



دانشگاه کردستان
University of Kurdistan
زانکۆی کوردستان

Nonlinear Analysis of Structures

Geometrically Nonlinear Analysis of Plane Frames

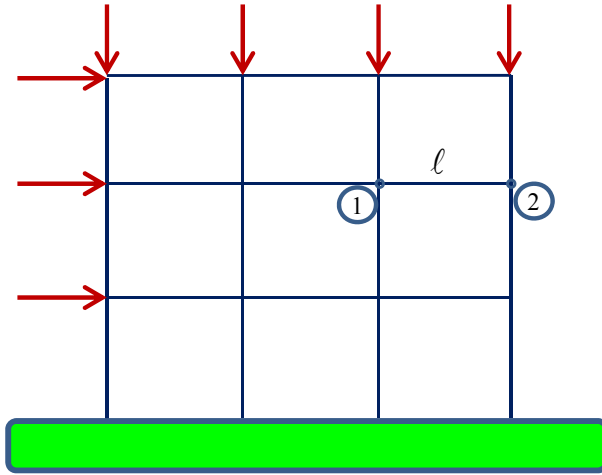
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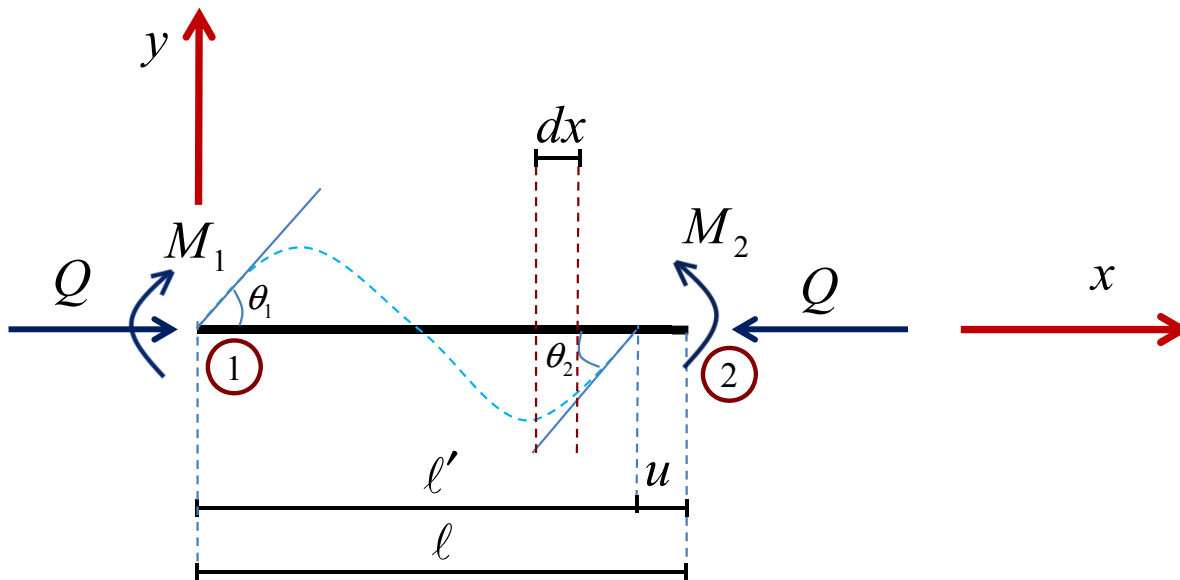
Geometrically Nonlinear Analysis of Plane Frames

□ Member, Force-Deformation Relationship in local Coordinate



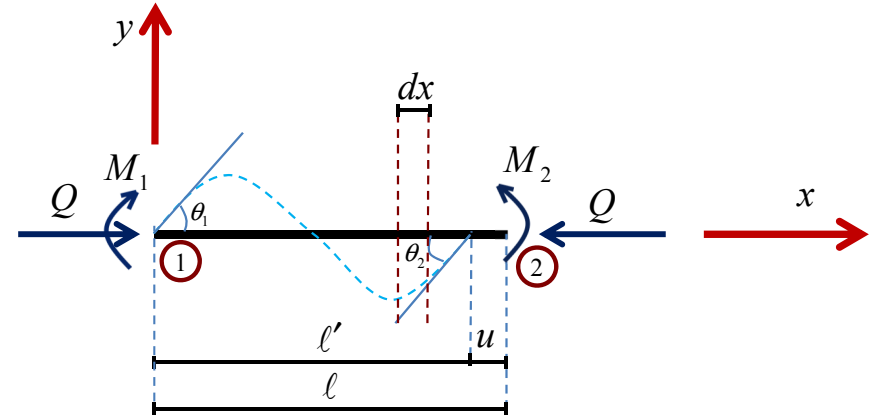
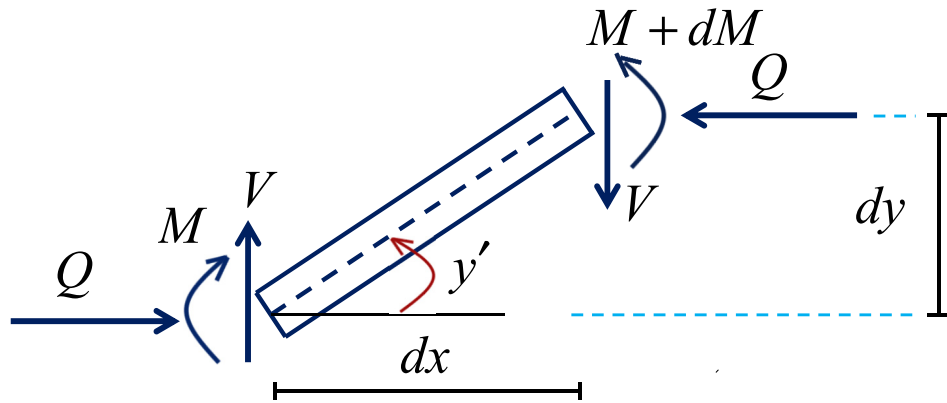
فرضیات این روش:

- سیستم Eulerian
- بار فقط به گره وارد می شود.
- افزایش بار همزمان انجام می شود.
- l فاصله مرکز تا مرکز است.
- l' طول محور عضو



Geometrically Nonlinear Analysis of Plane Frames

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$$\sum M = 0 \Rightarrow V = \frac{dM}{dx} + Q \frac{dy}{dx} \quad (1)$$

چون نیرو در گره‌ها وارد می‌شود بنابراین برش ثابت است.

$$V = cte \Rightarrow \frac{dV}{dx} = 0 \stackrel{(1)}{\Rightarrow} \frac{d^2 M}{dx^2} + Q \frac{d^2 y}{dx^2} = 0 \quad (2)$$

Geometrically Nonlinear Analysis of Plane Frames

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با فرض قابل توجه بودن شیب، رابطه کلی شعاع انحنا و لنگر به صورت زیر است (انحنای بزرگ)

$$\frac{M}{EI} = \frac{1}{\rho} = \frac{\frac{d^2 y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}$$

با فرض انحنای کوچک

$$\left(\frac{dy}{dx}\right)^2 \approx 0 \Rightarrow \frac{M}{EI} = \frac{d^2 y}{dx^2} \quad (3)$$

(2), (3) با فرض انحنای کوچک، EI ثابت و $\Rightarrow EI \frac{d^4 y}{dx^4} + Q \frac{d^2 y}{dx^2} = 0 \quad (4)$

رابطه (4) یک معادله دیفرانسیل مرتبه چهار است که به صورت زیر ساده می شود

$$\Rightarrow y'''' + k^2 y'' = 0, \quad \frac{Q}{EI} = k^2 \quad (4)$$

Geometrically Nonlinear Analysis of Plane Frames

□ Member, Force-Deformation Relationship in local Coordinate

حل معادله دیفرانسیل مرتبه چهارم جوابی به صورت زیر دارد

$$y = A \sin(kx) + B \cos(kx) + Cx + D \quad (5)$$

$$\begin{aligned} @ x = 0 \quad , \quad y = 0 \quad , \quad y' = \theta_1 \quad , \quad y'' = -\frac{M_1}{EI} \\ @ x = \ell' \quad , \quad y = 0 \quad , \quad y' = \theta_2 \quad , \quad y'' = -\frac{M_2}{EI} \end{aligned} \quad (6)$$

برای تعیین ثابت‌ها در رابطه (5) شرایط مرزی باید لحاظ گردد.

با جایگذاری رابطه (6) در (5) روابط نیرو-تغییرشکل در دستگاه مختصات محلی به دست می‌آید

$$M_1 = \frac{EI}{\ell'} [C_1 \theta_1 + C_2 \theta_2] \quad (7a)$$

$$M_2 = \frac{EI}{\ell'} [C_2 \theta_1 + C_1 \theta_2] \quad (7b)$$

$$Q = \frac{AE}{\ell} [u - \delta_b] = AE \left(\frac{u}{\ell} - C_b \right) \quad (7c)$$

u کوتاه شدگی در اثر نیروی محوری

(Axial deformation due to bowing)

δ_b کوتاه شدگی در اثر خم شدن

(Length correction factor due to bowing) $C_b = \frac{\delta_b}{\ell}$

(6) \rightarrow (5) \Rightarrow

Geometrically Nonlinear Analysis of Plane Frames

□ Member, Force-Deformation Relationship in local Coordinate

C_1 و C_2 توابع پایداری (Stability Functions) نام دارند که خود تابعی از Q است.

در صورتی که Q نیروی محوری فشاری باشد

$$\text{if } Q < 0 \quad \Rightarrow \quad q = \frac{Q}{Q_{cr=Euler}} > 0$$

$$C_1 = \frac{\phi \sin \phi - \phi^2 \cos \phi}{2 - 2 \cos \phi - \phi \sin \phi}, \quad C_2 = \frac{\phi^2 - \phi \sin \phi}{2 - 2 \cos \phi - \phi \sin \phi} \quad (8a)$$

که در آن

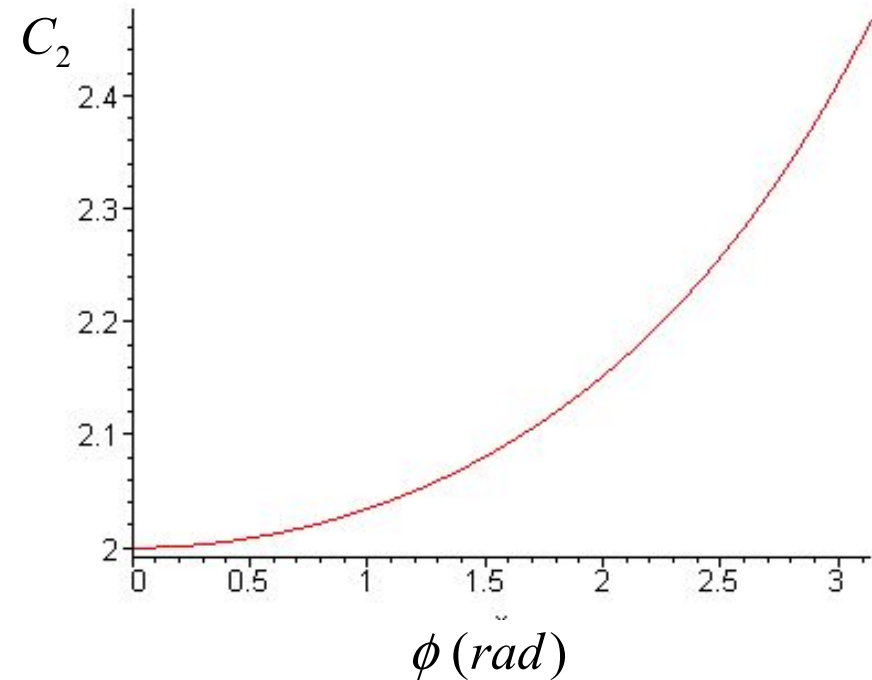
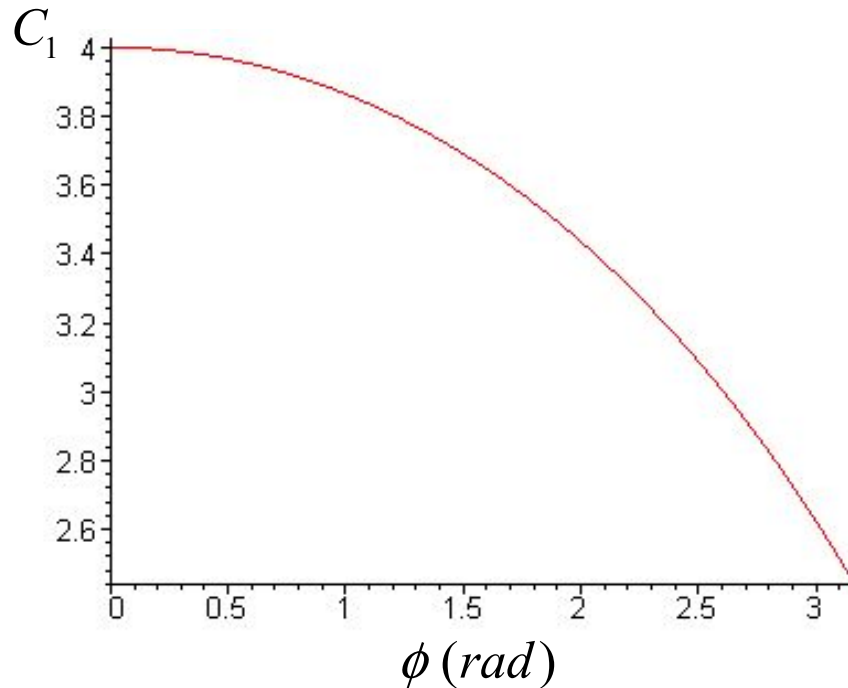
$$\phi^2 = k^2 \ell'^2 = \frac{Q \ell'^2}{EI}$$
$$q = \frac{Q}{Q_{cr=Euler}} = \frac{Q \ell^2}{\pi^2 EI} \quad \Rightarrow \quad \phi^2 = \pi^2 \left(\frac{\ell'}{\ell} \right)^2 q$$

Geometrically Nonlinear Analysis of Plane Frames

□ Member, Force-Deformation Relationship in local Coordinate

$$\text{if } Q=0 \text{ and } Q_{cr=Euler} < 0 \Rightarrow q = \frac{Q}{Q_{cr=Euler}} = 0$$

در صورتی که Q نیروی محوری برابر با صفر باشد



$$C_1 = 4 \quad , \quad C_2 = 2 \quad (8b)$$

Geometrically Nonlinear Analysis of Plane Frames

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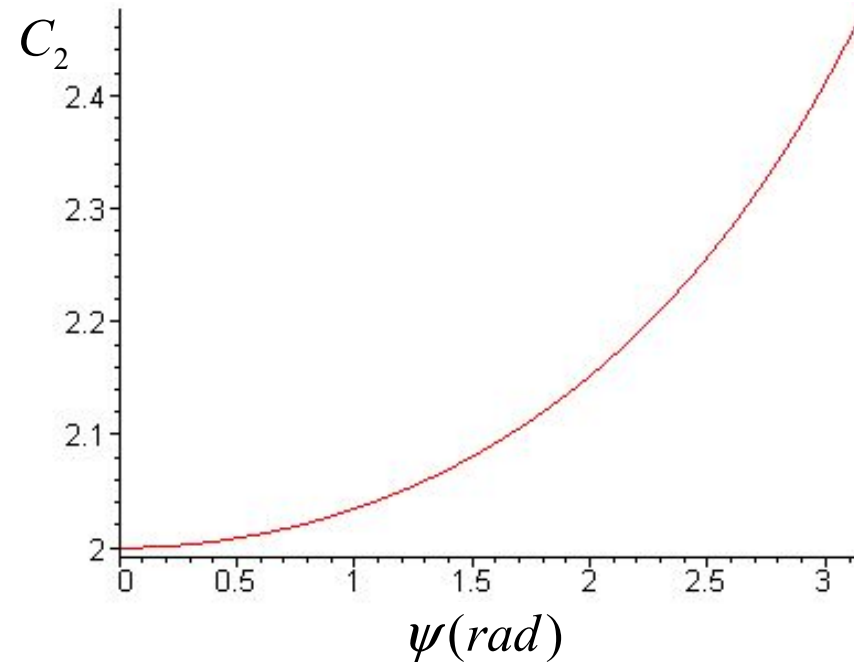
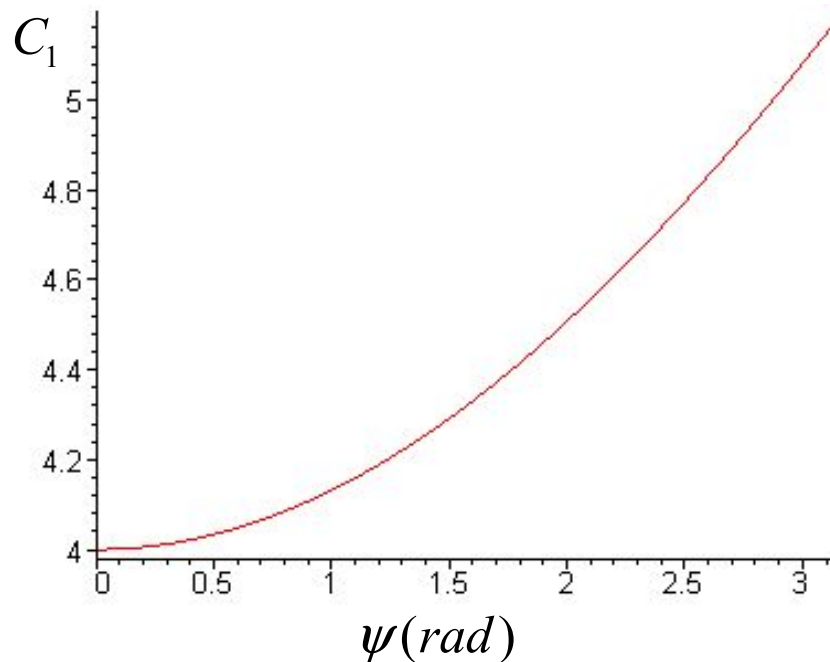
$$\text{if } Q > 0 \text{ and } Q_{cr=Euler} < 0 \Rightarrow q = \frac{Q}{Q_{cr=Euler}} < 0$$

در صورتی که Q نیروی محوری کششی باشد

$$C_1 = \frac{\psi^2 \cosh \psi - \psi \sinh \psi}{2 - 2 \cosh \psi + \psi \sinh \psi}, \quad C_2 = \frac{\psi \sinh \psi - \psi^2}{2 - 2 \cosh \psi + \psi \sinh \psi} \quad (8c)$$

که در آن

$$\psi^2 = -\pi^2 \left(\frac{l'}{l} \right)^2 q$$

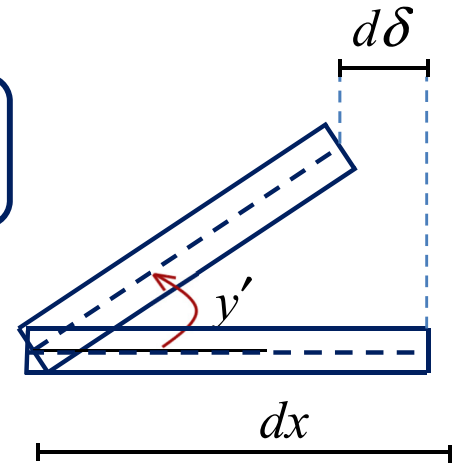


Geometrically Nonlinear Analysis of Plane Frames

□ Member, Force-Deformation Relationship in local Coordinate

$$d\delta = dx(1 - \cos y')$$

$$\cos \theta = \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!} \Rightarrow d\delta = dx \left[1 - \left(1 - \frac{y'^2}{2} + \frac{y'^4}{24} \dots \right) \right] \Rightarrow \boxed{d\delta = \frac{y'^2}{2} dx}$$



$$\Rightarrow \delta_b = \frac{1}{2} \int_0^{\ell} y'^2 dx \Rightarrow \boxed{C_b = \frac{1}{2\ell} \int_0^{\ell} y'^2 dx} \quad (9)$$

$$\boxed{C_b = b_1(\theta_1 + \theta_2)^2 + b_2(\theta_1 - \theta_2)^2} \quad (10)$$

$$b_1 = \frac{(C_1 + C_2)(C_2 - 2)}{8\pi^2 q}, \quad b_2 = \frac{C_2}{8(C_1 + C_2)}$$

(Bowing Functions) b_2 & b_1

θ_1 & θ_2 & u is known $\Rightarrow M_1$ & M_2 & Q to be determined

Geometrically Nonlinear Analysis of Plane Frames

□ Member, Force-Deformation Relationship in local Coordinate

$$Q = AE\left(\frac{u}{\ell} - C_b\right) \quad (7c)$$

با توجه به رابطه (7c) نیروی محوری عضو (Q) خود تابعی از Q است.

$$Q = f(C_b) \rightarrow C_b = g(b_1, b_2) \rightarrow b_1, b_2 = h(C_1, C_2) \rightarrow C_1, C_2 = S(Q)$$

Length correction
factor due to bowing

Bowing
Functions

Stability
Functions

برای حل معادله (7c) نیاز به یک روش تکرار شونده است. برای انجام این کار باید رابطه (7c) برحسب یک پارامتر نیروی محوری بدون بعد نوشته می‌شود.

$$Q = AE\left(\frac{u}{\ell} - C_b\right)$$
$$q = \frac{Q\ell^2}{\pi^2 EI} \Rightarrow Q = \frac{\pi^2 EI}{\ell^2} q \Rightarrow \frac{\pi^2}{\lambda^2} q = \frac{u}{\ell} - C_b, \quad \lambda = \frac{\ell}{\sqrt{I/A}} \quad (11)$$

$$\Rightarrow K(q) = \frac{\pi^2}{\lambda^2} q + C_b - \frac{u}{\ell} = 0 \quad (12)$$

Geometrically Nonlinear Analysis of Plane Frames

□ Member, Force-Deformation Relationship in local Coordinate

با استفاده از بسط سری تیلور

$$q_{i+1} = q_i + \Delta q_i$$

عملیات تکرار زمانی به سرانجام می‌رسد که $|\Delta q_i| \approx 0$

$$\Delta q_i = -\frac{K(q_i)}{K'(q_i)} \Rightarrow q_{i+1} = q_i - \frac{K(q_i)}{K'(q_i)} \quad (13)$$

$$K'(q) = \frac{d[K(q)]}{dq} \Rightarrow K'(q) = \frac{\pi^2}{\lambda^2} + b_1' (\theta_1 + \theta_2)^2 + b_2' (\theta_1 - \theta_2)^2 \quad (14)$$

که در آن

$$b_1' = -\frac{(b_1 - b_2)(C_1 + C_2) - 2C_2 b_1}{4q}, \quad b_2' = -\frac{\pi^2 (16b_1 b_2 - b_1 + b_2)}{4(C_1 + C_2)}$$

Geometrically Nonlinear Analysis of Plane Frames

□ Member, Force-Deformation Relationship in local Coordinate

$$\text{with iteration} \rightarrow q_i \cong q_{i+1} \rightarrow \overset{\checkmark}{q} \rightarrow \overset{\checkmark}{C_1}, \overset{\checkmark}{C_2} \rightarrow \overset{\checkmark}{M_1}, \overset{\checkmark}{M_2}, \overset{\checkmark}{Q}$$

رابطه (7) بر اساس گیردار بودن دو انتها می باشد

اگر انتهای (1) مفصلی باشد:

$$M_1 = \frac{EI}{\ell'} [C_1 \theta_1 + C_2 \theta_2] = 0 \Rightarrow \theta_1 = -\frac{C_2}{C_1} \theta_2 = f(\theta_2)$$

اگر انتهای (2) مفصلی باشد:

$$M_2 = \frac{EI}{\ell'} [C_2 \theta_1 + C_1 \theta_2] = 0 \Rightarrow \theta_2 = -\frac{C_2}{C_1} \theta_1 = f(\theta_1)$$

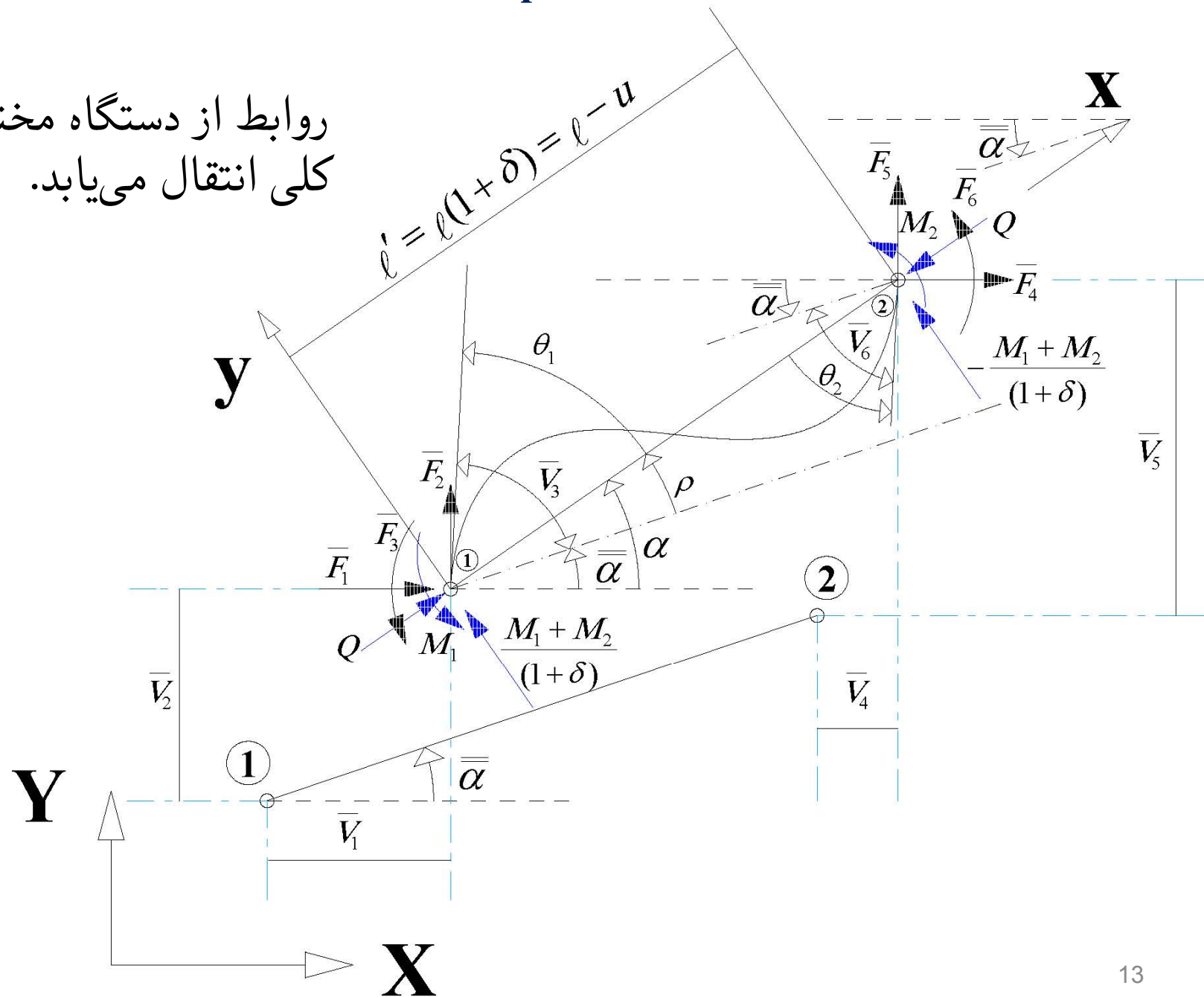
اگر دو انتها مفصلی باشد:

$$M_1 = M_2 = \theta_1 = \theta_2 = 0 \Rightarrow Q = \frac{AE}{\ell} u$$

Geometrically Nonlinear Analysis of Plane Frames

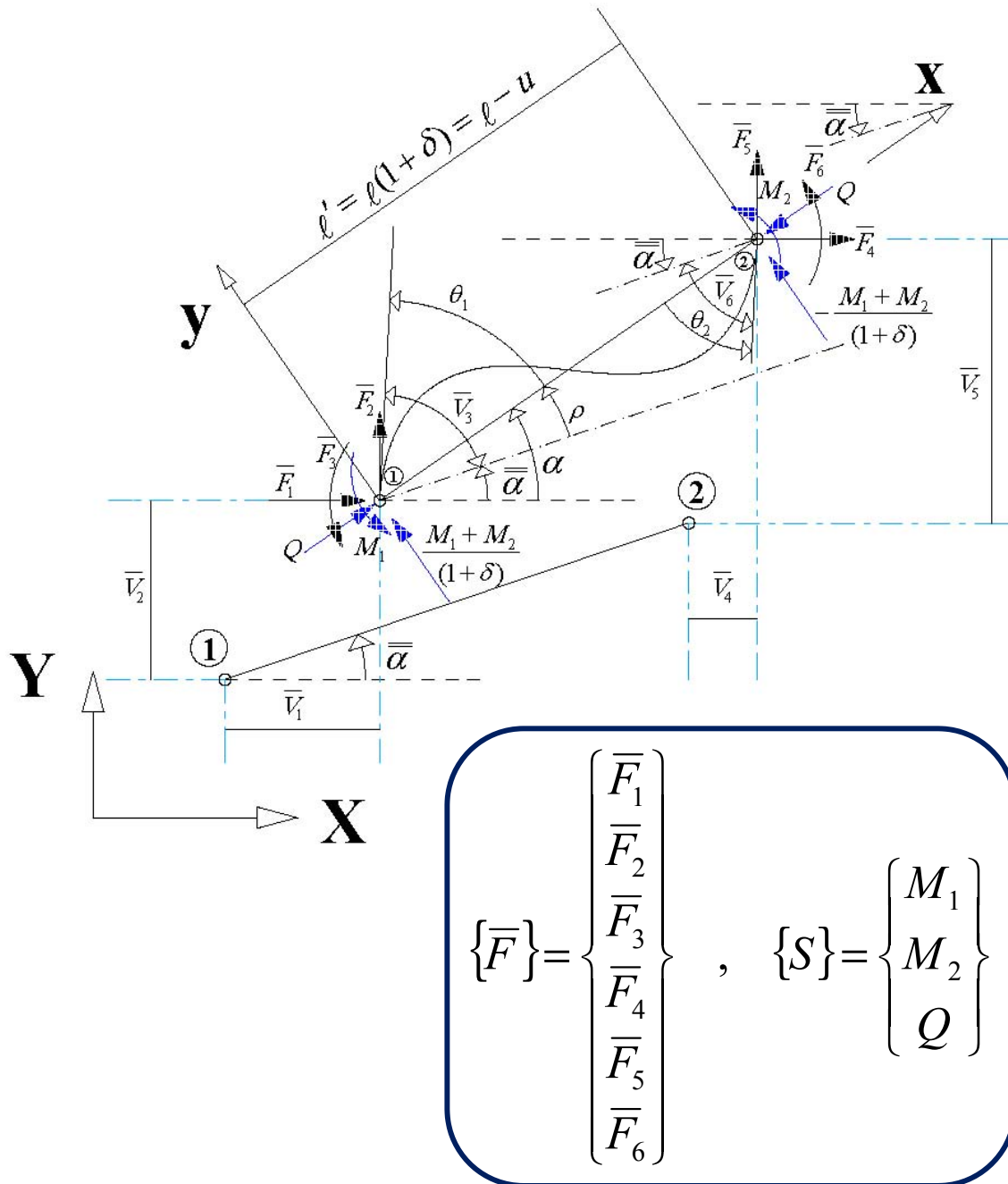
Member, Force-Deformation Relationship in Global Coordinate

روابط از دستگاه مختصات محلی به کلی انتقال می‌یابد.



Geometrically Nonlinear Analysis of Plane Frames

Member, Force-Deformation Relationship in Global Coordinate



$\bar{\alpha}$ موقعیت محور عضو در حالت اولیه
 α موقعیت محور عضو در حالت تغییر مکان یافته
 ρ دوران محور عضو
 l' طول کوتاه شدگی
 نیروها در دستگاه مختصات کلی از رابطه زیر به دست می‌آید.

$$\{\bar{F}\}_{6 \times 1} = [\bar{B}]_{6 \times 3} \{S\}_{3 \times 1} \quad (15)$$

که در آن

$\{\bar{F}\}$ نیروهای داخلی در دستگاه کلی

$\{S\}$ نیروهای داخلی در دستگاه محلی

$[\bar{B}]$ ماتریس انتقال (Transformation Matrix)

Geometrically Nonlinear Analysis of Plane Frames

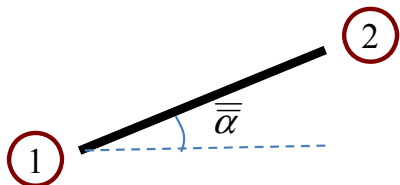
□ Member, Force-Deformation Relationship in Global Coordinate

$[\bar{B}]$ ماتریس انتقال (Transformation Matrix)

$$[\bar{B}] = \begin{bmatrix} -A_1 & -A_1 & m \\ A_2 & A_2 & n \\ 1 & 0 & 0 \\ A_1 & A_1 & -m \\ -A_2 & -A_2 & -n \\ 0 & 1 & 0 \end{bmatrix}, \quad A_1 = \frac{n}{\ell(1+\delta)}, \quad A_2 = \frac{m}{\ell(1+\delta)}, \quad m = \cos \alpha \\ n = \sin \alpha$$

پارامترهای θ_1 ، θ_2 و u به کمک رابطه زیر به دست می‌آیند

$$\theta_1 = \bar{V}_3 - \rho, \quad \theta_2 = \bar{V}_6 - \rho, \quad u = -\ell\delta \quad \text{or} \quad \frac{u}{\ell} = -\delta, \quad \rho = \alpha - \bar{\alpha} \quad (16)$$



$$\bar{\alpha} = \tan^{-1} \left(\frac{Y_2 - Y_1}{X_2 - X_1} \right)$$

Geometrically Nonlinear Analysis of Plane Frames

□ Member, Force-Deformation Relationship in Global Coordinate

مراحل روش محاسبه نیروهای داخلی در دستگاه کلی

$$\delta = \frac{l'}{l} - 1$$

(1) محاسبه δ (با استفاده از l' ، α و $\bar{\alpha}$)

(2) محاسبه θ_1 ، θ_2 و u به کمک رابطه (16)

(3) محاسبه M_1 ، M_2 و Q به کمک روابط (7) و (13) و تشکیل بردار $\{\bar{S}\}$

(4) نیروها از دستگاه مختصات محلی به دستگاه مختصات کلی با استفاده از رابطه (15) انتقال می‌یابد.

Geometrically Nonlinear Analysis of Plane Frames

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تعداد سیستم

$$f_i(x_1, \dots, x_n) = P_i \quad (17)$$

f_i برآیند نیروهای داخلی (Resultant Internal Forces)

P_i نیروهای خارجی (External Forces)

x_i مختصات تعمیم یافته شامل دوران و انتقال (Generalized Coordinates: translation & Rotation)

$$\{f(x)\}_{n \times 1} = \{P\}_{n \times 1} \quad (18)$$

$\{P^{(i)}\}$ Load level

در مرحله (Level) (i) بارگذاری و تغییر شکل به صورت زیر است

$\{x^{(i)}\}$ Corresponding deformed configuration

حالت مطلوب حالتی است که $\{x^{(i+1)}\}$ به $\{P^{(i+1)}\}$ برسیم.

Geometrically Nonlinear Analysis of Plane Frames

□ Member, Force-Deformation Relationship in Global Coordinate

$$\{P^{(i+1)}\} = \{P^{(i)}\} + \{\Delta P\} \quad (19)$$

$$\{f^{(i+1)}\} = \{f^{(i)}\} + \left[\frac{\partial f_i^{(i)}}{\partial x_j} \right] \{\Delta x\} + H.O.T \quad (18) \& (19) \Rightarrow \{f^{(i)}\} + \left[\frac{\partial f_i^{(i)}}{\partial x_j} \right] \{\Delta x\} = \{P^{(i)}\} + \{\Delta P\} \quad (20)$$

(20) \Rightarrow $\begin{cases} \{f^{(i)}\} = \{P^{(i)}\} \\ \{\Delta P\} = \left[\frac{\partial f_i^{(i)}}{\partial x_j} \right] \{\Delta x\} \end{cases} \quad (21)$

وضعیت تعادل برقرار است یعنی نیروهای خارجی مساوی با نیروهای داخلی می باشد.

(21) \Rightarrow $\{\Delta P\} = [\tau^{(i)}] \{\Delta x\}$ in which $[\tau^{(i)}] = \left[\frac{\partial f_i^{(i)}}{\partial x_j} \right] \quad (22)$

$[\tau^{(i)}]$ ماتریس سختی مماسی سیستم (System Tangent Stiffness Matrix) که در $\{x^{(i)}\}$ محاسبه می شود.

Geometrically Nonlinear Analysis of Plane Frames

□ Member, Force-Deformation Relationship in Global Coordinate

$$(21) \Rightarrow \{\Delta x\} = \left[\frac{\partial f_i^{(i)}}{\partial x_j} \right]^{-1} \{\Delta P\} \rightarrow x$$

✓ Linear Incremental Method Without N.R.It
 Linear Incremental Method With N.R.It

Newton Raphson Iteration (N.R.It)

- Simple
- Updated

Member Tangent Stiffness Matrix in Global Coordinates

$$\{\Delta \bar{F}\} = [T] \{\Delta \bar{V}\} \quad (23)$$

$$[T] = \left[\frac{\partial \bar{F}_i^{(i)}}{\partial \bar{V}_j} \right]$$

(24) M.T.S.M in Global Coordinate

$$\{\bar{V}\}_{6 \times 1} = \begin{Bmatrix} \bar{V}_1 \\ \bar{V}_2 \\ \bar{V}_3 \\ \bar{V}_4 \\ \bar{V}_5 \\ \bar{V}_6 \end{Bmatrix}$$

مشق \bar{F}_i نسبت به هر یک از درجات آزادی $\frac{\partial \bar{F}_i^{(i)}}{\partial \bar{V}_j}$

$\{\bar{V}\}$ تغییر مکان گره‌ای در دستگاه مختصات کلی

Geometrically Nonlinear Analysis of Plane Frames

□ Member, Force-Deformation Relationship in Global Coordinate

Member Tangent Stiffness Matrix in Local Coordinates

$$\{\Delta S\} = [t]\{\Delta u\} \quad (25)$$

تغییر مکان گره‌ای در دستگاه محلی $\{u\} = \begin{Bmatrix} \theta_1 \\ \theta_2 \\ u \end{Bmatrix}$

$$[t] = \left[\frac{\partial S_i}{\partial u_j} \right] \quad (26) \quad \text{M.T.S.M in Local Coordinate}$$

$$\{\bar{F}\} = [\bar{B}]\{S\} \quad (15) \quad \text{یادآوری}$$

$$\Rightarrow \{\Delta \bar{F}\} = [\bar{B}]\{\Delta S\} + [\Delta \bar{B}]\{S\} \quad (27)$$

$$[t] = \frac{EI}{\ell} \begin{bmatrix} C_1 + \frac{G_1^2}{\pi^2 H} & C_2 + \frac{G_1 G_2}{\pi^2 H} & \frac{G_1}{\ell H} \\ C_2 + \frac{G_1 G_2}{\pi^2 H} & C_1 + \frac{G_2}{\pi^2 H} & \frac{G_2}{\ell H} \\ \frac{G_1}{\ell H} & \frac{G_2}{\ell H} & \frac{\pi^2}{\ell^2 H} \end{bmatrix} \quad (28)$$

Geometrically Nonlinear Analysis of Plane Frames

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که در رابطه (28)

$$\begin{aligned}
 G_1 &= C_1' \theta_1 + C_2' \theta_2 \\
 G_2 &= C_2' \theta_1 + C_1' \theta_2 \\
 H &= \frac{\pi^2}{\lambda^2} + b_1' (\theta_1 + \theta_2)^2 + b_2' (\theta_1 - \theta_2)^2 \\
 \lambda &= \frac{\ell}{\sqrt{\frac{I}{A}}}
 \end{aligned}
 \tag{29}$$

$$\begin{aligned}
 C_1' &= -2\pi^2 (b_1 + b_2) \\
 C_2' &= -2\pi^2 (b_1 - b_2)
 \end{aligned}$$

$$\text{Local} \leftarrow \boxed{\{\Delta u\} = [\bar{B}]^T [\Delta \bar{V}]} \tag{30}$$

Global \nearrow

می‌توان نشان داد که

$$(25), (27), (30) \Rightarrow \boxed{\{\Delta \bar{F}\} = [\bar{B}][t] [\bar{B}]^T [\Delta \bar{V}] + [\Delta \bar{B}]\{S\}} \tag{31}$$

خصوصیات مواد \nearrow
خصوصیات هندسه \nearrow

Geometrically Nonlinear Analysis of Plane Frames

□ Member, Force-Deformation Relationship in Global Coordinate

$$[\Delta \bar{B}] \{S\} = \sum_{k=1}^3 S_k [g^{(k)}] \{\Delta \bar{V}\} \quad (32)$$

خصوصیات
هندسه

$$[g^{(1)}] = [g^{(2)}] = \frac{1}{\ell'} \begin{bmatrix} -2Z_1 & Z_3 - Z_2 & 0 & 2Z_1 & -(Z_3 - Z_2) & 0 \\ Z_3 - Z_2 & 2Z_1 & 0 & -(Z_3 - Z_2) & -2Z_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2Z_1 & -(Z_3 - Z_2) & 0 & -2Z_1 & Z_3 - Z_2 & 0 \\ -(Z_3 - Z_2) & -2Z_1 & 0 & Z_3 - Z_2 & 2Z_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[g^{(3)}] = \frac{1}{\ell'} \begin{bmatrix} -Z_2 & Z_1 & 0 & Z_2 & -Z_1 & 0 \\ Z_1 & Z_3 & 0 & -Z_1 & Z_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ Z_2 & -Z_1 & 0 & -Z_2 & Z_1 & 0 \\ -Z_1 & Z_3 & 0 & Z_1 & -Z_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Z_1 = mn \quad , \quad Z_2 = n^2 \quad , \quad Z_3 = m^2$$

Geometrically Nonlinear Analysis of Plane Frames

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$$(23), (31), (32) \Rightarrow \boxed{\{\Delta \bar{F}\} = [\bar{B}][t][\bar{B}]^T [\Delta \bar{V}] + \sum_{k=1}^3 S_k [g^{(k)}] \{\Delta \bar{V}\} = [T] \{\Delta \bar{V}\}} \quad (33)$$

$$(33) \Rightarrow \boxed{[T] = [\bar{B}][t][\bar{B}]^T + \sum_{k=1}^3 S_k [g^{(k)}]} \quad (34)$$

M.T.S.M in Global Coordinate

Geometrically Nonlinear Analysis of Plane Frames

□ Member, Force-Deformation Relationship

Series Expressions

$$C_1 = 4 - \frac{2}{15} \pi^2 q - \frac{11}{6300} \pi^4 q^2 - \frac{1}{27000} \pi^6 q^3$$

$$C_2 = 2 + \frac{1}{30} \pi^2 q + \frac{13}{12600} \pi^4 q^2 + \frac{11}{378000} \pi^6 q^3$$

$$C_1' = -\frac{2}{15} \pi^2 - \frac{11}{3150} \pi^4 q - \frac{1}{9000} \pi^6 q^2$$

$$C_2' = \frac{1}{30} \pi^2 + \frac{13}{6300} \pi^4 q + \frac{11}{126000} \pi^6 q^2$$

$$b_1 = \frac{1}{40} + \frac{1}{2800} \pi^2 q + \frac{1}{168000} \pi^4 q^2 + \frac{37}{388080000} \pi^6 q^3$$

$$b_2 = \frac{1}{24} + \frac{1}{720} \pi^2 q + \frac{1}{20160} \pi^4 q^2 + \frac{1}{604800} \pi^6 q^3$$

$$b_1' = \frac{1}{2800} \pi^2 + \frac{1}{84000} \pi^4 q + \frac{37}{129360000} \pi^6 q^2$$

$$b_2' = \frac{1}{720} \pi^2 + \frac{1}{10080} \pi^4 q + \frac{1}{201600} \pi^6 q^2$$

این عبارات برحسب سری، که برای هر دو حالت نیروی محوری کششی و فشاری معتبر است، برای مقادیر کوچک q صادق است.

Geometrically Nonlinear Analysis of Plane Frames

□ Member, Force-Deformation Relationship in Global Coordinate

$$[\tau] = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ T_{11}^{(1)} & T_{12}^{(1)} & T_{13}^{(1)} & T_{14}^{(1)} & T_{15}^{(1)} & T_{16}^{(1)} & & & & & & \\ T_{21}^{(1)} & T_{22}^{(1)} & T_{23}^{(1)} & T_{24}^{(1)} & T_{25}^{(1)} & T_{26}^{(1)} & & & & & & \\ T_{31}^{(1)} & T_{32}^{(1)} & T_{33}^{(1)} & T_{34}^{(1)} & T_{35}^{(1)} & T_{36}^{(1)} & & & & & & \\ T_{41}^{(1)} & T_{42}^{(1)} & T_{43}^{(1)} & T_{44}^{(1)} + T_{44}^{(2)} & T_{45}^{(1)} + T_{45}^{(2)} & T_{46}^{(1)} + T_{46}^{(2)} & T_{47}^{(2)} & T_{48}^{(2)} & T_{49}^{(2)} & & & \\ T_{51}^{(1)} & T_{52}^{(1)} & T_{53}^{(1)} & T_{54}^{(1)} + T_{54}^{(2)} & T_{55}^{(1)} + T_{55}^{(2)} & T_{56}^{(1)} + T_{56}^{(2)} & T_{57}^{(2)} & T_{58}^{(2)} & T_{59}^{(2)} & & & \\ T_{61}^{(1)} & T_{62}^{(1)} & T_{63}^{(1)} & T_{64}^{(1)} + T_{64}^{(2)} & T_{65}^{(1)} + T_{65}^{(2)} & T_{66}^{(1)} + T_{66}^{(2)} & T_{67}^{(2)} & T_{68}^{(2)} & T_{69}^{(2)} & & & \\ & & & T_{74}^{(2)} & T_{75}^{(2)} & T_{76}^{(2)} & T_{77}^{(2)} + T_{77}^{(3)} & T_{78}^{(2)} + T_{78}^{(3)} & T_{79}^{(2)} + T_{79}^{(3)} & T_{7,10}^{(3)} & T_{7,11}^{(3)} & T_{7,12}^{(3)} \\ & & & T_{84}^{(2)} & T_{85}^{(2)} & T_{86}^{(2)} & T_{87}^{(2)} + T_{87}^{(3)} & T_{88}^{(2)} + T_{88}^{(3)} & T_{89}^{(2)} + T_{89}^{(3)} & T_{8,10}^{(3)} & T_{8,11}^{(3)} & T_{8,12}^{(3)} \\ & & & T_{94}^{(2)} & T_{95}^{(2)} & T_{96}^{(2)} & T_{97}^{(2)} + T_{97}^{(3)} & T_{98}^{(3)} + T_{98}^{(3)} & T_{99}^{(2)} + T_{99}^{(3)} & T_{9,10}^{(3)} & T_{9,11}^{(3)} & T_{9,12}^{(3)} \\ & & & & & & T_{10,7}^{(3)} & T_{10,8}^{(3)} & T_{10,9}^{(3)} & T_{10,10}^{(3)} & T_{10,11}^{(3)} & T_{10,12}^{(3)} \\ & & & & & & T_{11,7}^{(3)} & T_{11,8}^{(3)} & T_{11,9}^{(3)} & T_{11,10}^{(3)} & T_{11,11}^{(3)} & T_{11,12}^{(3)} \\ & & & & & & T_{12,7}^{(3)} & T_{12,8}^{(3)} & T_{12,9}^{(3)} & T_{12,10}^{(3)} & T_{12,11}^{(3)} & T_{12,12}^{(3)} \end{bmatrix}$$

$$\begin{bmatrix} \{\Delta P\}_{12 \times 1} \\ \{\Delta R\}_{6 \times 1} \end{bmatrix}_{12 \times 1} = [\tau]_{12 \times 12} \begin{bmatrix} \{\Delta x\}_{12 \times 1} \\ \{\Delta x_{Support}\}_{6 \times 1} \end{bmatrix}_{12 \times 1} \Rightarrow \begin{bmatrix} \{\Delta P\}_{6 \times 1} \\ \{\Delta R\}_{6 \times 1} \end{bmatrix}_{12 \times 1} = \begin{bmatrix} 7 & 8 & 9 & 4 & 5 & 6 & 1 & 2 & 3 & 10 & 11 & 12 \\ & [\tau_{PP}]_{6 \times 6} & & & & & & & [\tau_{PR}]_{6 \times 6} & & & \\ & & [\tau_{RP}]_{6 \times 6} & & & & & & & [\tau_{RR}]_{6 \times 6} & & \\ & & & & & & & & & & & \end{bmatrix}_{12 \times 12} \begin{bmatrix} \{\Delta x\}_{6 \times 1} \\ \{\Delta x_{Support}\}_{6 \times 1} \end{bmatrix}_{12 \times 1}$$

Geometrically Nonlinear Analysis of Plane Frames

□ Member, Force-Deformation Relationship in Global Coordinate

$$\begin{Bmatrix} \{\Delta P\}_{6 \times 1} \\ \{\Delta R\}_{6 \times 1} \end{Bmatrix}_{12 \times 1} = \begin{bmatrix} 7 & 8 & 9 & 4 & 5 & 6 & 1 & 2 & 3 & 10 & 11 & 12 \\ \tau_{PP} & & & & & & \tau_{PR} & & & & & \\ \tau_{RP} & & & & & & \tau_{RR} & & & & & \end{bmatrix}_{12 \times 12} \begin{Bmatrix} \{\Delta x\}_{6 \times 1} \\ \{\Delta x_{Support}\}_{6 \times 1} \end{Bmatrix}_{12 \times 1}$$

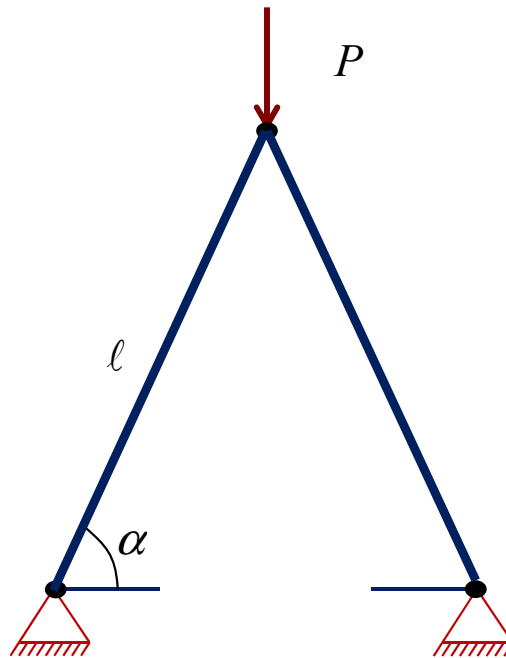
$$\Rightarrow \{\Delta P\}_{6 \times 1} = [\tau_{PP}]_{6 \times 6} \{\Delta x\}_{6 \times 1} + [\tau_{PR}]_{6 \times 6} \{\Delta x_{Support}\}_{6 \times 1} \Rightarrow \{\Delta x\}_{6 \times 1} \quad \checkmark$$

$$\Rightarrow \{\Delta R\}_{6 \times 1} = [\tau_{RP}]_{6 \times 6} \{\Delta x\}_{6 \times 1} + [\tau_{RR}]_{6 \times 6} \{\Delta x_{Support}\}_{6 \times 1} \Rightarrow \{\Delta R\}_{6 \times 1} \quad \checkmark$$

Geometrically Nonlinear Analysis of Plane Frames

□ Example

مثال: در سازه نشان داده شده در شکل زیر مطلوب است تعیین الف) بار بحرانی و ب) جابجایی نقطه اثر بار $P=1500$ kips.



$$\alpha = 30^{\circ}$$

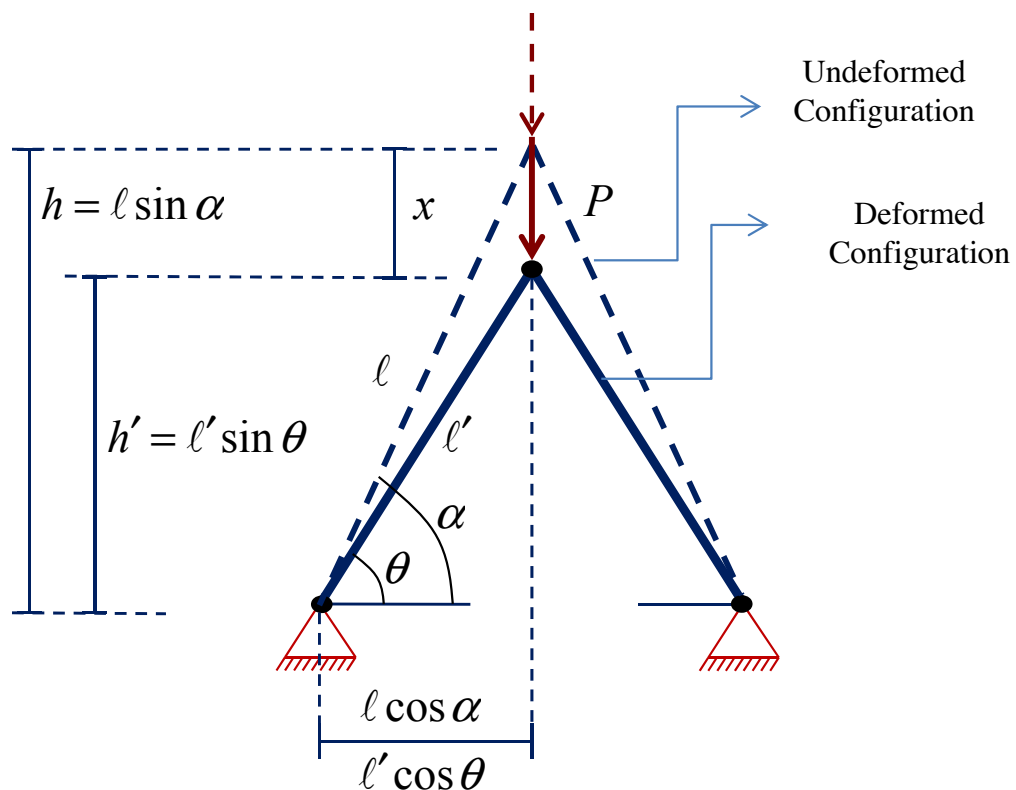
$$l = 100 \text{ (in.)}$$

$$E = 30000 \text{ (ksi)}$$

$$A = 1 \text{ (in.}^2\text{)}$$

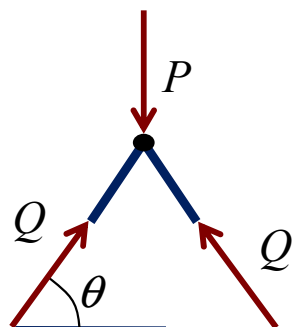
Geometrically Nonlinear Analysis of Plane Frames

□ Example



اگر θ معیار تغییر شکل باشد.
با توجه به شکل

اگر دو انتها مفصلی باشد:



Geometrically Nonlinear Analysis of Plane Frames

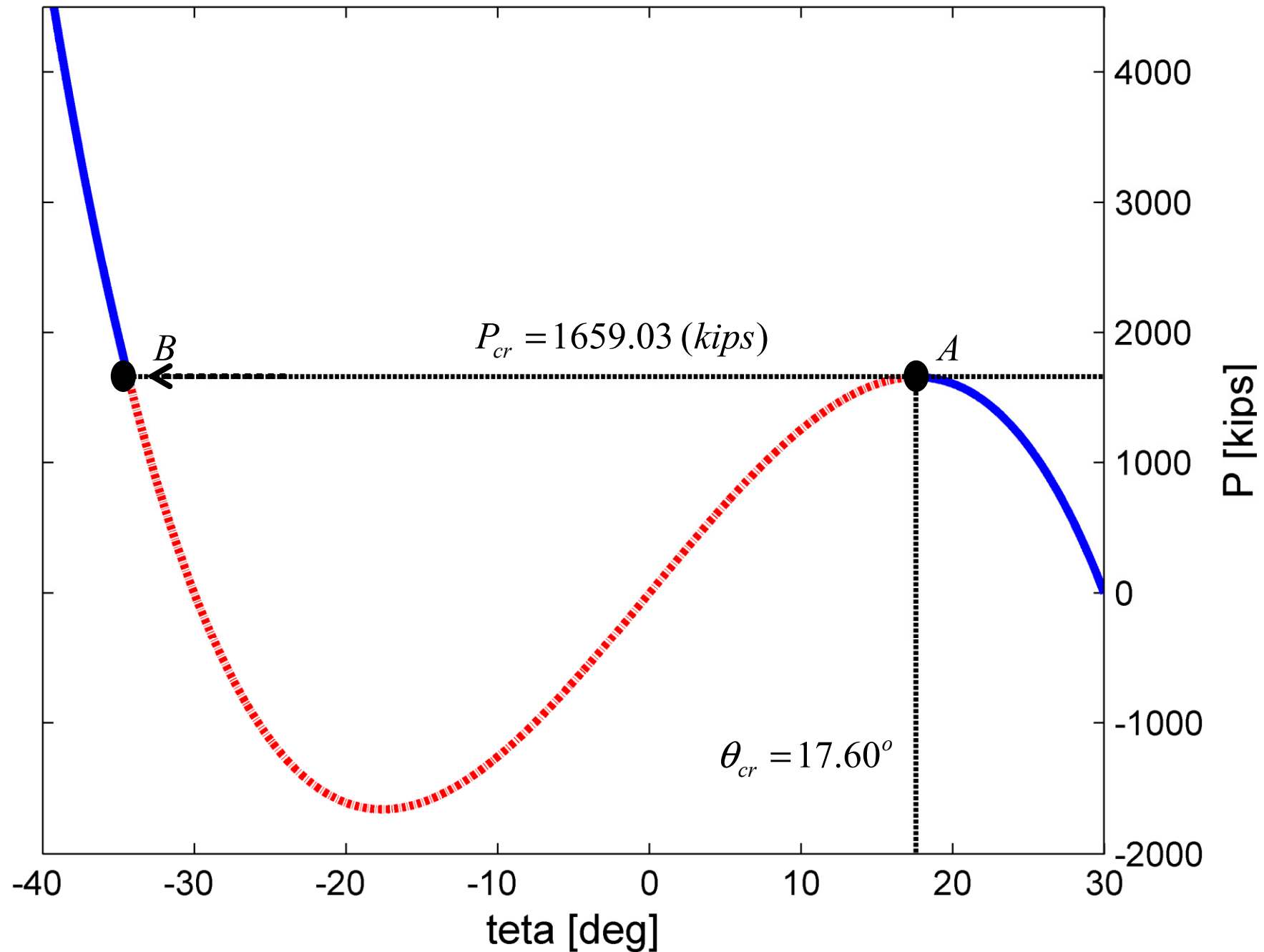
□ Example

تعیین بار بحرانی P_{cr}

بار بحرانی زمانی اتفاق می افتد که سختی مماسی صفر شود.

Geometrically Nonlinear Analysis of Plane Frames

□ Example



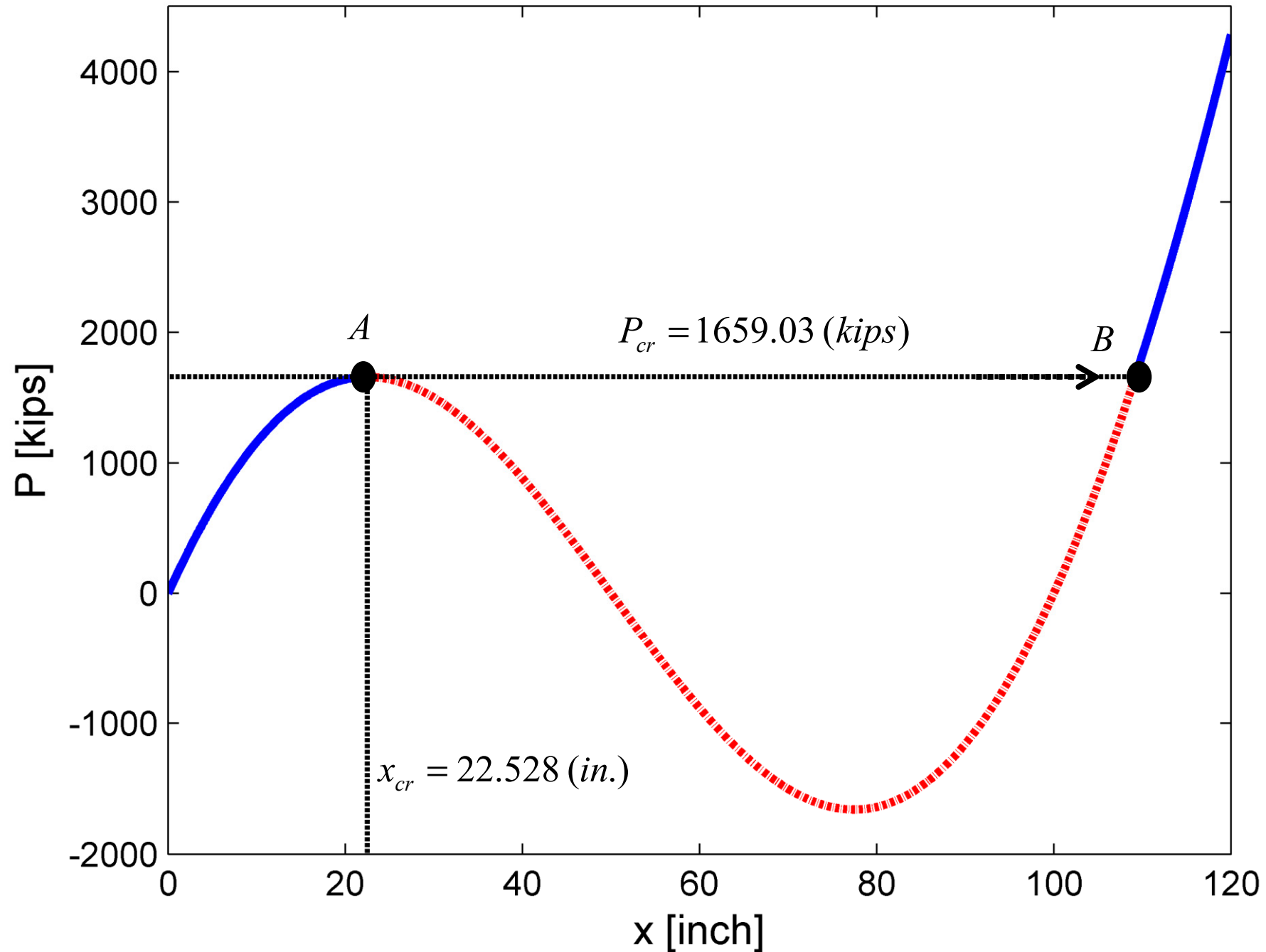
Geometrically Nonlinear Analysis of Plane Frames

□ Example

اگر x معیار تغییر شکل باشد.
با توجه به شکل

Geometrically Nonlinear Analysis of Plane Frames

□ Example



Geometrically Nonlinear Analysis of Plane Frames

□ Example

Geometrically Nonlinear Analysis of Plane Frames

□ Example

تعیین جابجایی نقطه اثر بار $P=1500$ kips

روش اول: Linear Incremental Method with no Iteration

Geometrically Nonlinear Analysis of Plane Frames

□ Example

$$\begin{aligned} \textcircled{2} \quad P^{(2)} &= P^{(1)} + \Delta P = 300 + 300 &\Rightarrow & \boxed{P^{(2)} = 600 \text{ (kips)}} &\stackrel{(11)}{\Rightarrow} & \boxed{\tau^{(2)} = 121.42 \text{ (kips / in)}} \\ x^{(2)} &= x^{(1)} + \Delta x^{(1)} = 2 + 2.20 && \boxed{x^{(2)} = 4.20 \text{ (in)}} && \end{aligned}$$

$$\begin{aligned} &\stackrel{\text{(Equation 22)}}{\Rightarrow} \Delta x^{(2)} = \frac{\Delta P}{\tau^{(2)}} = \frac{300}{121.42} &\Rightarrow & \boxed{\Delta x^{(2)} = 2.47 \text{ (in.)}} \end{aligned}$$

$$\begin{aligned} \textcircled{3} \quad P^{(3)} &= P^{(2)} + \Delta P = 600 + 300 &\Rightarrow & \boxed{P^{(3)} = 900 \text{ (kips)}} &\stackrel{(11)}{\Rightarrow} & \boxed{\tau^{(3)} = 104.46 \text{ (kips / in)}} \\ x^{(3)} &= x^{(2)} + \Delta x^{(2)} = 4.20 + 2.47 && \boxed{x^{(3)} = 6.67 \text{ (in)}} && \end{aligned}$$

$$\begin{aligned} &\stackrel{\text{(Equation 22)}}{\Rightarrow} \Delta x^{(3)} = \frac{\Delta P}{\tau^{(3)}} = \frac{300}{104.46} &\Rightarrow & \boxed{\Delta x^{(3)} = 2.87 \text{ (in.)}} \end{aligned}$$

Geometrically Nonlinear Analysis of Plane Frames

□ Example

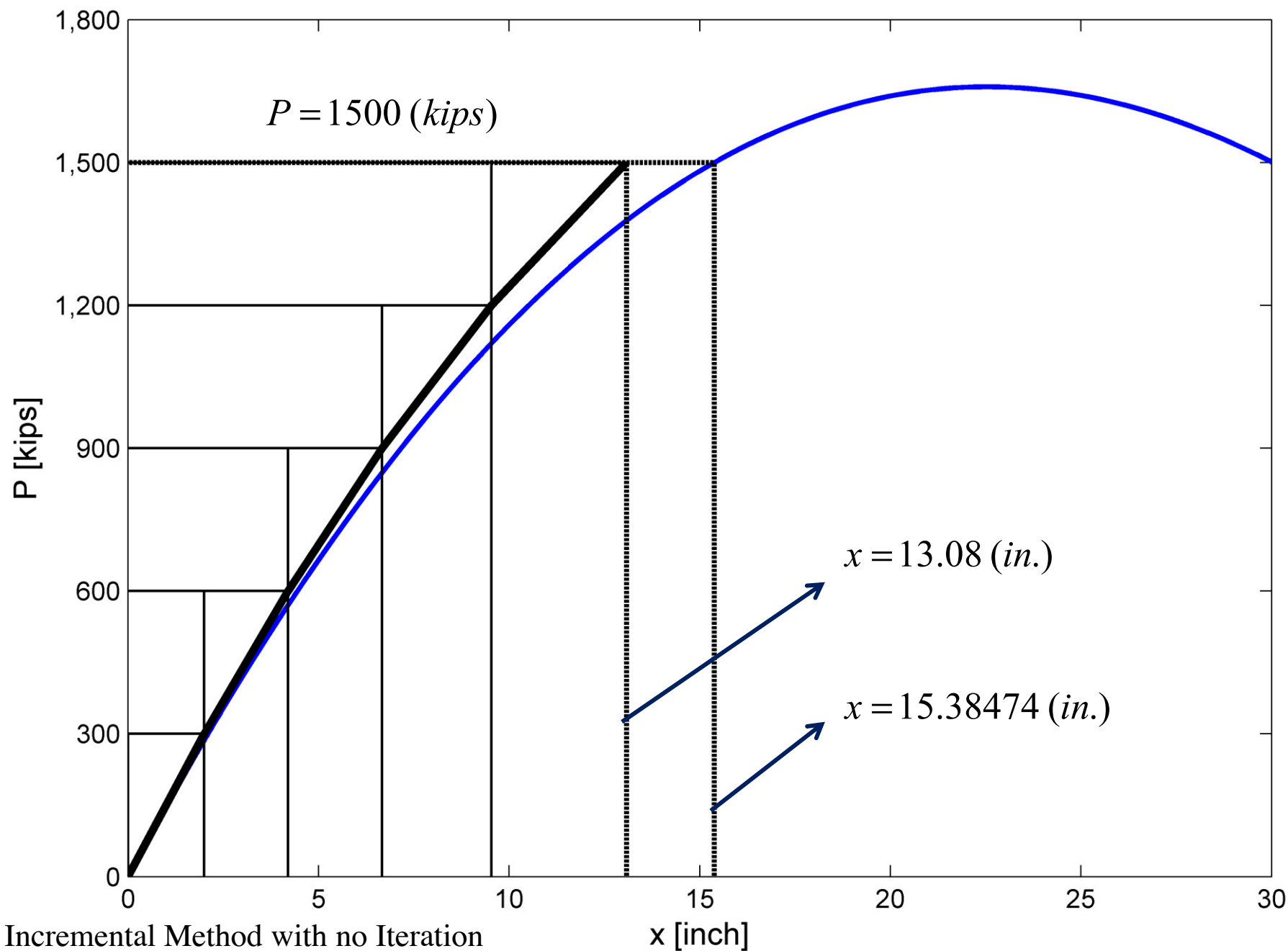
$$\textcircled{4} \quad \begin{aligned} P^{(4)} &= P^{(3)} + \Delta P = 900 + 300 \\ x^{(4)} &= x^{(3)} + \Delta x^{(3)} = 6.67 + 2.87 \end{aligned} \Rightarrow \begin{aligned} P^{(4)} &= 1200 \text{ (kips)} \\ x^{(4)} &= 9.54 \text{ (in)} \end{aligned} \stackrel{(11)}{\Rightarrow} \tau^{(4)} = 84.76 \text{ (kips / in)}$$

$$\stackrel{\text{(Equation 22)}}{\Rightarrow} \Delta x^{(4)} = \frac{\Delta P}{\tau^{(4)}} = \frac{300}{84.76} \Rightarrow \Delta x^{(4)} = 3.54 \text{ (in.)}$$

ΔP (kips)	Iteration	x (in.)
150	10	13.97
100	15	14.35
50	30	14.81
10	150	15.26
1	1500	15.37

Geometrically Nonlinear Analysis of Plane Frames

□ Example



Linear Incremental Method with no Iteration

Geometrically Nonlinear Analysis of Plane Frames

□ Example

روش دوم: Linear Incremental Method with Iteration

a) Simple Newton Raphson Iteration (Simple N.R.It)

$$\Delta P = 1500 \text{ (kips)}$$

Geometrically Nonlinear Analysis of Plane Frames

□ Example

$$x^{(2)} = x^{(1)} + \Delta x^{(1)} = 10 + 2.274 \Rightarrow x^{(2)} = 12.274 \text{ (in)} \stackrel{(10)}{\Rightarrow} P_{x^{(2)}}^{(2)} = 1326.836 \text{ (kips)}$$

$$\textcircled{2} \quad \Delta Q^{(2)} = 1500 - P_{x^{(2)}}^{(2)} = 1500 - 1326.836 \Rightarrow \Delta Q^{(2)} = 173.164 \text{ (kips)}$$

$$\stackrel{\text{(Equation 22)}}{\Rightarrow} \Delta x^{(2)} = \frac{\Delta Q^{(2)}}{\tau^{(0)}} = \frac{173.164}{150} \Rightarrow \Delta x^{(2)} = 1.154 \text{ (in.)}$$

$$x^{(3)} = x^{(2)} + \Delta x^{(2)} = 12.274 + 1.154 \Rightarrow x^{(3)} = 13.429 \text{ (in)} \stackrel{(10)}{\Rightarrow} P_{x^{(3)}}^{(3)} = 1398.684 \text{ (kips)}$$

$$\textcircled{3} \quad \Delta Q^{(3)} = 1500 - P_{x^{(3)}}^{(3)} = 1500 - 1398.684 \Rightarrow \Delta Q^{(3)} = 101.316 \text{ (kips)}$$

$$\stackrel{\text{(Equation 22)}}{\Rightarrow} \Delta x^{(3)} = \frac{\Delta Q^{(3)}}{\tau^{(0)}} = \frac{101.316}{150} \Rightarrow \Delta x^{(3)} = 0.675 \text{ (in.)}$$

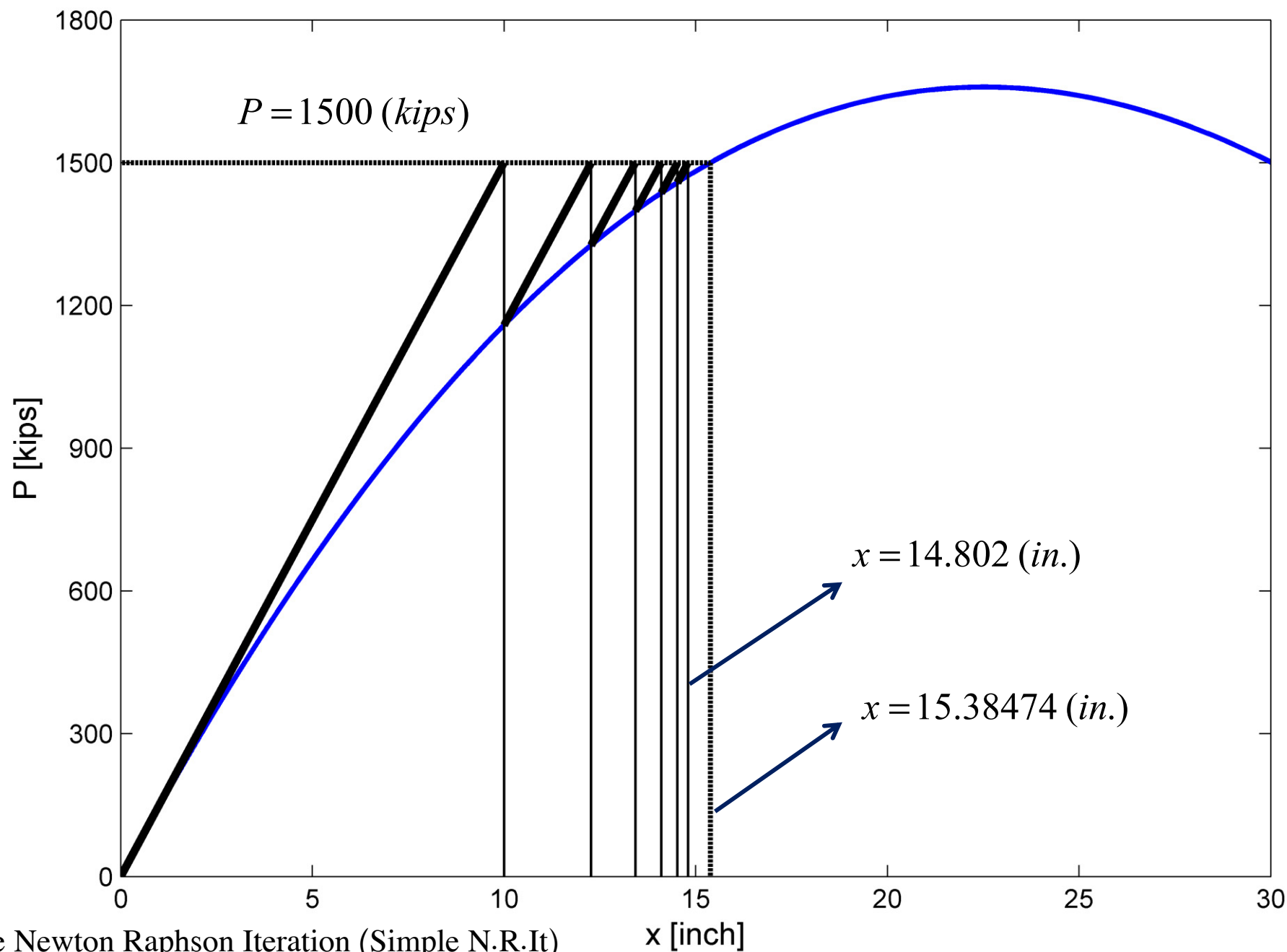
Geometrically Nonlinear Analysis of Plane Frames

□ Example

Iteration	$P^{(i)}$ (kips)	$x^{(i)}$ (in.)	$\tau^{(i)}$ (kips/in.)	$\Delta Q^{(i)}$ (kips)	$\Delta x^{(i)}$ (in.)	$x^{(i+1)}$ (in.)
0	0.0	0.000	150.00	1500	10	10
1	1158.8	10.000	150.00	341.164	2.274426	12.274
2	1326.8	12.274	150.00	173.164	1.154427	13.429
3	1398.7	13.429	150.00	101.316	0.675443	14.104
4	1436.6	14.104	150.00	63.439	0.422925	14.527
5	1458.7	14.527	150.00	41.277	0.275181	14.802
6	1472.5	14.802	150.00	27.498	0.183323	14.986
7	1481.4	14.986	150.00	18.599	0.123994	15.110
8	1487.3	15.110	150.00	12.706	0.084709	15.194
9	1491.3	15.194	150.00	8.739	0.058263	15.253
10	1494.0	15.253	150.00	6.039	0.040257	15.293
11	1495.8	15.293	150.00	4.186	0.027903	15.321
12	1497.1	15.321	150.00	2.907	0.019383	15.340
13	1498.0	15.340	150.00	2.023	0.013484	15.354
14	1498.6	15.354	150.00	1.409	0.009391	15.363
15	1499.0	15.363	150.00	0.982	0.006544	15.370
16	1499.3	15.370	150.00	0.684	0.004563	15.374
17	1499.5	15.374	150.00	0.477	0.003183	15.377
18	1499.7	15.377	150.00	0.333	0.002221	15.380
19	1499.8	15.380	150.00	0.232	0.00155	15.381
20	1499.8	15.381	150.00	0.162	0.001081	15.382
21	1499.9	15.382	150.00	0.113	0.000755	15.383
22	1499.9	15.383	150.00	0.079	0.000527	15.384
23	1499.9	15.384	150.00	0.055	0.000368	15.384
24	1500.0	15.384	150.00	0.039	0.000257	15.384

Geometrically Nonlinear Analysis of Plane Frames

□ Example



Simple Newton Raphson Iteration (Simple N.R.It)

Geometrically Nonlinear Analysis of Plane Frames

□ Example

روش دوم: Linear Incremental Method with Iteration

b) Updated Newton Raphson Iteration (Updated N.R.It)

$$\Delta P = 1500 \text{ (kips)}$$

Geometrically Nonlinear Analysis of Plane Frames

□ Example

$$x^{(2)} = x^{(1)} + \Delta x^{(1)} = 10 + 4.18 \Rightarrow \boxed{x^{(2)} = 14.18 \text{ (in)}} \stackrel{(10)}{\Rightarrow} \boxed{P_{x^{(2)}}^{(2)} = 1440.62 \text{ (kips)}}$$

$$\textcircled{2} \quad \Delta Q^{(2)} = 1500 - P_{x^{(2)}}^{(2)} = 1500 - 1440.62 \Rightarrow \boxed{\Delta Q^{(2)} = 59.381 \text{ (kips)}}$$

$$\stackrel{(11)}{\Rightarrow} \boxed{\tau_{x^{(2)}}^{(2)} = 53.308 \text{ (kips / in)}} \stackrel{\text{(Equation 22)}}{\Rightarrow} \Delta x^{(2)} = \frac{\Delta Q^{(2)}}{\tau^{(2)}} = \frac{59.381}{53.308} \Rightarrow \boxed{\Delta x^{(2)} = 1.1139 \text{ (in.)}}$$

$$x^{(3)} = x^{(2)} + \Delta x^{(2)} = 14.18 + 1.1139 \Rightarrow \boxed{x^{(3)} = 15.294 \text{ (in)}} \stackrel{(10)}{\Rightarrow} \boxed{P_{x^{(3)}}^{(3)} = 1495.86 \text{ (kips)}}$$

$$\textcircled{3} \quad \Delta Q^{(3)} = 1500 - P_{x^{(3)}}^{(3)} = 1500 - 1495.86 \Rightarrow \boxed{\Delta Q^{(3)} = 4.137 \text{ (kips)}}$$

$$\stackrel{(11)}{\Rightarrow} \boxed{\tau_{x^{(3)}}^{(3)} = 45.890 \text{ (kips / in)}} \stackrel{\text{(Equation 22)}}{\Rightarrow} \Delta x^{(3)} = \frac{\Delta Q^{(3)}}{\tau^{(3)}} = \frac{4.137}{45.890} \Rightarrow \boxed{\Delta x^{(3)} = 0.0902 \text{ (in.)}}$$

Geometrically Nonlinear Analysis of Plane Frames

□ Example

Iteration	$P^{(i)}$ (kips)	$x^{(i)}$ (in.)	$\tau^{(i)}$ (kips/in.)	$\Delta Q^{(i)}$ (kips)	$\Delta x^{(i)}$ (in.)	$x^{(i+1)}$ (in.)
0	0	0	150.000	1500	10	10
1	1158.836	10	81.617	341.1639	4.1800	14.18004
2	1440.618	14.18004	53.308	59.38181	1.1139	15.29399
3	1495.862	15.29399	45.890	4.137854	0.0902	15.38416
4	1499.973	15.38416	45.292	0.026937	0.0006	15.38475
5	1500	15.38475	45.288	1.17E-06	0.0000	15.38475

Geometrically Nonlinear Analysis of Plane Frames

□ Example

