forming a body. These forces can be replaced by a single equivalent force equal to the weight of the body and applied at the *center of gravity* for the body.
- The *centroid of an area* is analogous to the center of gravity of a body. The concept of the *first moment of an area* is used to locate the centroid.
- Determination of the area of a *surface of revolution* and the volume of a *body of revolution* are accomplished with the *Theorems of Pappus-Guldinus*.

# Distributed Forces: Centroids and Centers of Gravity

## □ Center of Gravity of a 2D Body

- Center of gravity of a plate



$$\sum F_z : \Rightarrow W = \Delta W_1 + \Delta W_1 + \dots + \Delta W_n$$

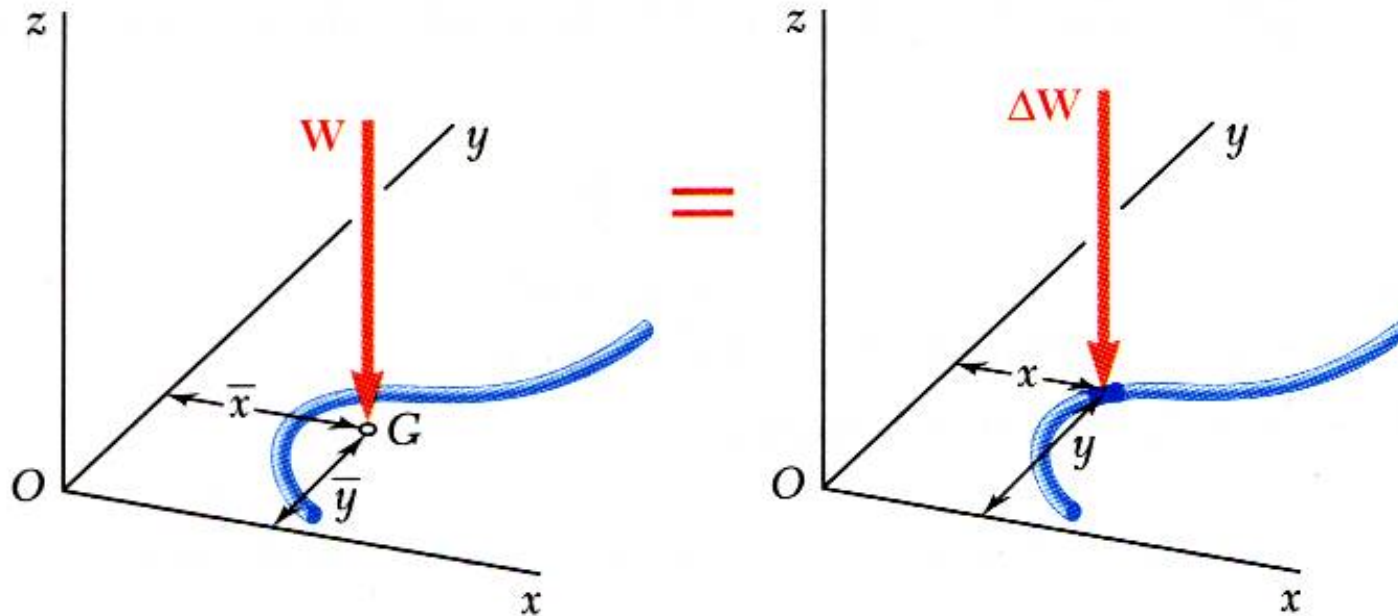
$$\sum M_y : \Rightarrow \bar{x}W = \sum x\Delta W \Rightarrow \boxed{\bar{x} = \frac{\sum x\Delta W}{W} \quad \text{or} \quad \bar{x} = \frac{\int x dW}{W}}$$

$$\sum M_x : \Rightarrow \bar{y}W = \sum y\Delta W \Rightarrow \boxed{\bar{y} = \frac{\sum y\Delta W}{W} \quad \text{or} \quad \bar{y} = \frac{\int y dW}{W}}$$

# Distributed Forces: Centroids and Centers of Gravity

## □ Center of Gravity of a 2D Body

- Center of gravity of a wire



$$\sum M_y : \Rightarrow \bar{x}W = \sum x\Delta W \Rightarrow$$

$$\bar{x} = \frac{\sum x\Delta W}{W} \quad \text{or} \quad \bar{x} = \frac{\int x dW}{W}$$

$$\sum M_x : \Rightarrow \bar{y}W = \sum y\Delta W \Rightarrow$$

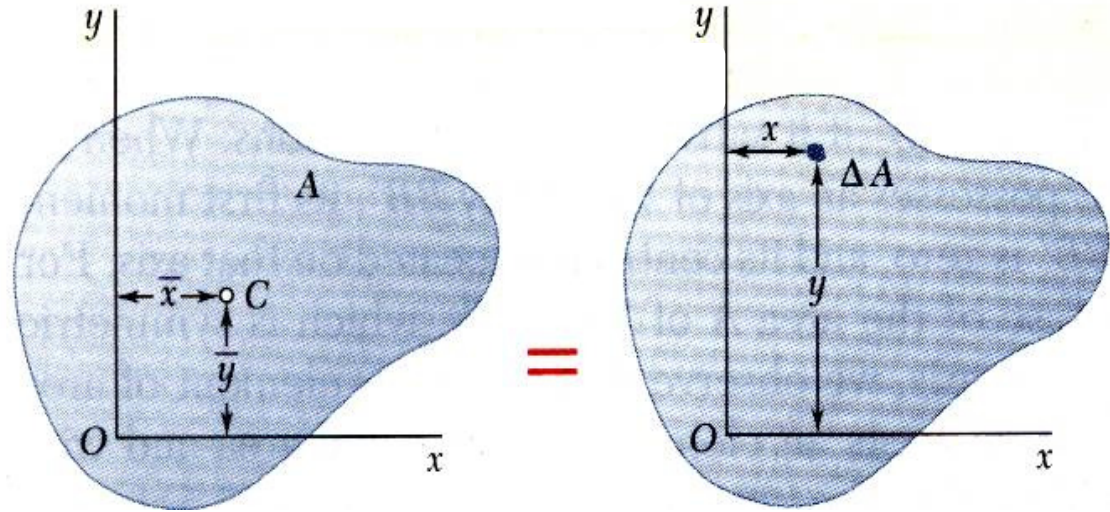
$$\bar{y} = \frac{\sum y\Delta W}{W} \quad \text{or} \quad \bar{y} = \frac{\int y dW}{W}$$

# Distributed Forces: Centroids and Centers of Gravity

## □ Centroids and First Moments of Areas

- Centroid of an area

$$\Delta W = \gamma t \Delta A \Rightarrow \boxed{W = \gamma t A}$$



$$\bar{x} = \frac{\int x dW}{W} \Rightarrow \bar{x} = \frac{\int x(\gamma t) dA}{(\gamma t A)} \Rightarrow \boxed{\bar{x} = \frac{\int x dA}{A}}$$

$$\text{if } \boxed{Q_y = \int x dA \Rightarrow Q_y = \bar{x} A}$$

$Q_y$  : First moment with respect to y axis

$$\bar{y} = \frac{\int y dW}{W} \Rightarrow \bar{y} = \frac{\int y(\gamma t) dA}{(\gamma t A)} \Rightarrow \boxed{\bar{y} = \frac{\int y dA}{A}}$$

$$\text{if } \boxed{Q_x = \int y dA \Rightarrow Q_x = \bar{y} A}$$

$Q_x$  : First moment with respect to x axis

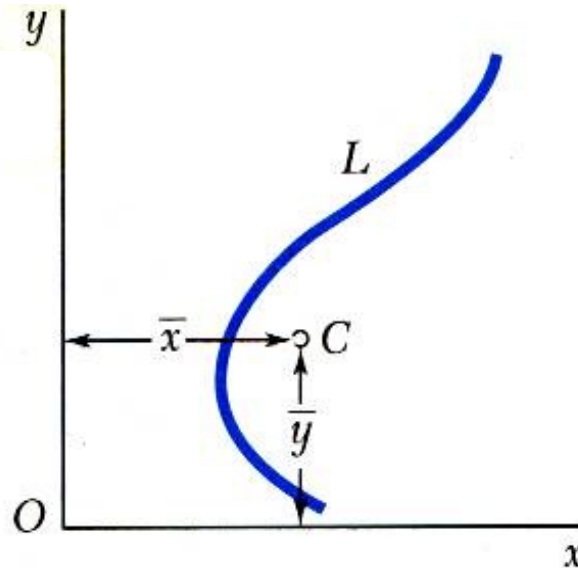
# Distributed Forces: Centroids and Centers of Gravity

## Centroids and First Moments of Lines

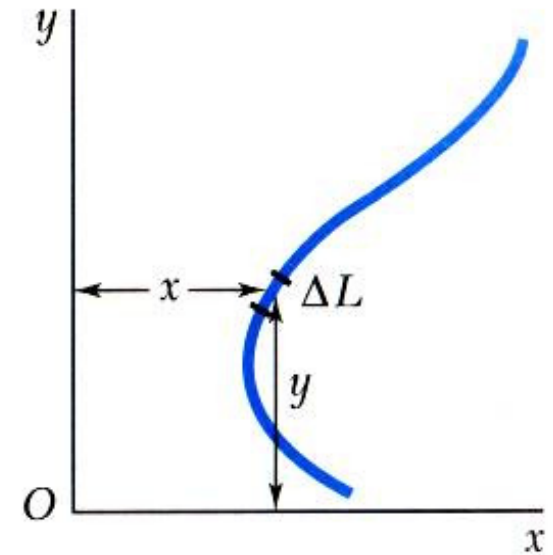
- Centroid of a line

$$\Delta W = \gamma a \Delta L$$

$$\Rightarrow W = \gamma a L$$



=



$a$ : Cross section area

$$\bar{x} = \frac{\int x dW}{W}$$

$\Rightarrow$

$$\bar{x} = \frac{\int x(\gamma a) dL}{(\gamma La)}$$

$\Rightarrow$

$$\bar{x} = \frac{\int x dL}{L}$$

$$\bar{y} = \frac{\int y dW}{W}$$

$\Rightarrow$

$$\bar{y} = \frac{\int y(\gamma a) dL}{(\gamma La)}$$

$\Rightarrow$

$$\bar{y} = \frac{\int y dL}{L}$$

$$dL = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

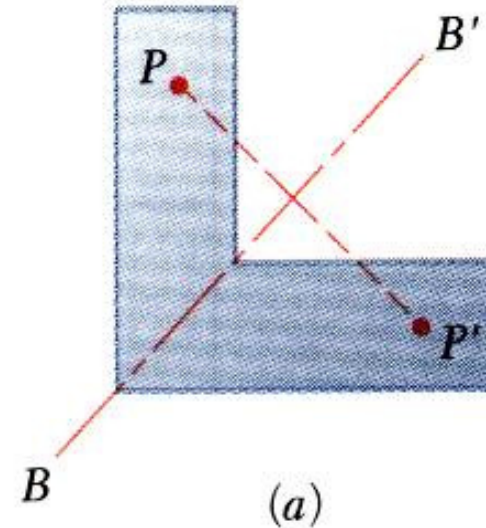
$$dL = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

$$dL = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

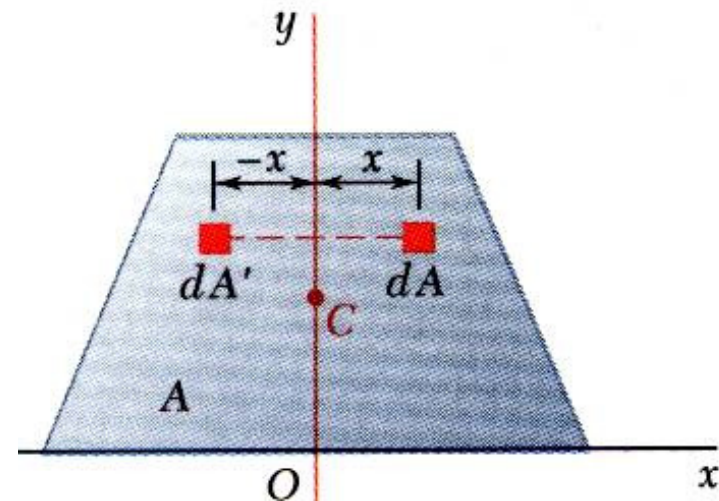
# Distributed Forces: Centroids and Centers of Gravity

## □ First Moments of Areas and Lines

- An area is symmetric with respect to an axis  $BB'$  if for every point  $P$  there exists a point  $P'$  such that  $PP'$  is perpendicular to  $BB'$  and is divided into two equal parts by  $BB'$ .



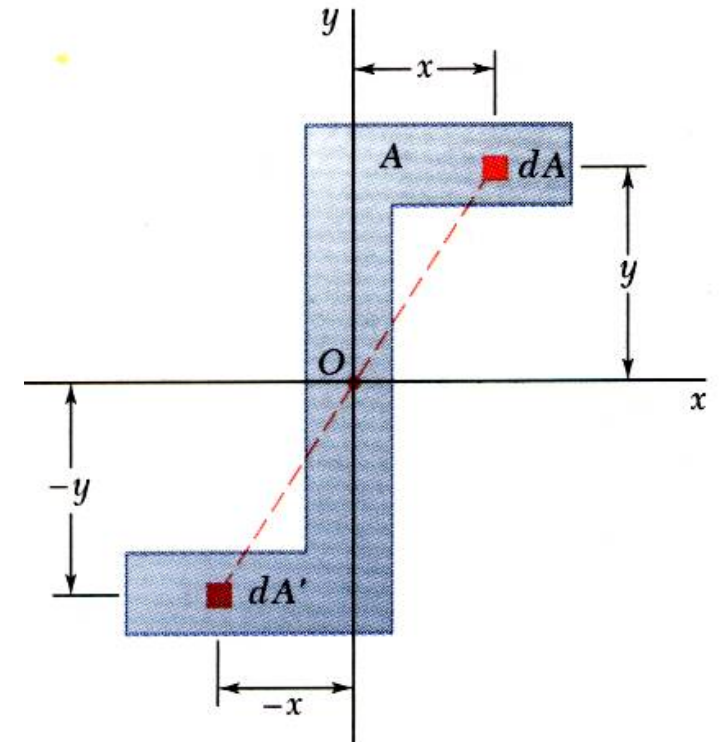
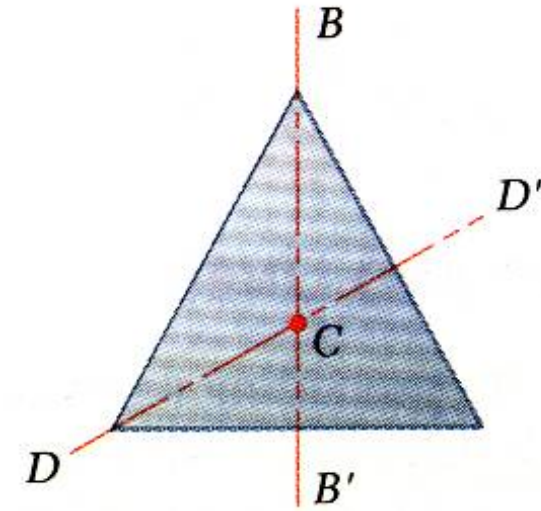
- The first moment of an area with respect to a line of symmetry is zero.
- If an area possesses a line of symmetry, its centroid lies on that axis



# Distributed Forces: Centroids and Centers of Gravity

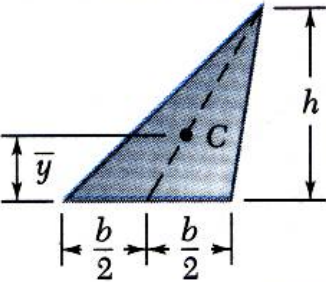
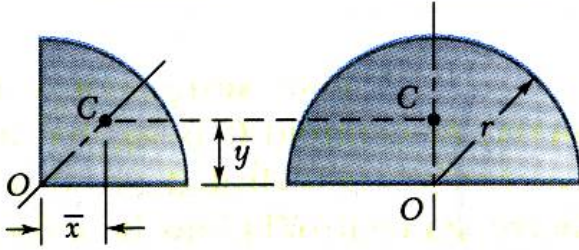
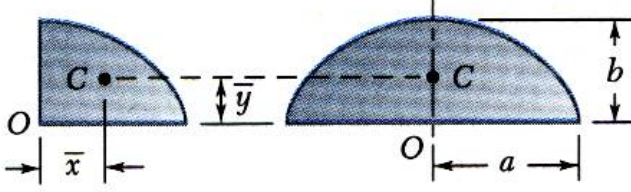
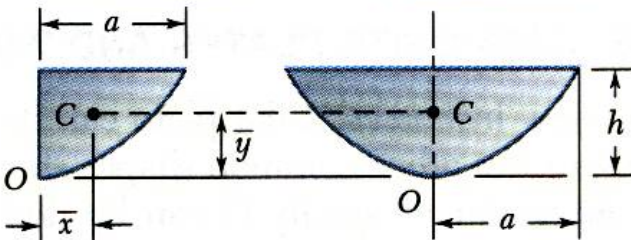
## □ First Moments of Areas and Lines

- If an area possesses two lines of symmetry, its centroid lies at their intersection.
- An area is symmetric with respect to a center  $O$  if for every element  $dA$  at  $(x,y)$  there exists an area  $dA'$  of equal area at  $(-x,-y)$ .
- The centroid of the area coincides with the center of symmetry.



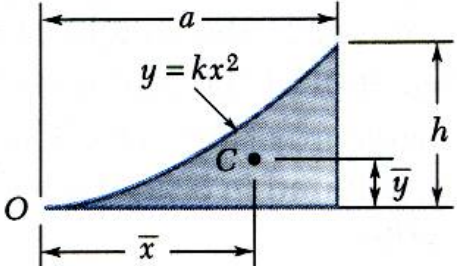
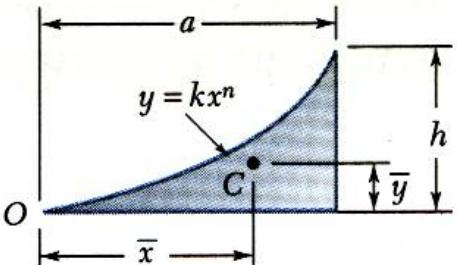
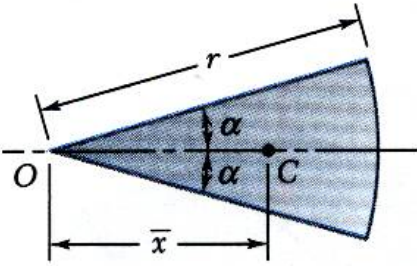
# Distributed Forces: Centroids and Centers of Gravity

## Centroids of Common Shapes of Areas

Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$

# Distributed Forces: Centroids and Centers of Gravity

## □ Centroids of Common Shapes of Areas

Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
General spandrel		$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$	$\frac{ah}{n+1}$
Circular sector		$\frac{2r \sin \alpha}{3\alpha}$	0	$\alpha r^2$

# Distributed Forces: Centroids and Centers of Gravity

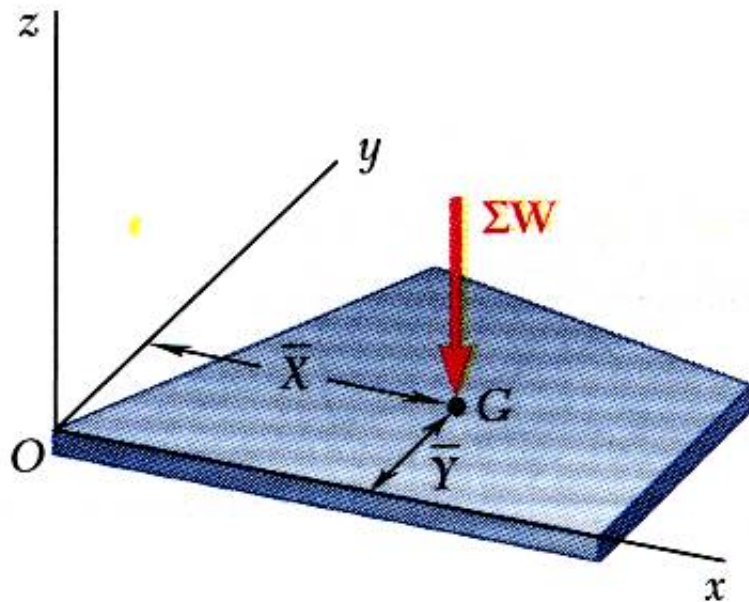
## □ Centroids of Common Shapes of Lines

Shape		$\bar{x}$	$\bar{y}$	Length
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc		0	$\frac{2r}{\pi}$	$\pi r$
Arc of circle		$\frac{r \sin \alpha}{\alpha}$	0	$2\alpha r$

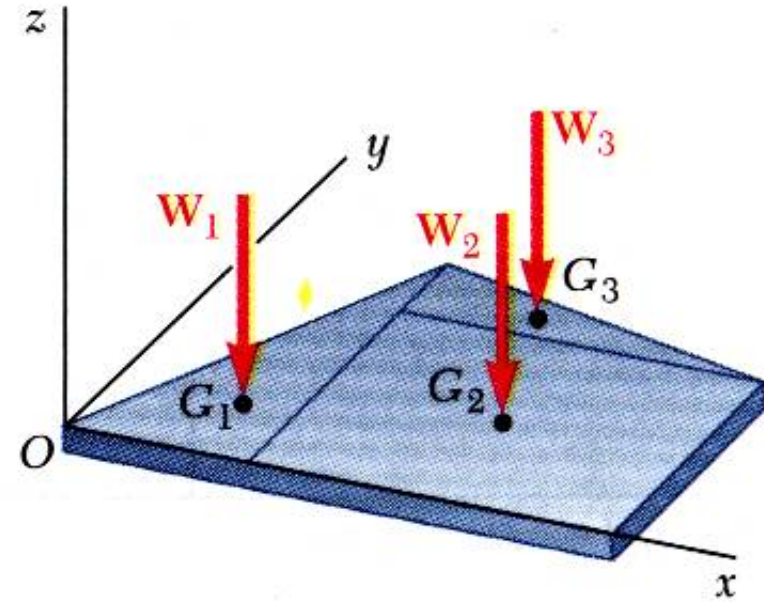
# Distributed Forces: Centroids and Centers of Gravity

## □ Composite Plates

- Composite plates



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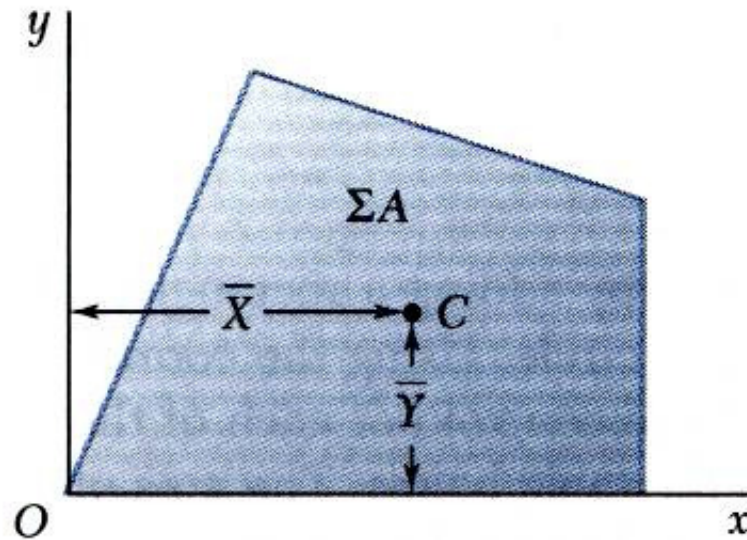
$$\bar{X} = \frac{\sum \bar{x}W}{\sum W}$$

$$\bar{Y} = \frac{\sum \bar{y}W}{\sum W}$$

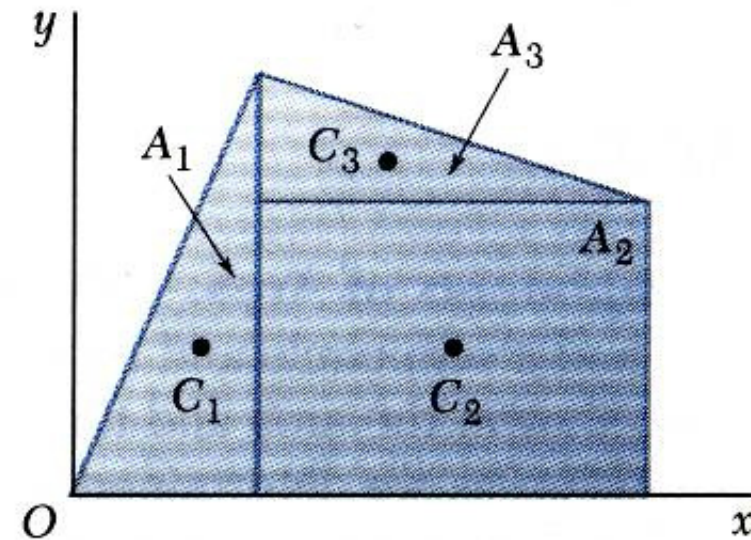
# Distributed Forces: Centroids and Centers of Gravity

## □ Composite Areas

- Composite area



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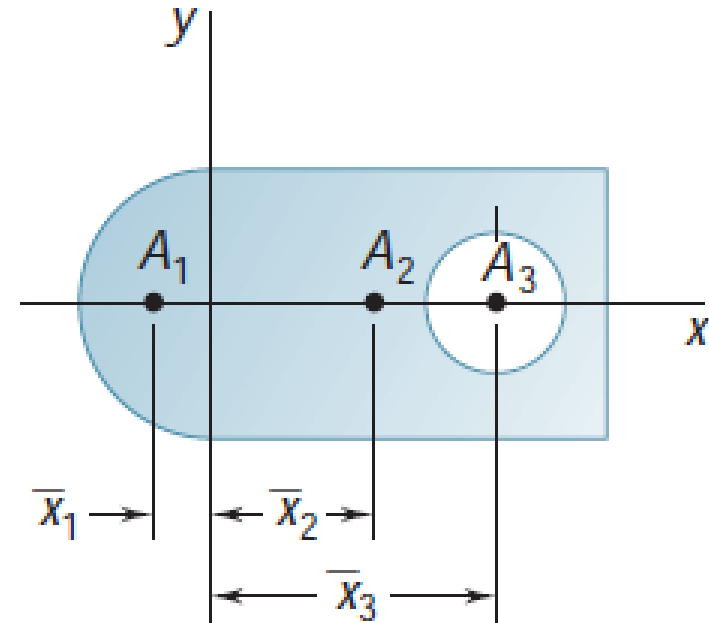
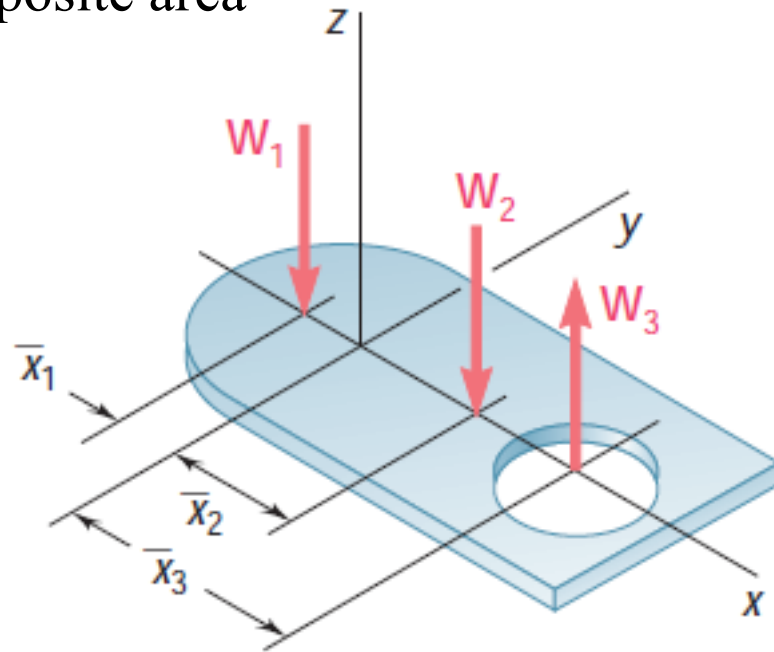
$$\bar{X} = \frac{\sum \bar{x} A}{\sum A}$$

$$\bar{Y} = \frac{\sum \bar{y} A}{\sum A}$$

# Distributed Forces: Centroids and Centers of Gravity

## □ Composite Areas

- Composite area

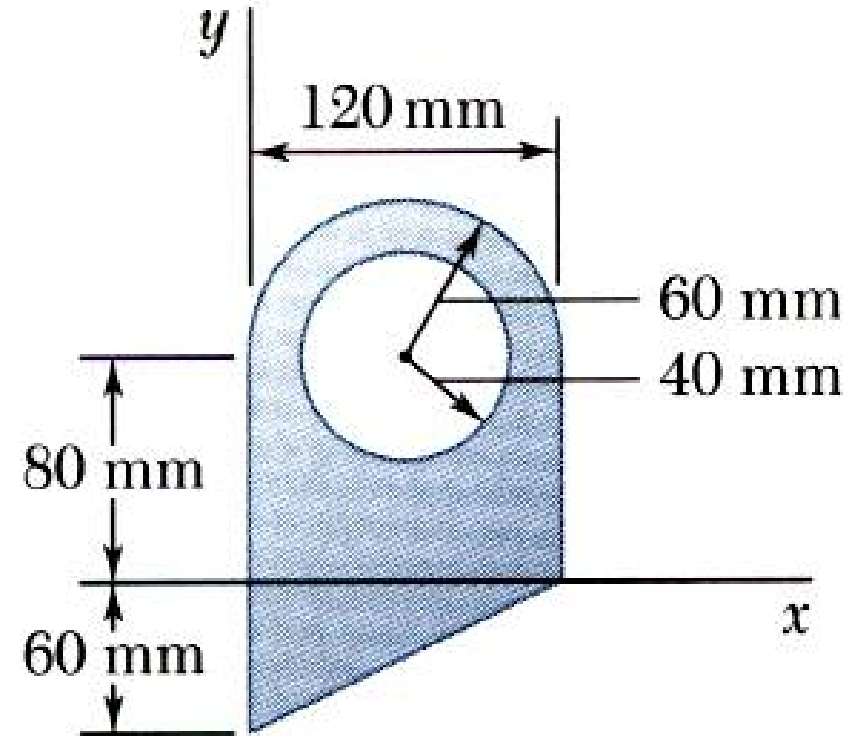


	$\bar{x}$	$A$	$\bar{x}A$
$A_1$ Semicircle	-	+	-
$A_2$ Full rectangle	+	+	+
$A_3$ Circular hole	+	-	-

# Distributed Forces: Centroids and Centers of Gravity

## □ Sample Problem 01

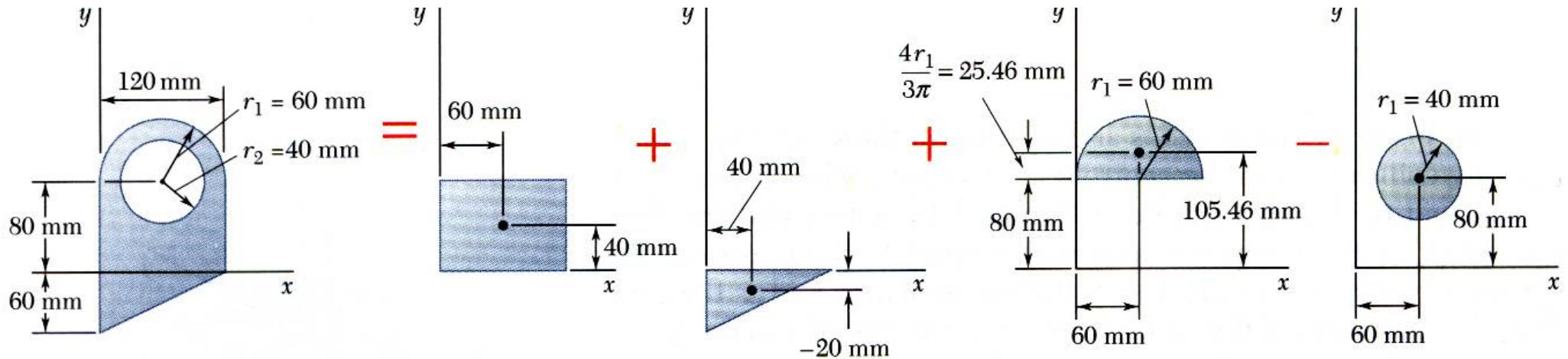
For the plane area shown, determine the first moments with respect to the  $x$  and  $y$  axes and the location of the centroid.



# Distributed Forces: Centroids and Centers of Gravity

## □ Sample Problem 01

SOLUTION:



Component	A, mm <sup>2</sup>	$\bar{x}$ , mm	$\bar{y}$ , mm	$\bar{x}A$ , mm <sup>3</sup>	$\bar{y}A$ , mm <sup>3</sup>
Rectangle					
Triangle					
Semicircle					
Circle					

- Find the total area and first moments of the triangle, rectangle, and semicircle. Subtract the area and first moment of the circular cutout.

$$Q_x = +506.2 \times 10^3 \text{ mm}^3$$

$$Q_y = +757.7 \times 10^3 \text{ mm}^3$$

# Distributed Forces: Centroids and Centers of Gravity

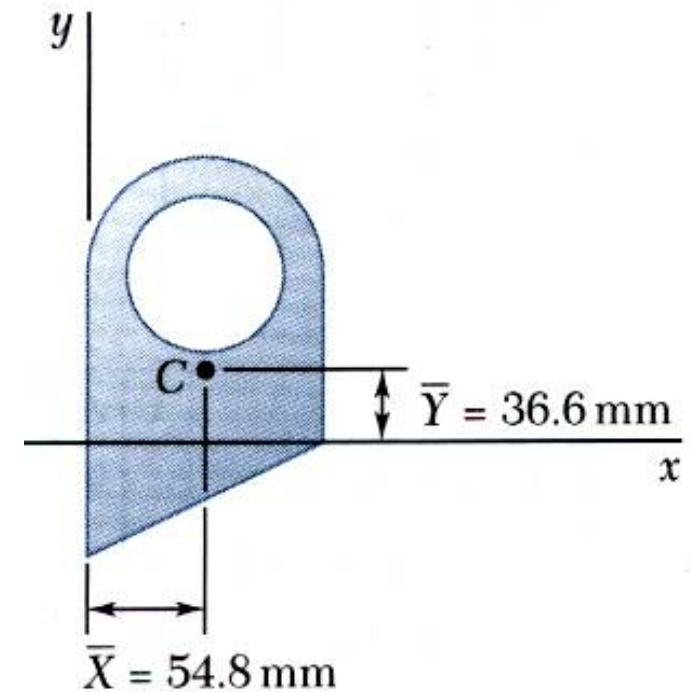
## □ Sample Problem 01

SOLUTION:

- Compute the coordinates of the area centroid by dividing the first moments by the total area.

$$\bar{X} = 54.8 \text{ (mm)}$$

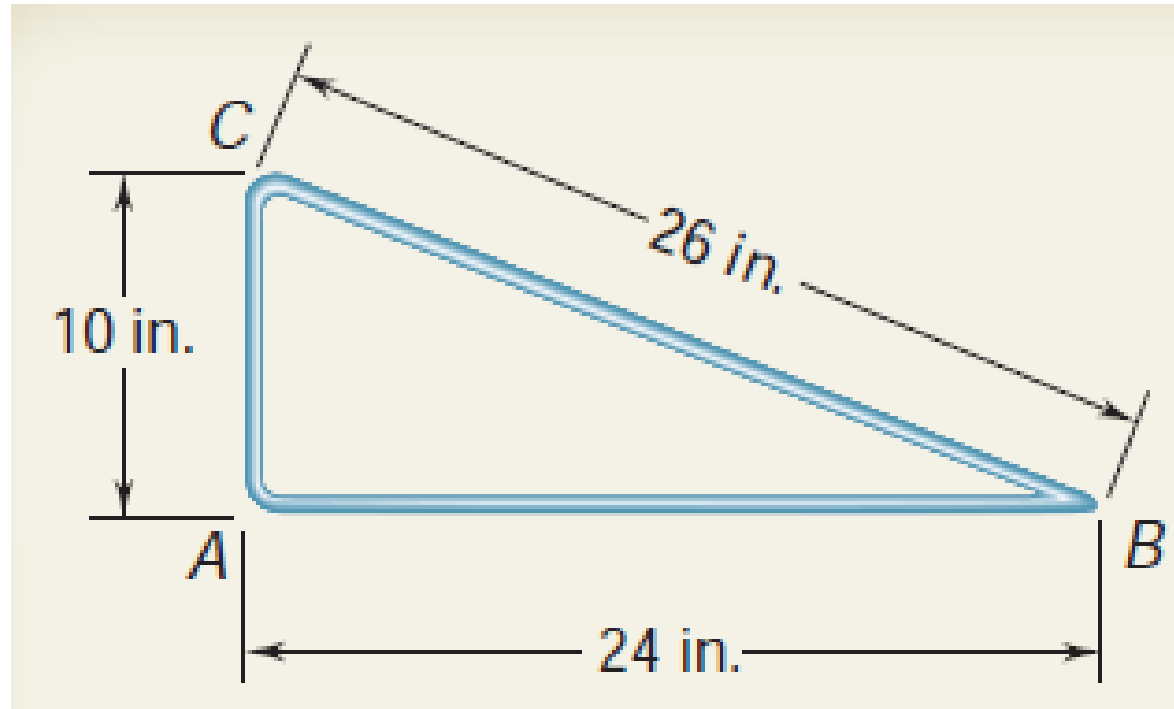
$$\bar{Y} = 36.6 \text{ (mm)}$$



# Distributed Forces: Centroids and Centers of Gravity

## □ Sample Problem 02

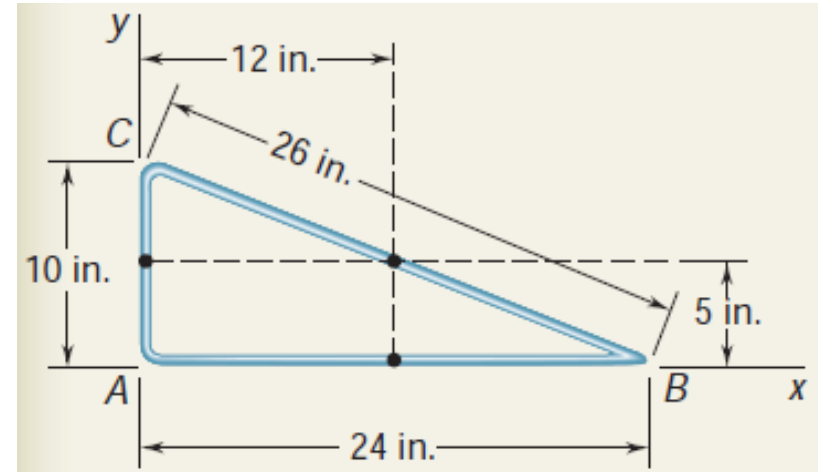
The figure shown is made from a piece of thin, homogeneous wire. Determine the location of its center of gravity.



# Distributed Forces: Centroids and Centers of Gravity

## □ Sample Problem 02

SOLUTION:



Segment	$L$ , in.	$\bar{x}$ , in.	$\bar{y}$ , in.	$\bar{x}L$ , in <sup>2</sup>	$\bar{y}L$ , in <sup>2</sup>
AB					
BC					
CA					

$$\bar{X} = 10 \text{ (in.)}$$

$$\bar{Y} = 3 \text{ (in.)}$$

# Distributed Forces: Centroids and Centers of Gravity

## □ Determination of Centroids by Integration

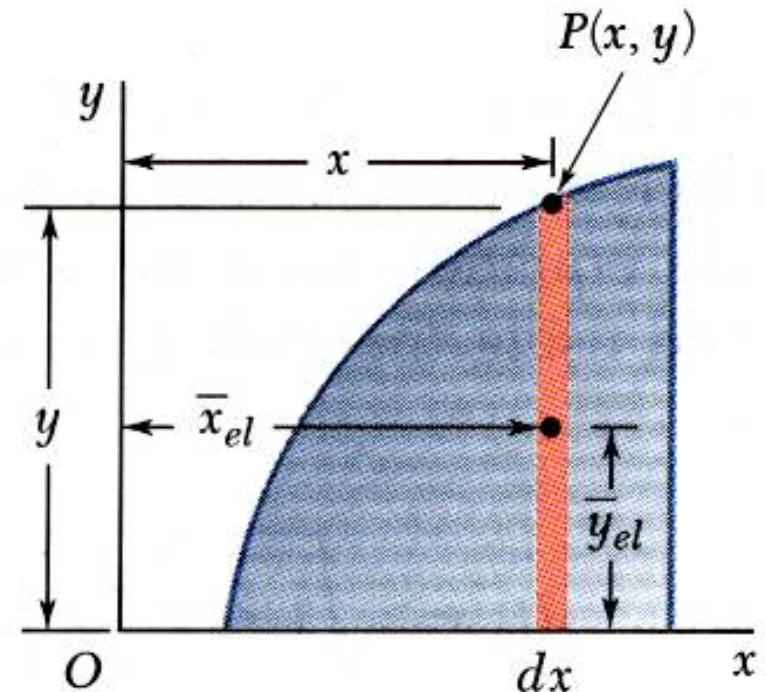
- Double integration to find the first moment may be avoided by defining  $dA$  as a thin rectangle or strip.

$$Q_y = \bar{x}A = \int x dA \Rightarrow Q_y = \iint x dx dy \quad \text{or} \quad Q_y = \int \bar{x}_{el} dA$$

$$Q_x = \bar{y}A = \int y dA \Rightarrow Q_x = \iint y dx dy \quad \text{or} \quad Q_x = \int \bar{y}_{el} dA$$

$$Q_y = \bar{x}A = \int \bar{x}_{el} dA \Rightarrow Q_y = \int x (y dx)$$

$$Q_x = \bar{y}A = \int \bar{y}_{el} dA \Rightarrow Q_x = \int \frac{y}{2} (y dx)$$

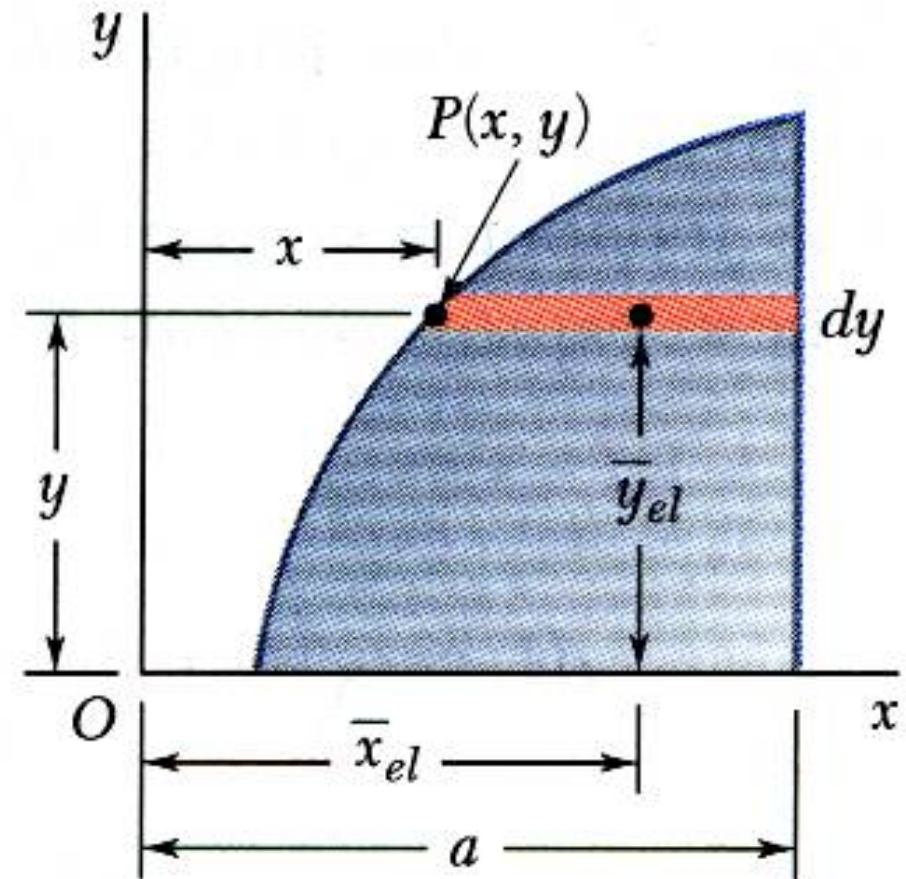


# Distributed Forces: Centroids and Centers of Gravity

## □ Determination of Centroids by Integration

$$Q_y = \bar{x}A = \int \bar{x}_{el} dA \Rightarrow Q_y = \int \frac{a+x}{2} [(a-x) dy]$$

$$Q_x = \bar{y}A = \int \bar{y}_{el} dA \Rightarrow Q_x = \int y [(a-x) dy]$$

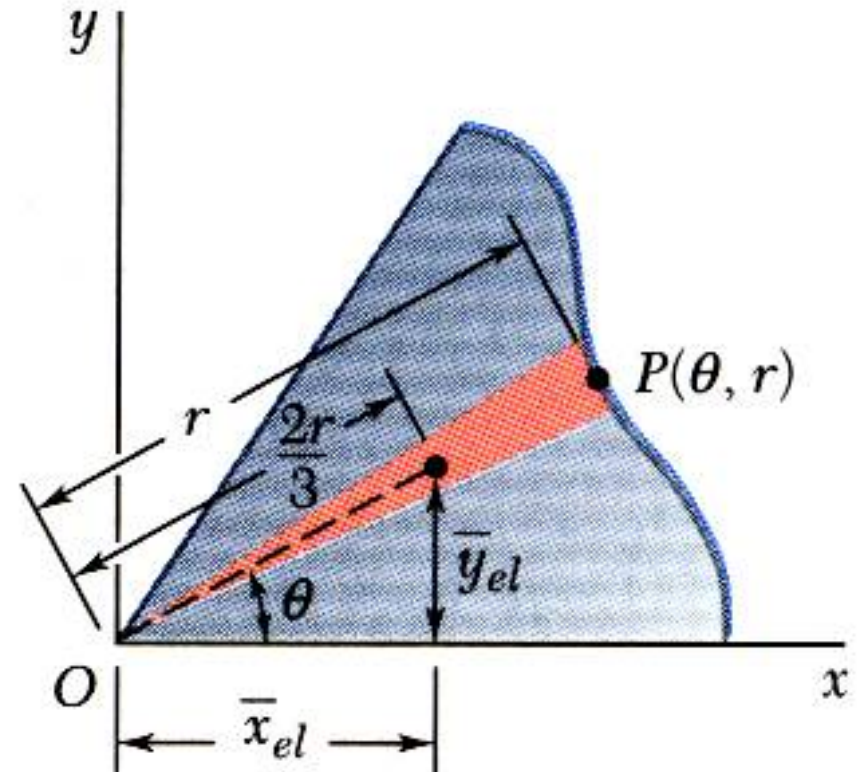


# Distributed Forces: Centroids and Centers of Gravity

## □ Determination of Centroids by Integration

$$Q_y = \bar{x}A = \int \bar{x}_{el} dA \Rightarrow Q_y = \int \frac{2r}{3} \cos \theta \left( \frac{1}{2} r^2 d\theta \right)$$

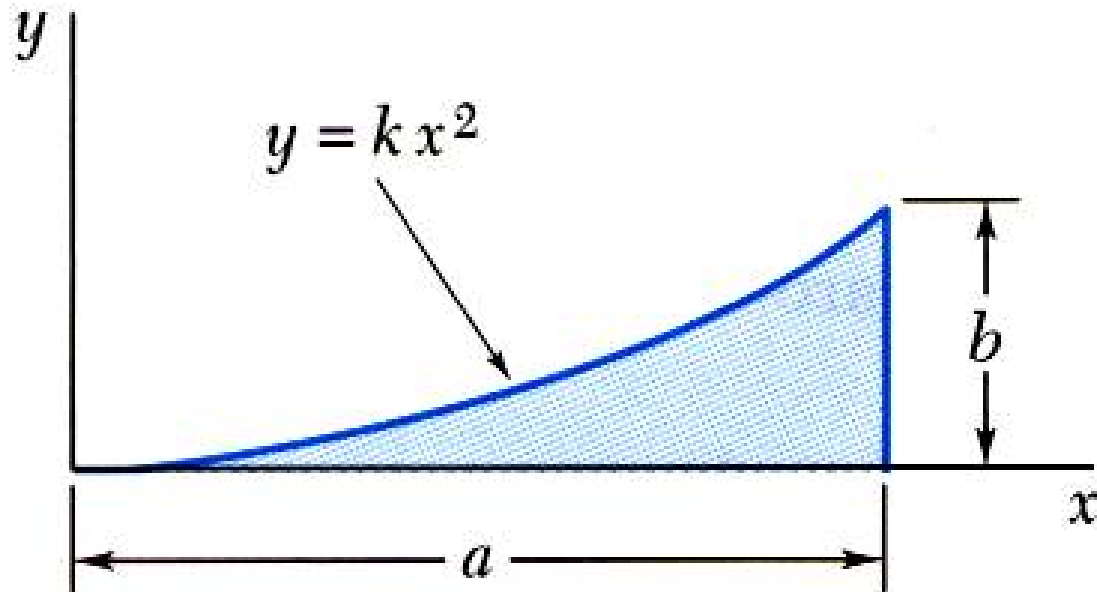
$$Q_x = \bar{y}A = \int \bar{y}_{el} dA \Rightarrow Q_x = \int \frac{2r}{3} \sin \theta \left( \frac{1}{2} r^2 d\theta \right)$$



# Distributed Forces: Centroids and Centers of Gravity

## □ Sample Problem 03

Determine by direct integration the location of the centroid of a parabolic spandrel.



# Distributed Forces: Centroids and Centers of Gravity

## □ Sample Problem 03

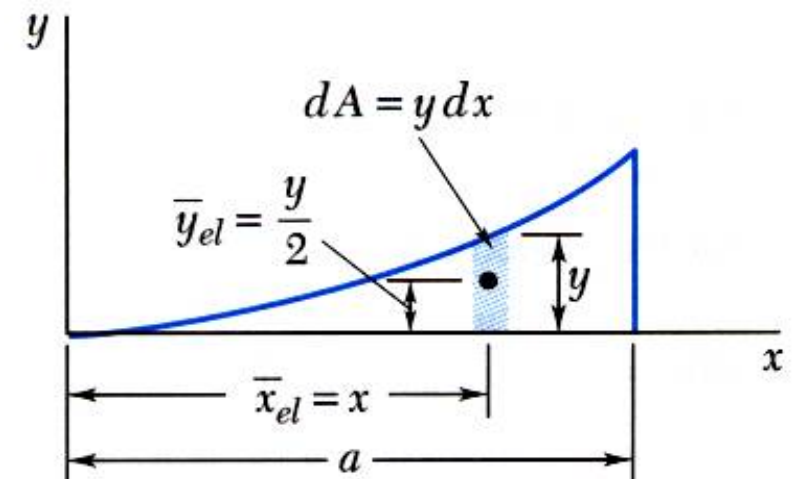
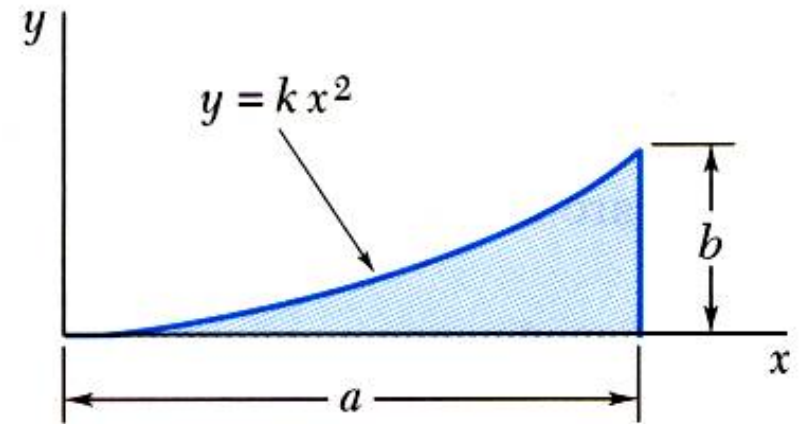
SOLUTION:

- Determine the constant  $k$ .

$$\Rightarrow y = \frac{b}{a^2} x^2 \quad \text{or} \quad x = \frac{a}{b^{1/2}} y^{1/2}$$

- Evaluate the total area.

$$A = \frac{ab}{3}$$



# Distributed Forces: Centroids and Centers of Gravity

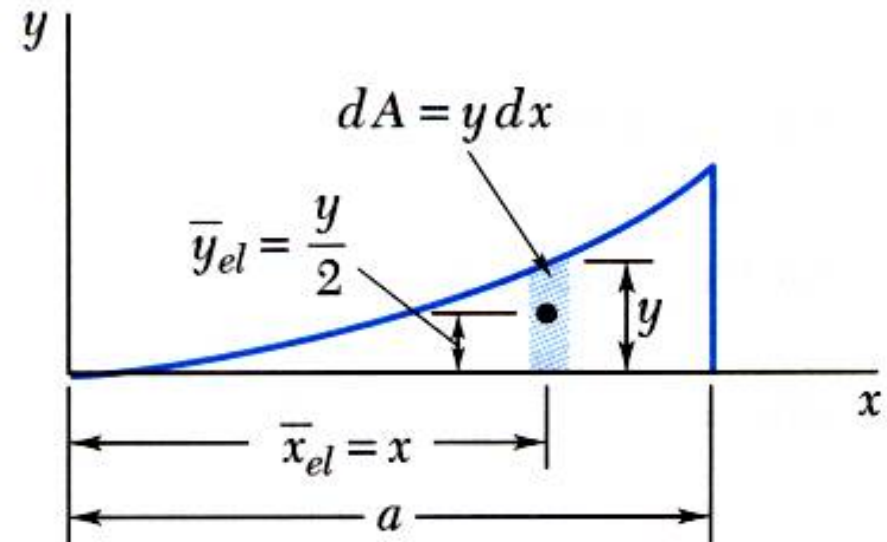
## □ Sample Problem 03

SOLUTION:

- Using vertical strips, perform a single integration to find the first moments.

$$Q_y = \frac{a^2 b}{4}$$

$$Q_x = \frac{ab^2}{10}$$



# Distributed Forces: Centroids and Centers of Gravity

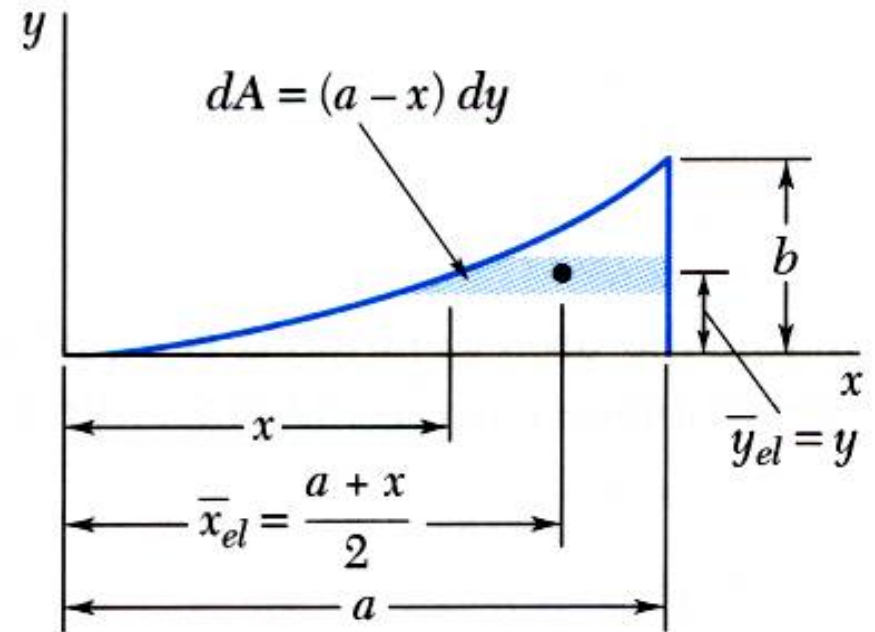
## □ Sample Problem 03

SOLUTION:

- Or, using horizontal strips, perform a single integration to find the first moments.

$$Q_y = \frac{a^2 b}{4}$$

$$Q_x = \frac{ab^2}{10}$$



# Distributed Forces: Centroids and Centers of Gravity

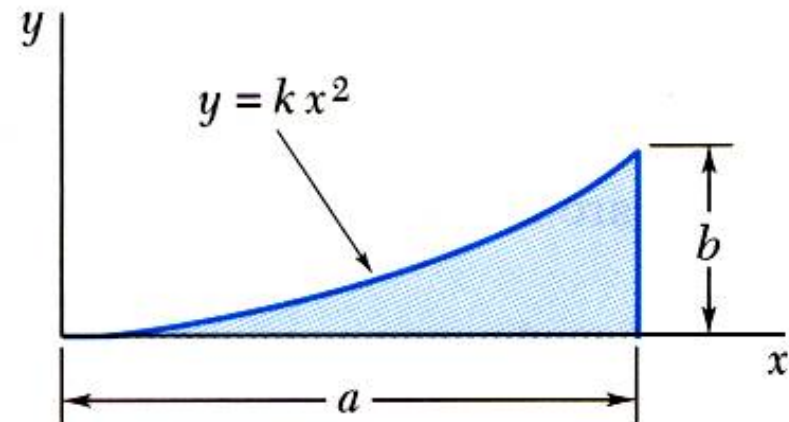
## □ Sample Problem 03

SOLUTION:

- Evaluate the centroid coordinates.

$$\bar{x} = \frac{3}{4}a$$

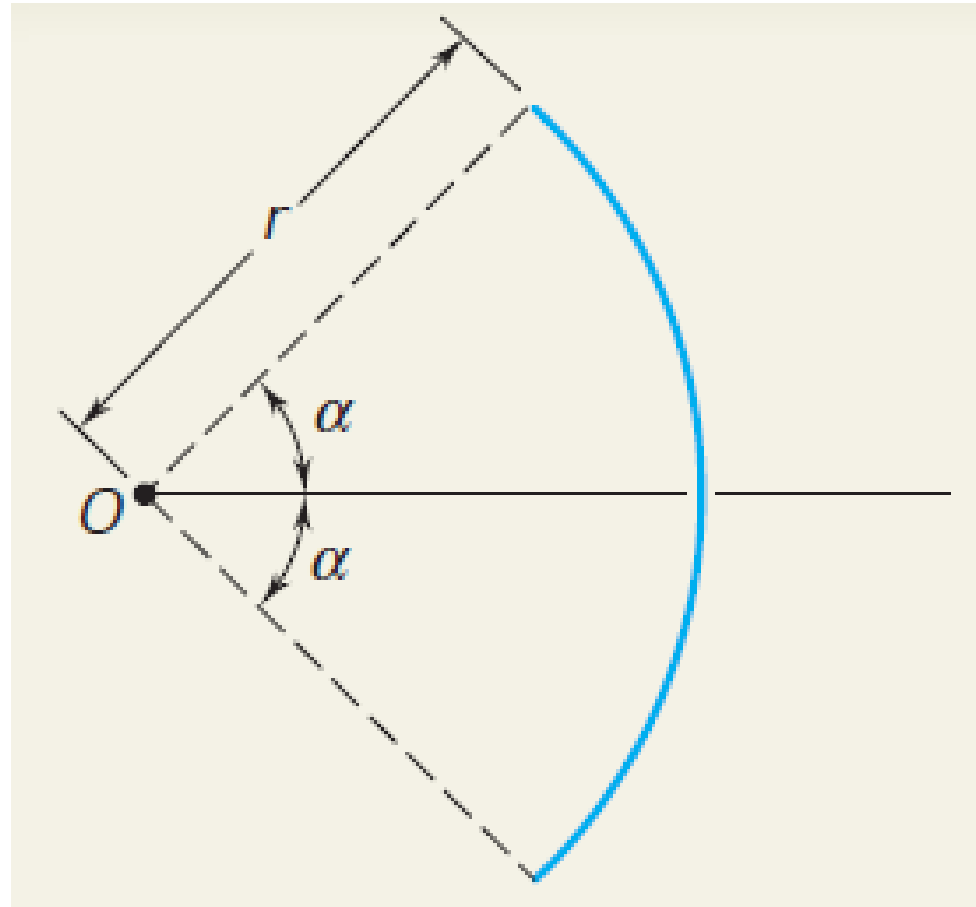
$$\bar{y} = \frac{3}{10}b$$



# Distributed Forces: Centroids and Centers of Gravity

## □ Sample Problem 04

Determine the location of the centroid of the circular arc shown.



# Distributed Forces: Centroids and Centers of Gravity

## □ Sample Problem 04

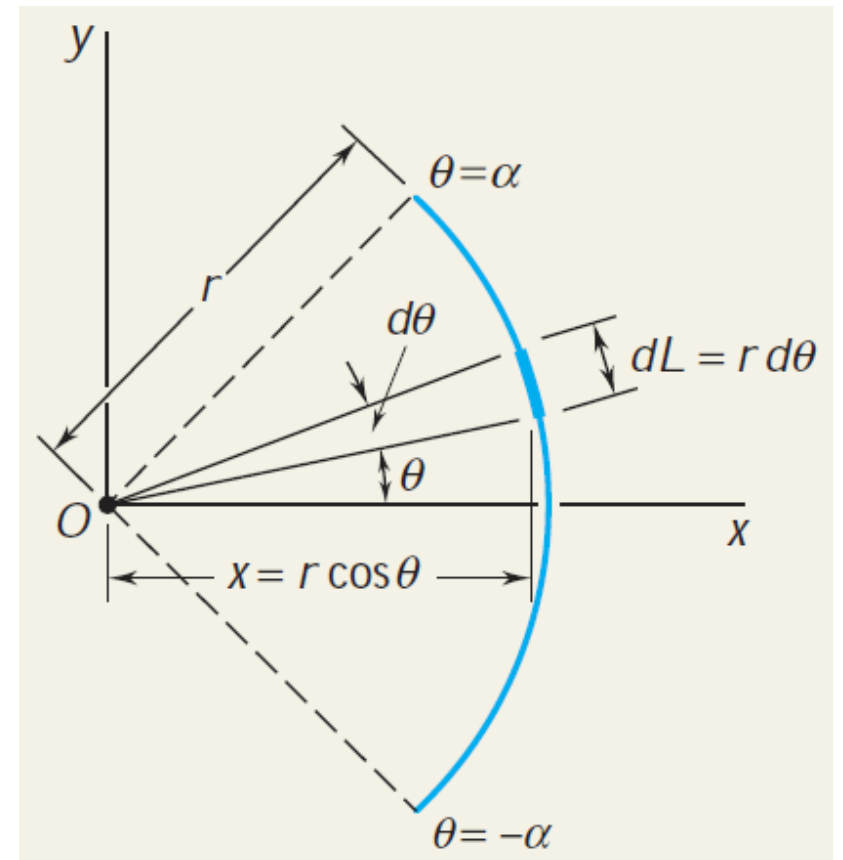
SOLUTION:

- Since the arc is symmetrical with respect to the x axis,  $\bar{y} = 0$ . A differential element is chosen as shown, and the length of the arc  $L$  determined by integration

$$L = 2r\alpha$$

$$\Rightarrow Q_y = 2r^2 \sin \alpha$$

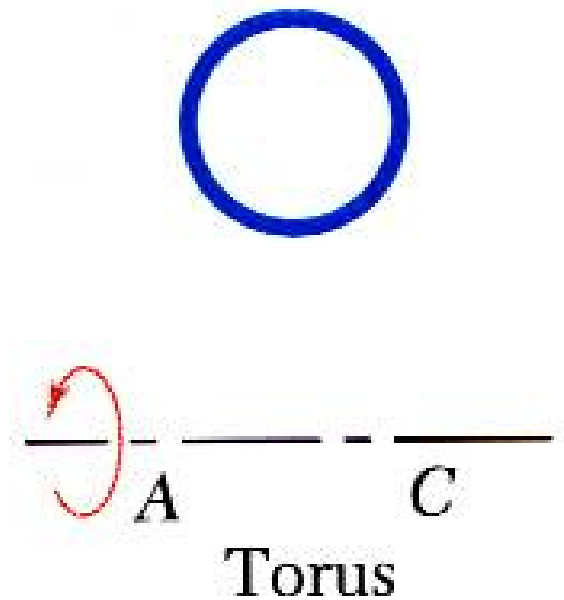
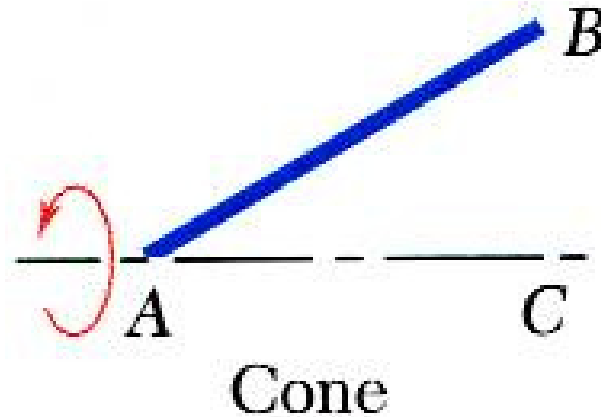
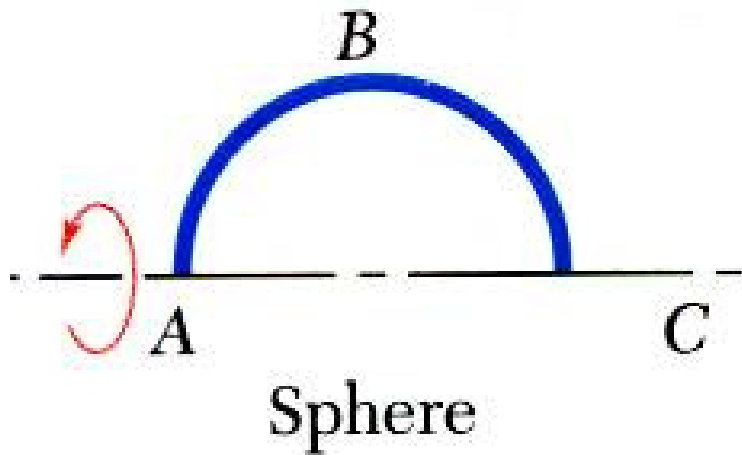
$$\bar{x} = \frac{r \sin \alpha}{\alpha}$$



# Distributed Forces: Centroids and Centers of Gravity

## □ Theorems of Pappus-Guldinus

- *Surface of revolution* is generated by rotating a plane curve about a fixed axis.



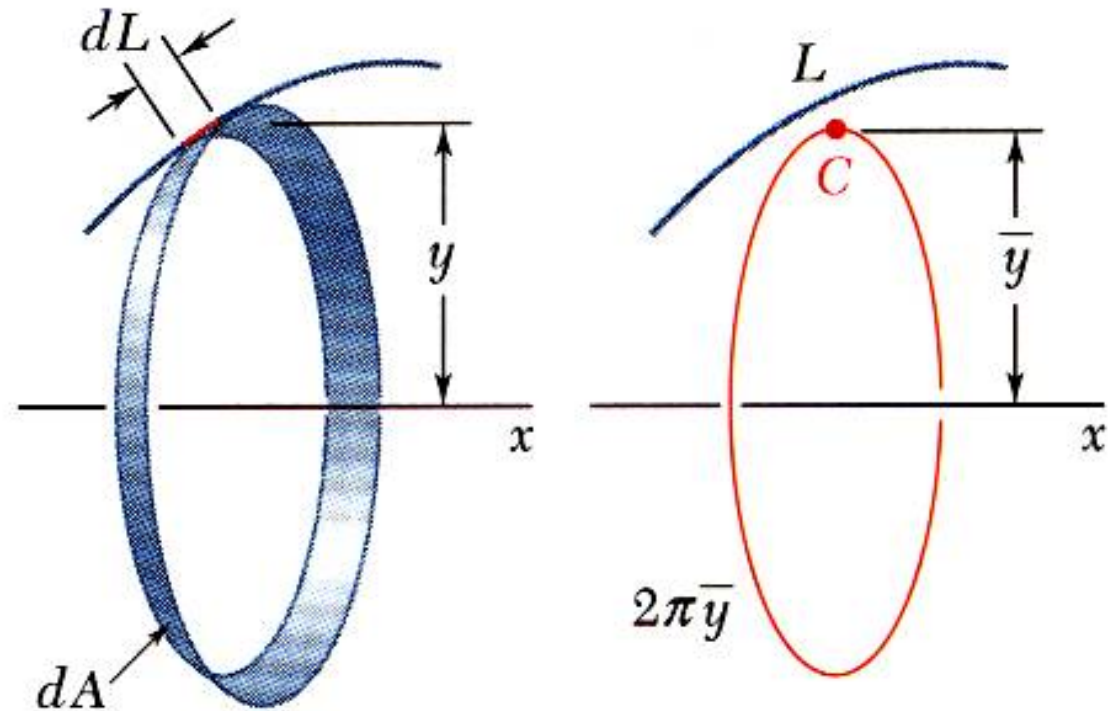
# Distributed Forces: Centroids and Centers of Gravity

## □ Theorems of Pappus-Guldinus

- *Area of a surface of revolution is equal to the length of the generating curve times the distance traveled by the centroid through the rotation.*

$$dA = 2\pi y dL \Rightarrow A = 2\pi \int y dL$$

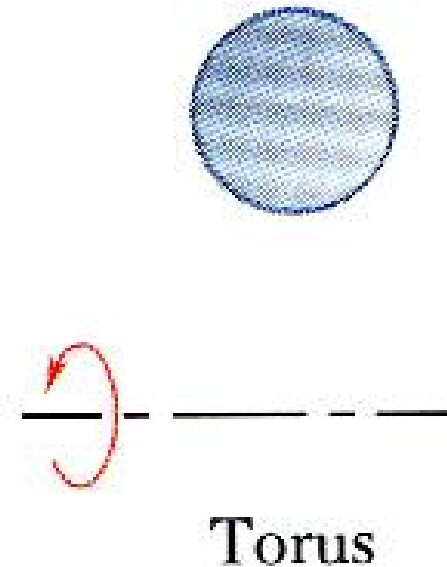
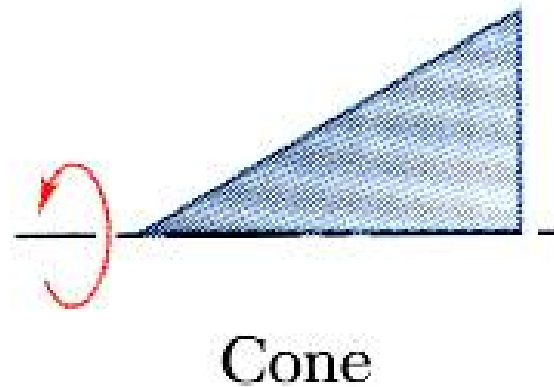
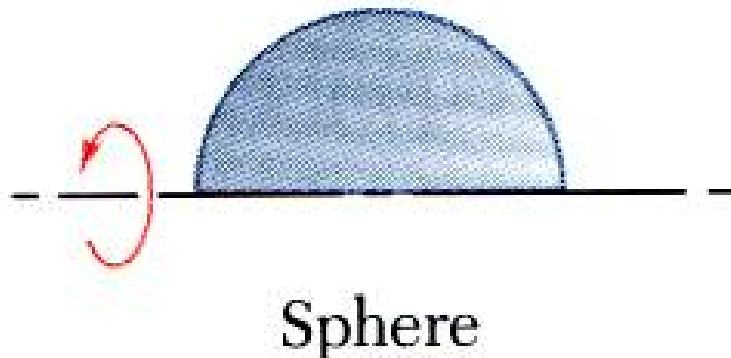
$$\Rightarrow A = 2\pi \bar{y}L$$



# Distributed Forces: Centroids and Centers of Gravity

## □ Theorems of Pappus-Guldinus

- Body of revolution is generated by rotating a plane area about a fixed axis.



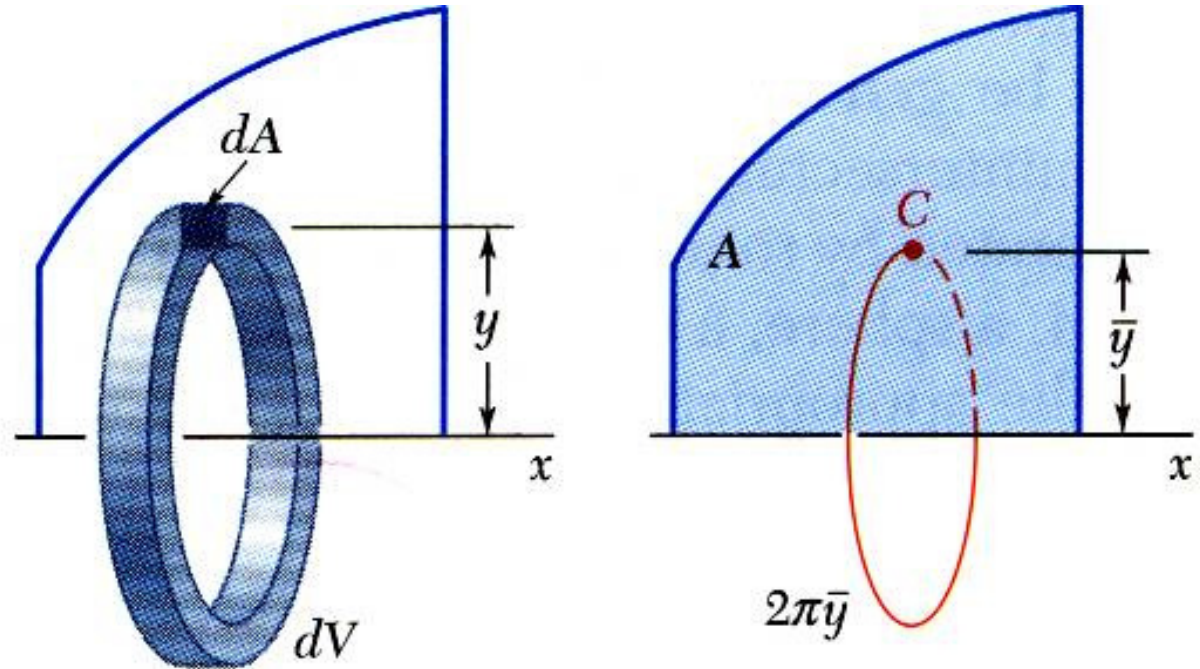
# Distributed Forces: Centroids and Centers of Gravity

## □ Theorems of Pappus-Guldinus

- *Volume of a body of revolution is equal to the generating area times the distance traveled by the centroid through the rotation.*

$$dV = 2\pi y dA \Rightarrow V = 2\pi \int y dA$$

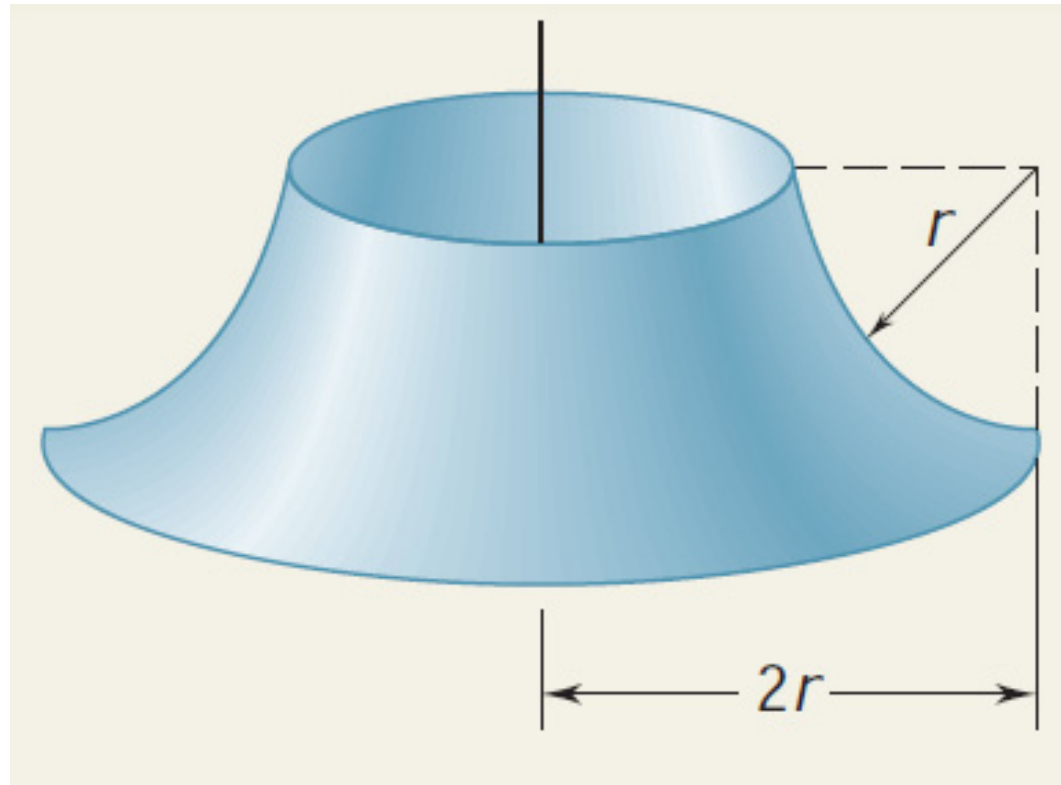
$$\Rightarrow V = 2\pi \bar{y} A$$



# Distributed Forces: Centroids and Centers of Gravity

## □ Sample Problem 05

Determine the area of the surface of revolution shown, which is obtained by rotating a quarter-circular arc about a vertical axis.



# Distributed Forces: Centroids and Centers of Gravity

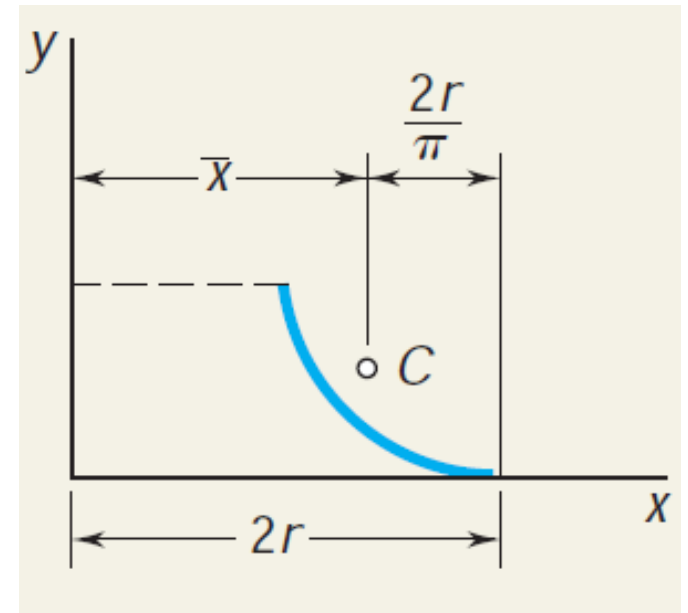
## □ Sample Problem 05

SOLUTION:

According to Theorem I of Pappu -Guldinus, the area generated is equal to the product of the length of the arc and the distance traveled by its centroid.

$$\bar{x} = 2r \left( 1 - \frac{1}{\pi} \right)$$

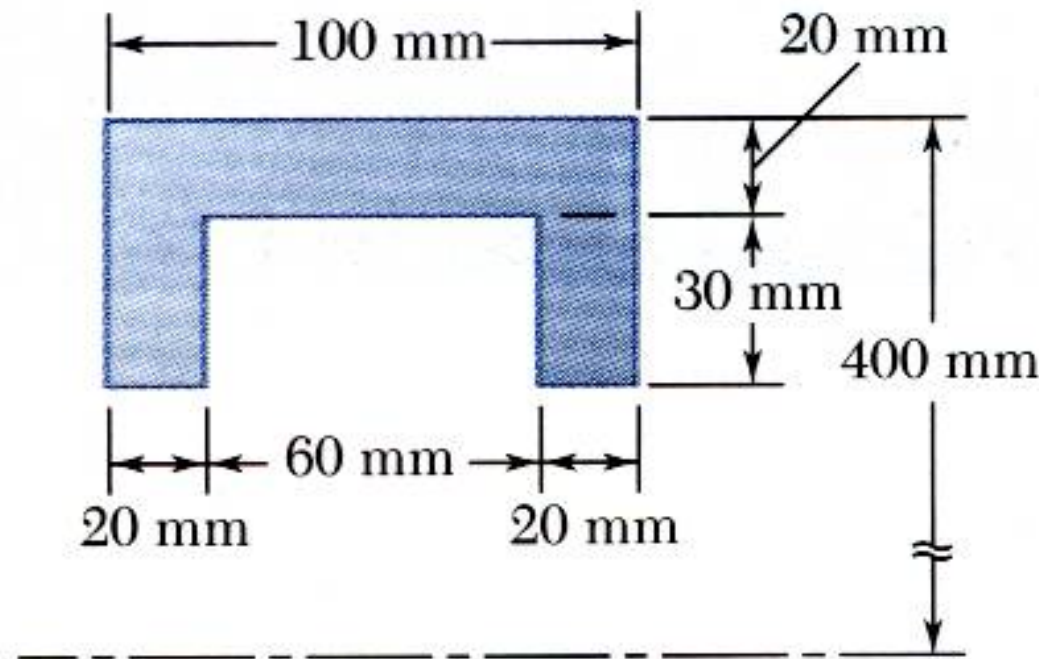
$$A = 2\pi r^2 (\pi - 1)$$



# Distributed Forces: Centroids and Centers of Gravity

## □ Sample Problem 06

The outside diameter of a pulley is 0.8 m, and the cross section of its rim is as shown. Knowing that the pulley is made of steel and that the density of steel is  $\rho = 7.85 \times 10^3 \text{ kg/m}^3$ , determine the mass and weight of the rim.

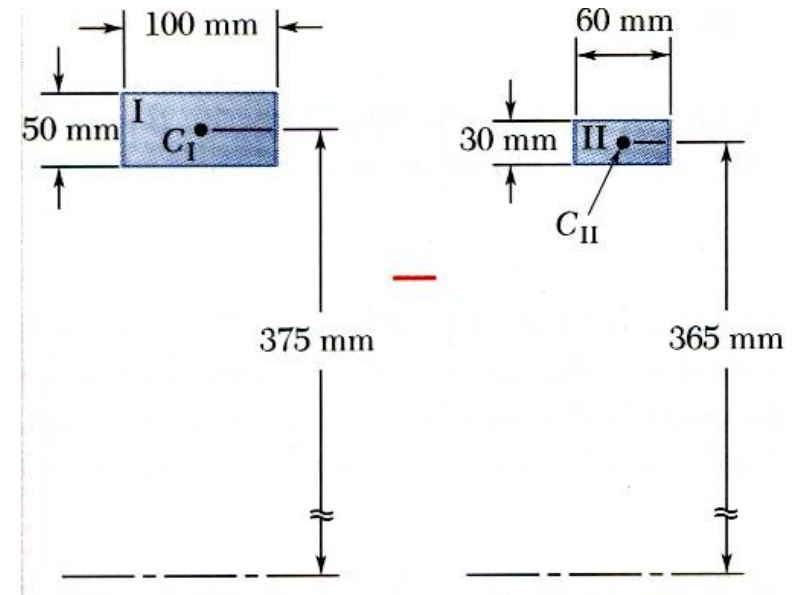


# Distributed Forces: Centroids and Centers of Gravity

## □ Sample Problem 06

SOLUTION:

- Apply the theorem of Pappus-Guldinus to evaluate the volumes or revolution for the rectangular rim section and the inner cutout section.
- Multiply by density and acceleration to get the mass and weight.



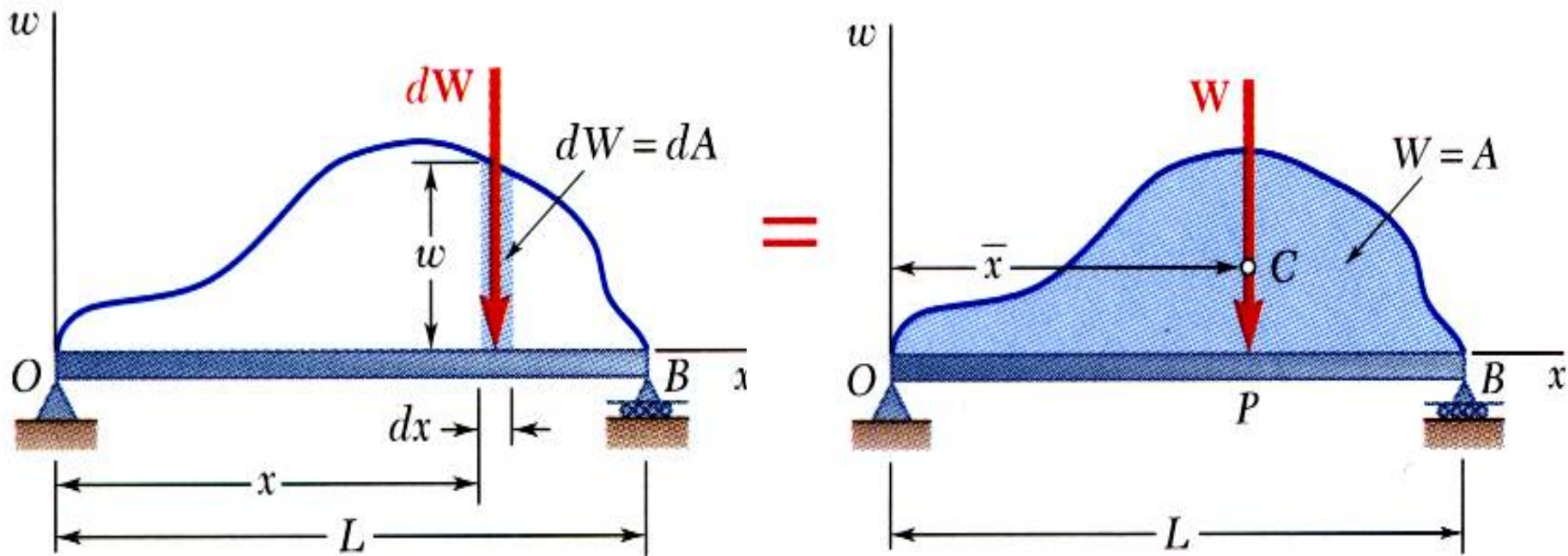
	Area, mm <sup>2</sup>	$\bar{y}$ , mm	Distance Traveled by $C$ , mm	Volume, mm <sup>3</sup>
I				
II				

$$m = 60.0 \text{ kg}$$

$$W = 589 \text{ N}$$

# Distributed Forces: Centroids and Centers of Gravity

## □ Distributed Loads on Beams



- A distributed load is represented by plotting the load per unit length,  $w$  (N/m). The total load is equal to the area under the load curve.

$$W = \int_0^L w dx = \int dA \Rightarrow \boxed{W = A}$$

- ***A distributed load can be replaced by a concentrated load with a magnitude equal to the area under the load curve and a line of action passing through the area centroid.***

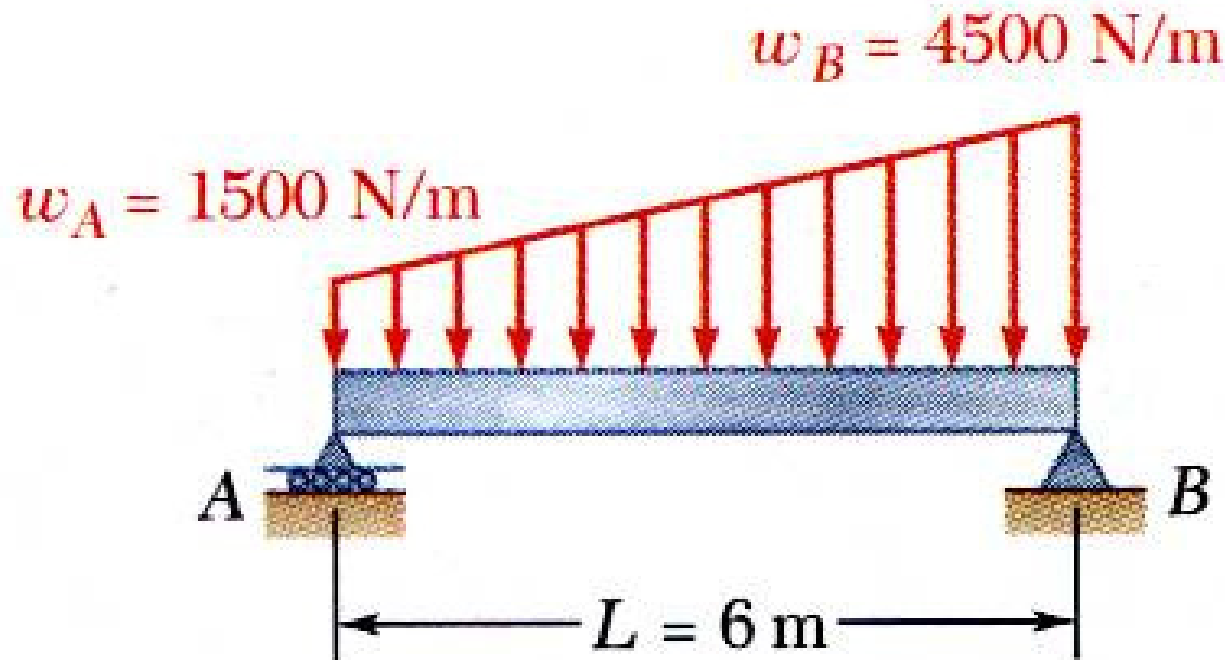
$$(OP)W = \int x dW \Rightarrow$$

$$(OP)A = \int_0^L x dA = \bar{x}A \Rightarrow \boxed{(OP) = \bar{x}}$$

# Distributed Forces: Centroids and Centers of Gravity

## □ Sample Problem 07

A beam supports a distributed load as shown. Determine the equivalent concentrated load and the reactions at the supports.



# Distributed Forces: Centroids and Centers of Gravity

## □ Sample Problem 07

SOLUTION:

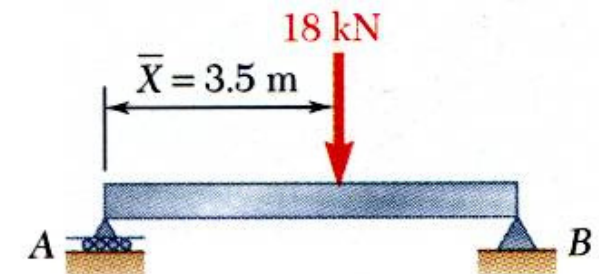
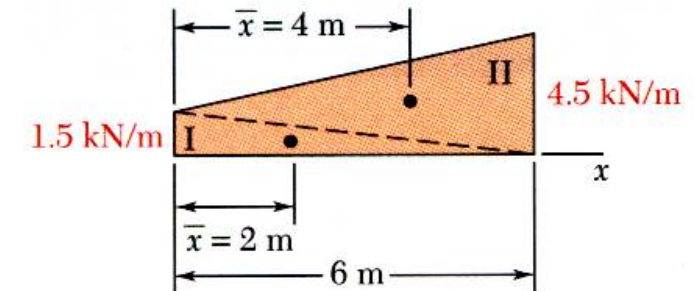
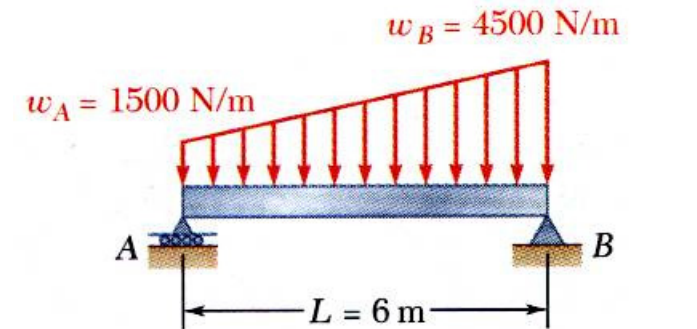
Component	A, kN	$\bar{x}$ , m	$\bar{x}A$ , kN·m
Triangle I			
Triangle II			

- The magnitude of the concentrated load is equal to the total load or the area under the curve.

$$F = 18.0 \text{ kN}$$

- The line of action of the concentrated load passes through the centroid of the area under the curve.

$$\bar{X} = 3.5 \text{ m}$$



# Distributed Forces: Centroids and Centers of Gravity

## □ Sample Problem 07

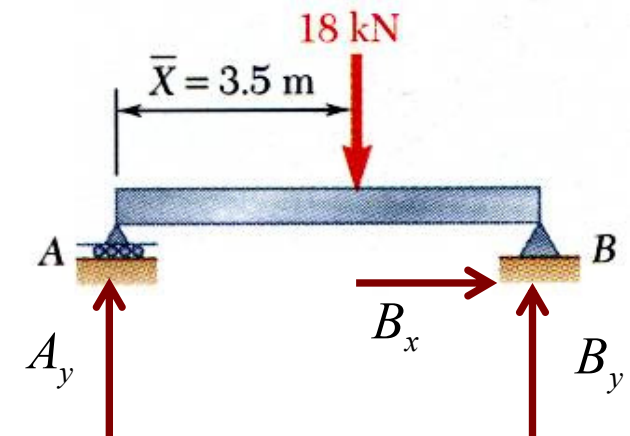
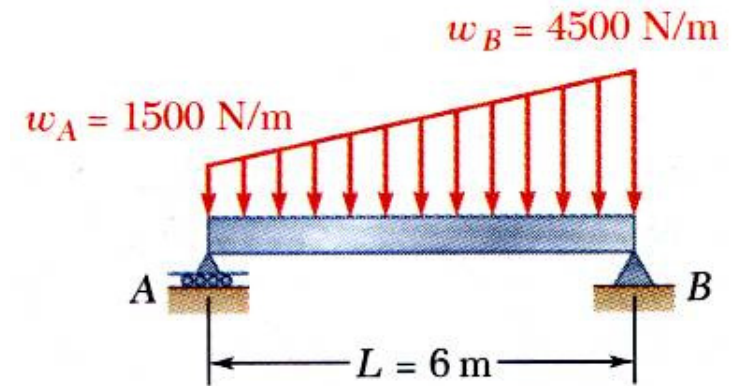
SOLUTION:

- Determine the support reactions by summing moments about the beam ends.

$$B_x = 0$$

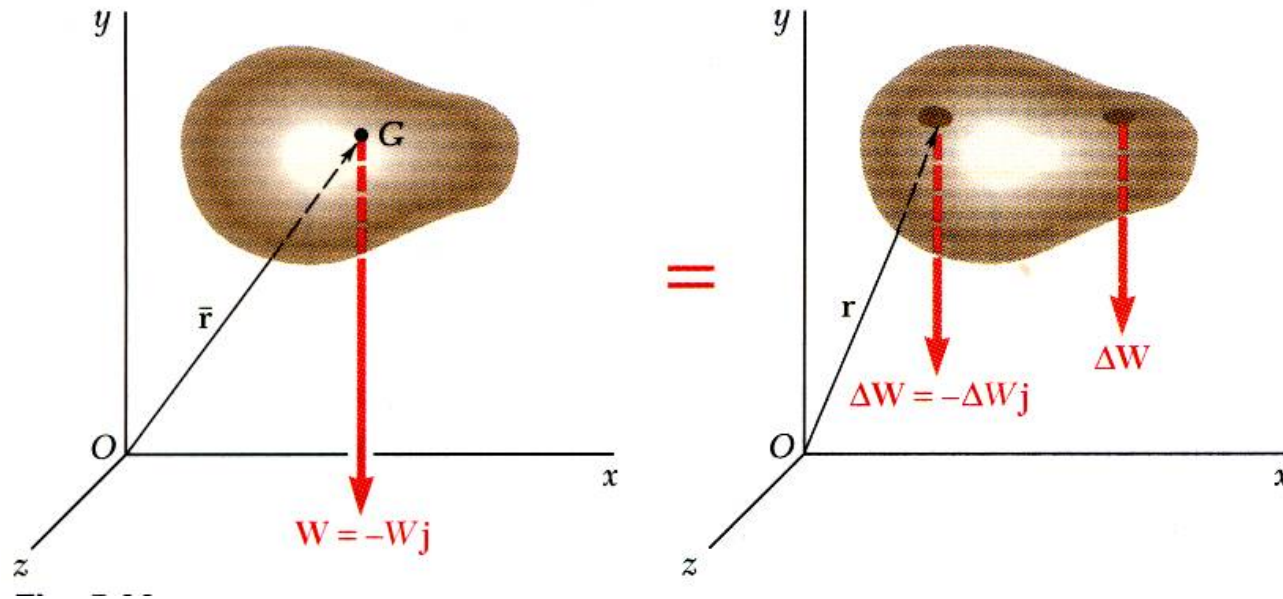
$$\Rightarrow B_y = 10.5 \text{ (kN)}$$

$$\Rightarrow A_y = 7.5 \text{ (kN)}$$



# Distributed Forces: Centroids and Centers of Gravity

## □ Center of Gravity of a 3D Body: Centroid of a Volume



- Center of gravity  $G$

$$-W\vec{j} = \sum (-\Delta W\vec{j})$$

$$\vec{r}_G \times (-W\vec{j}) = \sum [\vec{r} \times (-\Delta W\vec{j})]$$

$$\vec{r}_G W \times (-\vec{j}) = (\sum \vec{r} \Delta W) \times (-\vec{j})$$

$$W = \int dW \Rightarrow \boxed{\vec{r}_G W = \int \vec{r} dW}$$

- Results are independent of body orientation,

$$\boxed{\bar{x}W = \int x dW \quad \bar{y}W = \int y dW \quad \bar{z}W = \int z dW}$$

- For homogeneous bodies,

$$W = \gamma V \text{ and } dW = \gamma dV \Rightarrow$$

$$\boxed{\bar{x}V = \int x dV \quad \bar{y}V = \int y dV \quad \bar{z}V = \int z dV}$$

# Distributed Forces: Centroids and Centers of Gravity

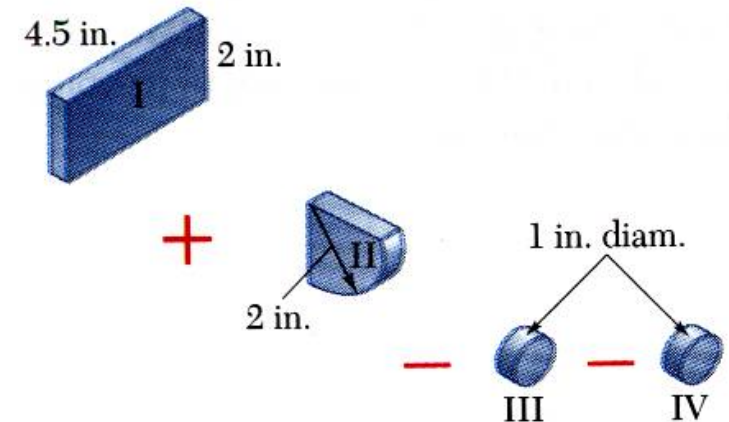
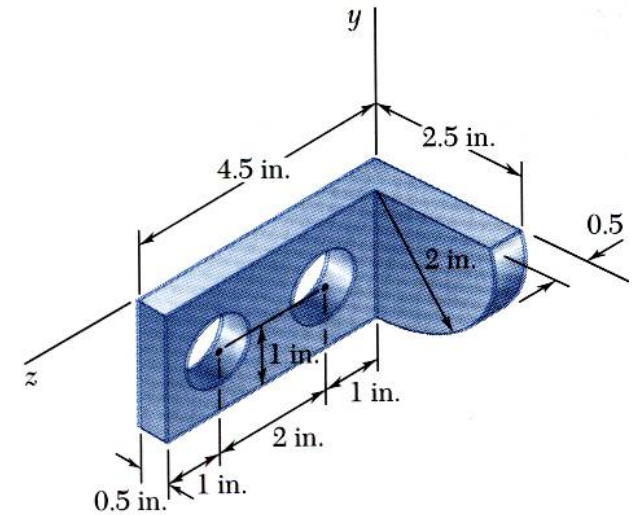
## □ Composite 3D Bodies

- Moment of the total weight concentrated at the center of gravity  $G$  is equal to the sum of the moments of the weights of the component parts.

$$\bar{X} \sum W = \sum \bar{x} W \quad \bar{Y} \sum W = \sum \bar{y} W \quad \bar{Z} \sum W = \sum \bar{z} W$$

- For homogeneous bodies,

$$\bar{X} \sum V = \sum \bar{x} V \quad \bar{Y} \sum V = \sum \bar{y} V \quad \bar{Z} \sum V = \sum \bar{z} V$$

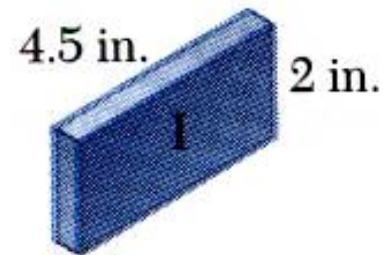
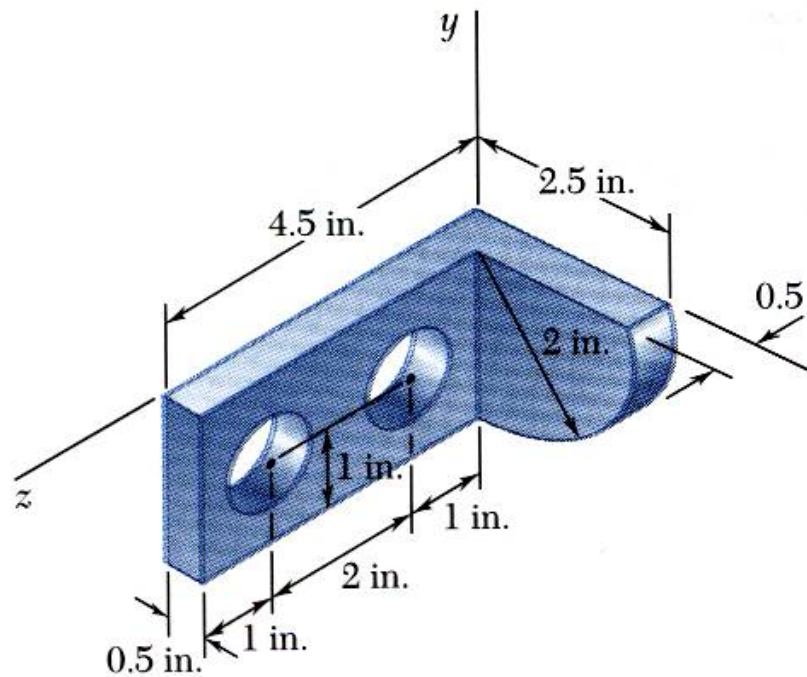




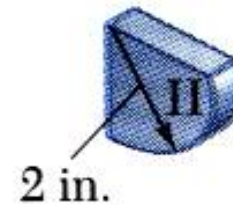
# Distributed Forces: Centroids and Centers of Gravity

## □ Sample Problem 08

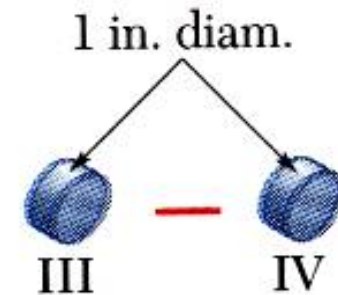
SOLUTION:



+



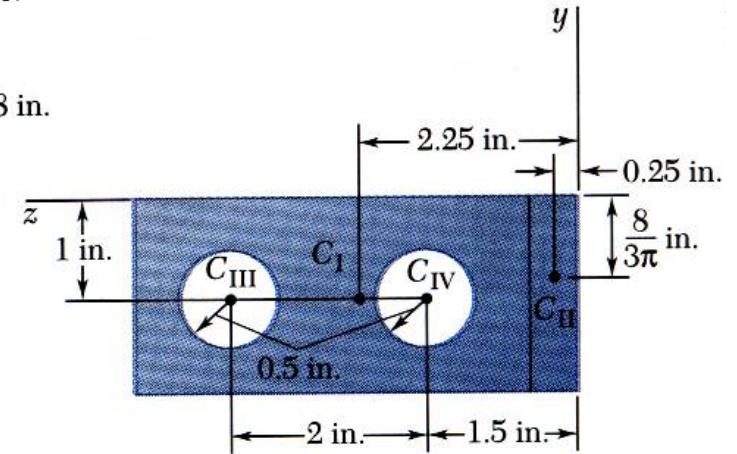
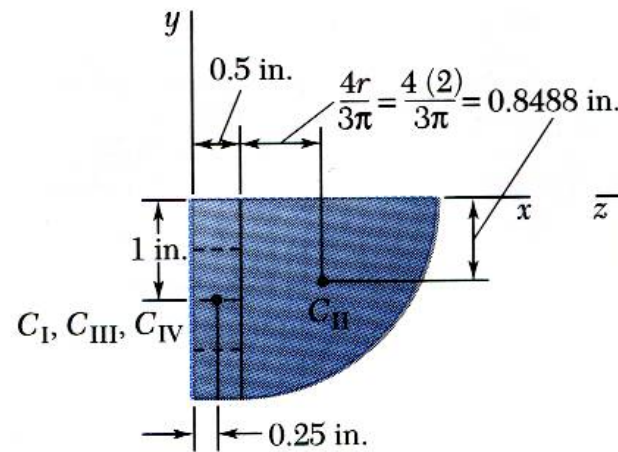
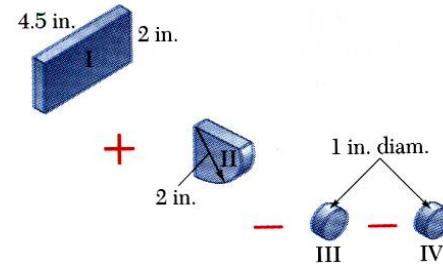
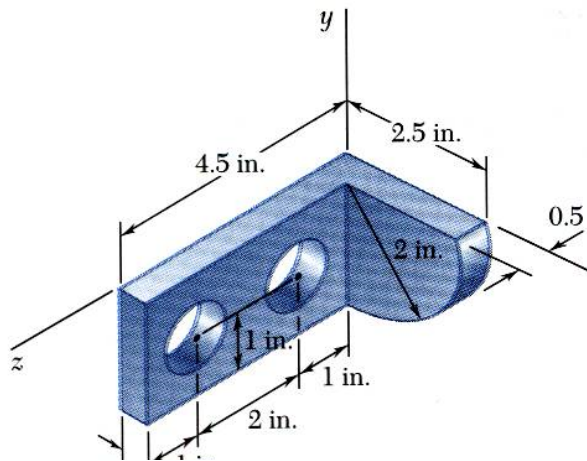
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# Distributed Forces: Centroids and Centers of Gravity

## □ Sample Problem 08

SOLUTION:



	$V, \text{in}^3$	$\bar{x}, \text{in.}$	$\bar{y}, \text{in.}$	$\bar{z}, \text{in.}$	$\bar{x}V, \text{in}^4$	$\bar{y}V, \text{in}^4$	$\bar{z}V, \text{in}^4$
I							
II							
III							
IV							

# Distributed Forces: Centroids and Centers of Gravity

## □ Sample Problem 08

SOLUTION:

	$V, \text{in}^3$	$\bar{x}, \text{in.}$	$\bar{y}, \text{in.}$	$\bar{z}, \text{in.}$	$\bar{x}V, \text{in}^4$	$\bar{y}V, \text{in}^4$	$\bar{z}V, \text{in}^4$
I	$(4.5)(2)(0.5) = 4.5$	0.25	-1	2.25	1.125	-4.5	10.125
II	$\frac{1}{4}\pi(2)^2(0.5) = 1.571$	1.3488	-0.8488	0.25	2.119	-1.333	0.393
III	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	-1	3.5	-0.098	0.393	-1.374
IV	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	-1	1.5	-0.098	0.393	-0.589
	$\Sigma V = 5.286$				$\Sigma \bar{x}V = 3.048$	$\Sigma \bar{y}V = -5.047$	$\Sigma \bar{z}V = 8.555$

$$\bar{X} = 0.577 \text{ in.}$$

$$\bar{Y} = 0.577 \text{ in.}$$

$$\bar{Z} = 0.577 \text{ in.}$$