Mechanics of Materials

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Ferdinand P.Beer, E.Russel Johnston, Jr., John T.Dewolf

Other Reference:
J.Wat Oler "Lectures notes on Mechanics od Materials"
Ibrahim A.Assakkaf "Lectures notes on Mechanics od Materials"

Energy Methods

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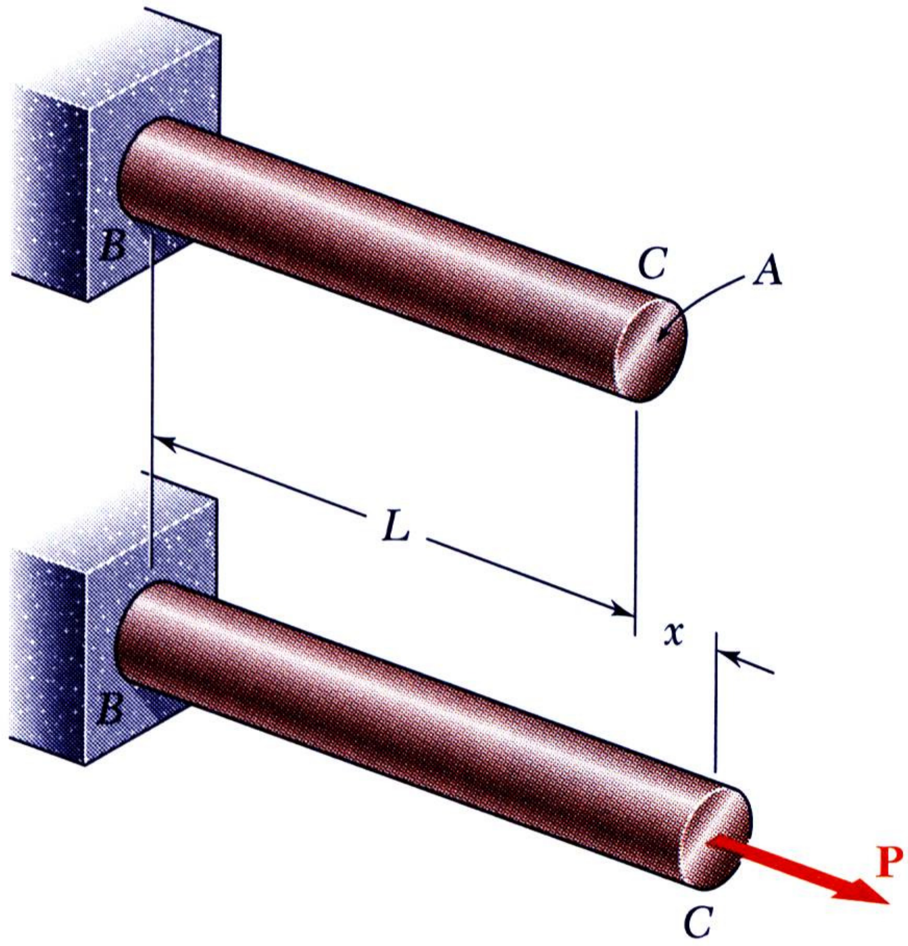
Energy Methods

❑ Strain Energy



- A uniform rod is subjected to a slowly increasing load

- The ***elementary work*** done by the load P as the rod elongates by a small *dx* is

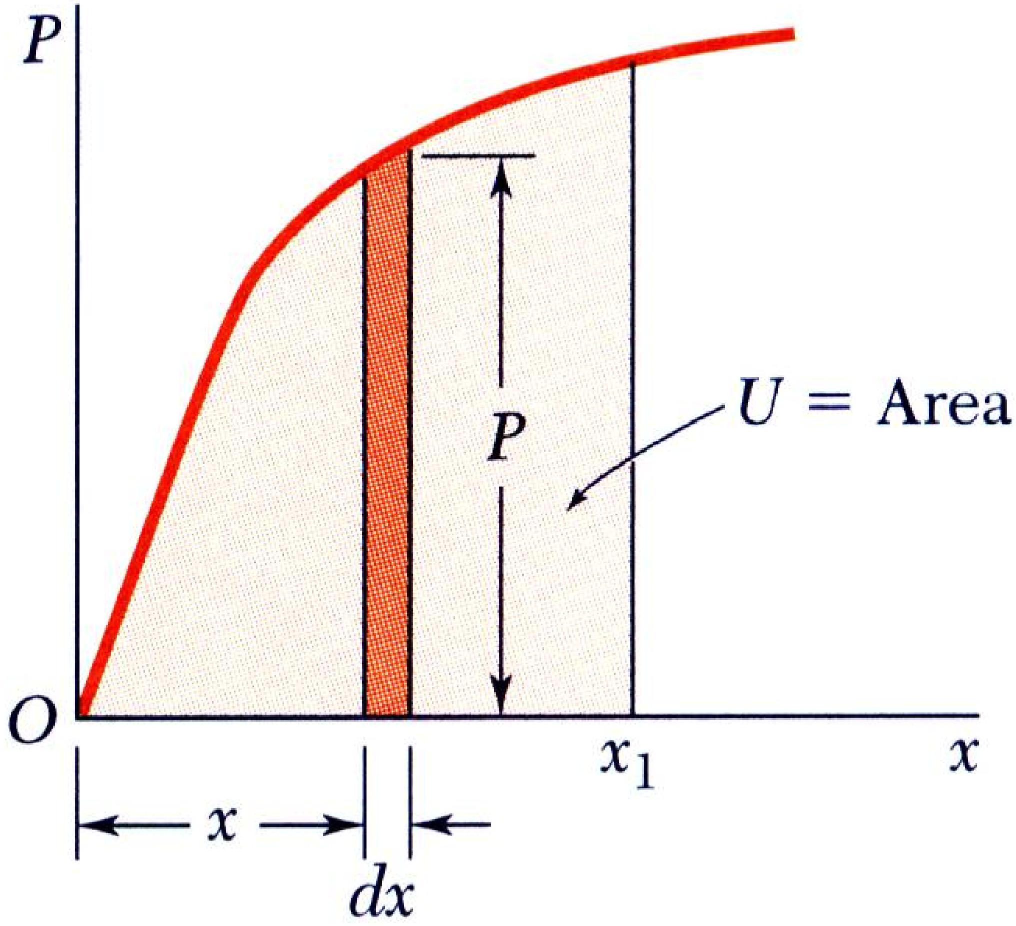
$$dU = P\,dx = elementary\ work$$

Energy Methods

❑ Strain Energy

$$dU = P\,dx = elementary\ work$$

which is equal to the area of width *dx* under the load-deformation diagram.

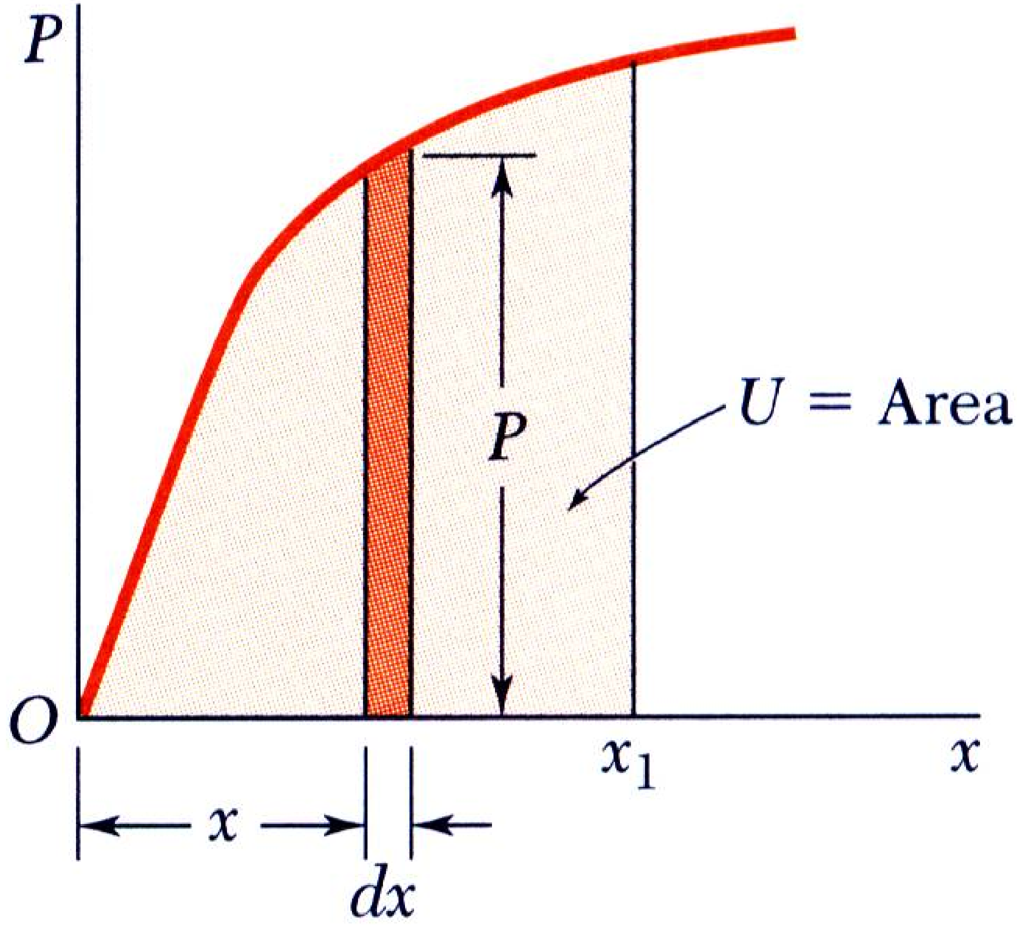


❑ Strain Energy

- The *total work* done by the load for a deformation x_1,

$$U = \int_0^{x_1} P\,dx = total\ work = strain\ energy$$

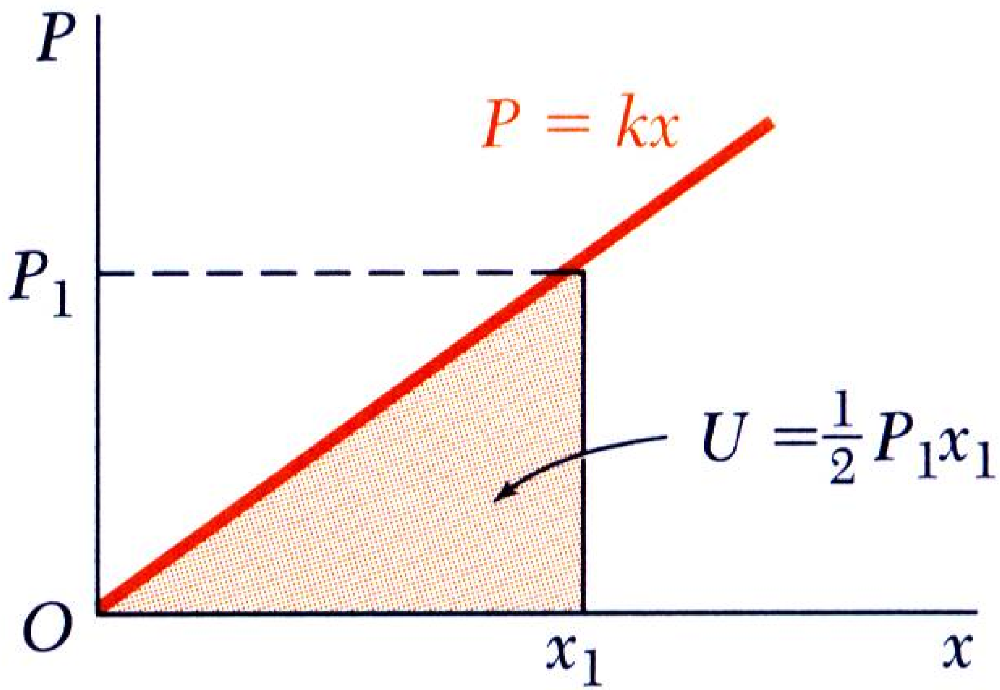
which results in an increase of *strain energy* in the rod.



❑ Strain Energy

- In the case of a linear elastic deformation,

$$U = \int_0^{x_1} kx\,dx = \frac{1}{2}kx_1^2 = \frac{1}{2}P_1x_1$$



❑ Strain Energy Density

- To eliminate the effects of size, evaluate the strain- energy per unit volume,

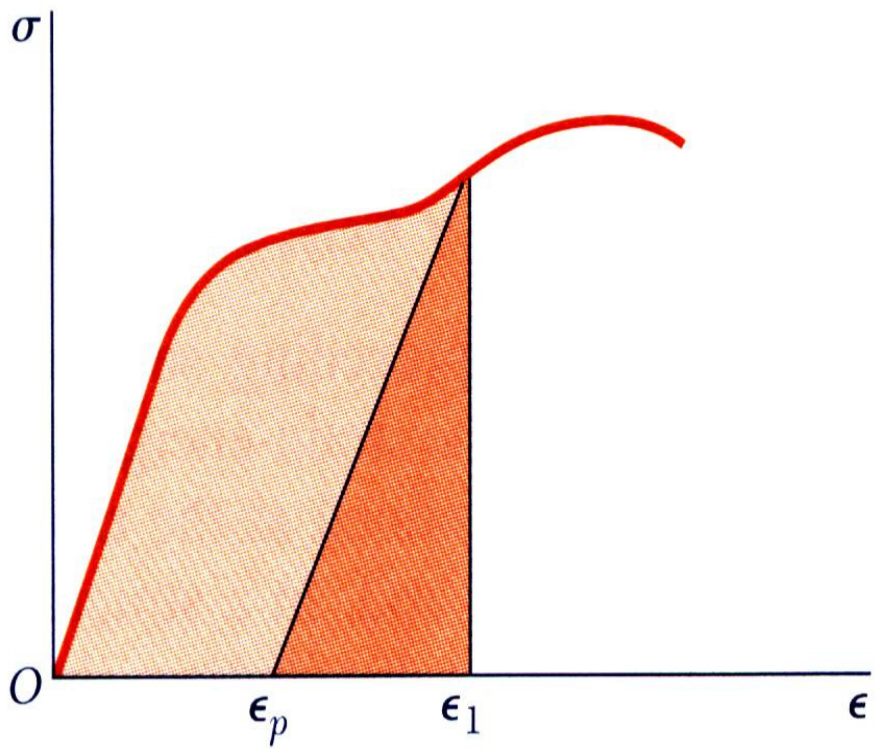
$$\frac{U}{V} = \int_0^{x_1} \frac{P}{A}\frac{dx}{L}$$

$$u = \int_0^{\varepsilon_1} \sigma_x \, d\varepsilon = strain\ \ energy\ density$$

- The total strain energy density resulting from the deformation is equal to the area under the curve to ε_1.

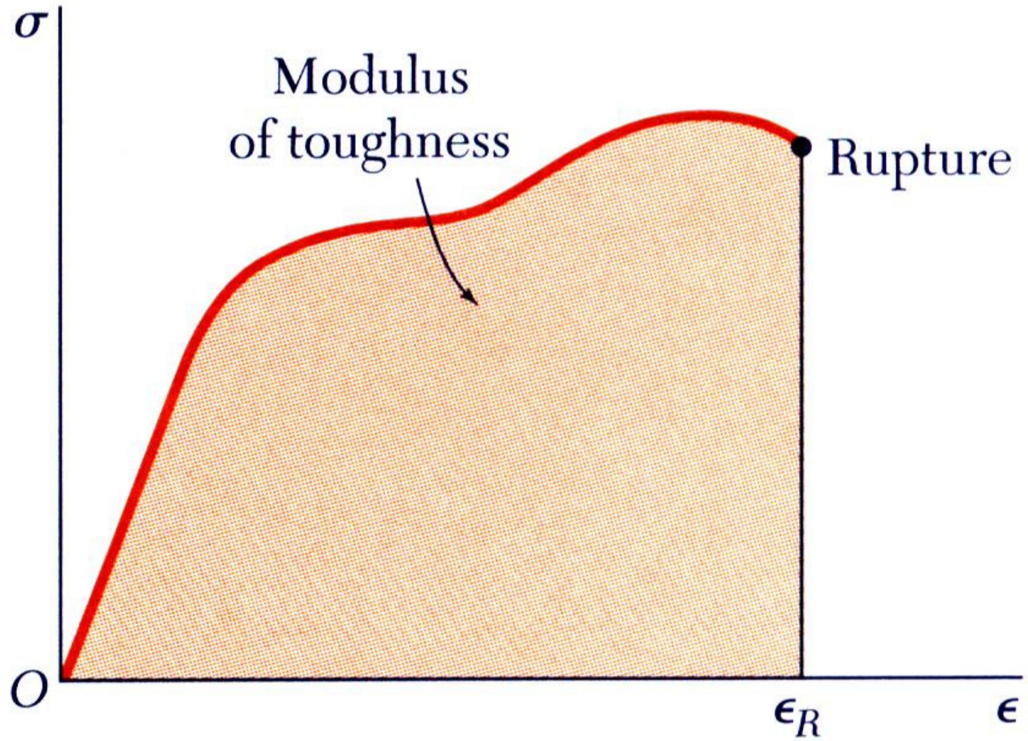
- As the material is unloaded, the stress returns to zero but there is a permanent deformation. Only the strain energy represented by the triangular area is recovered.

- Remainder of the energy spent in deforming the material is dissipated as heat.



❑ Strain Energy Density

- The strain energy density resulting from setting $\varepsilon_1 = \varepsilon_R$ is the *modulus of toughness.*



- The energy per unit volume required to cause the material to rupture is related to its ductility as well as its ultimate strength.

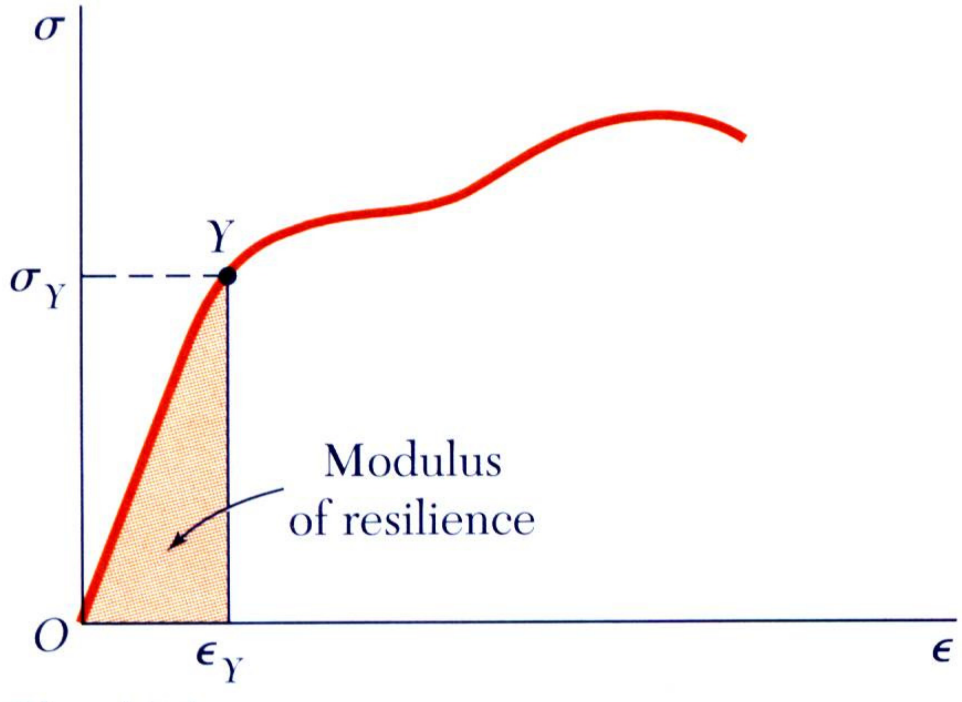
❑ Strain Energy Density

- If the stress remains within the proportional limit,

$$u = \int E\varepsilon_x \, d\varepsilon_x = \frac{E\varepsilon_x^2}{2} = \frac{\sigma_x^2}{2E}$$

- The strain energy density resulting from setting $\sigma_1 = \sigma_Y$ is the *modulus of resilience.*

$$u_Y = \frac{\sigma_Y^2}{2E} = modulus\ of\ resilience$$



Energy Methods

❑ Elastic Strain Energy for Normal Stresses

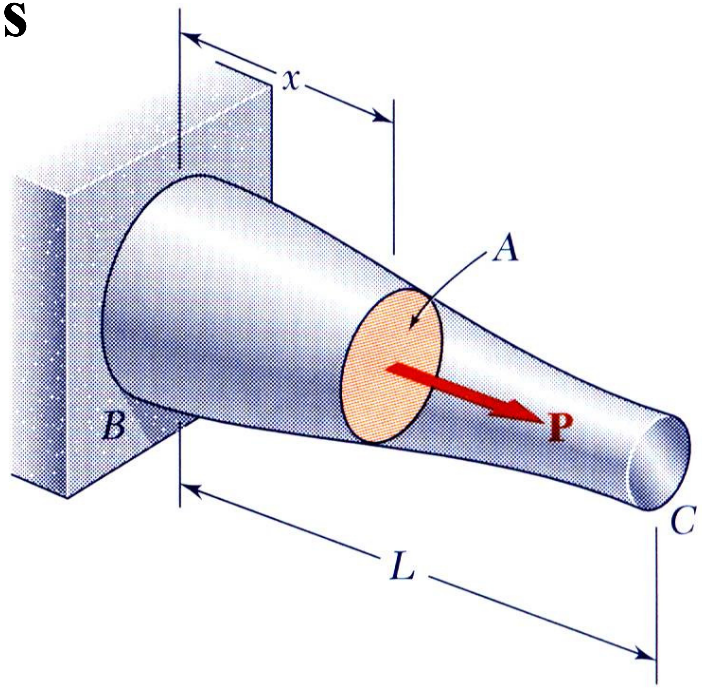
- In an element with a Nonuniform stress distribution,

$$u = \lim_{\Delta V \to 0} \frac{\Delta U}{\Delta V} = \frac{dU}{dV} \qquad U = \int u \, dV = \text{total strain energy}$$

- For values of $u < u_Y$, i.e., below the proportional limit,

$$U = \int \frac{\sigma_x^2}{2E} dV = \textit{elastic strain energy}$$

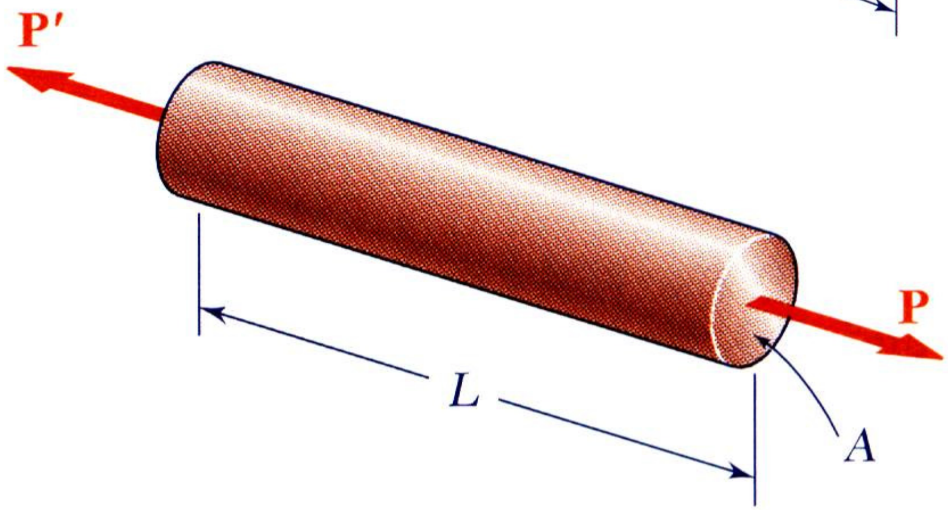
❑ Elastic Strain Energy for Normal Stresses



- Under axial loading,

$$\sigma_x = P/A \qquad dV = A\,dx \quad \Rightarrow \quad U = \int_0^L \frac{P^2}{2AE}dx$$

- For a rod of uniform cross-section,



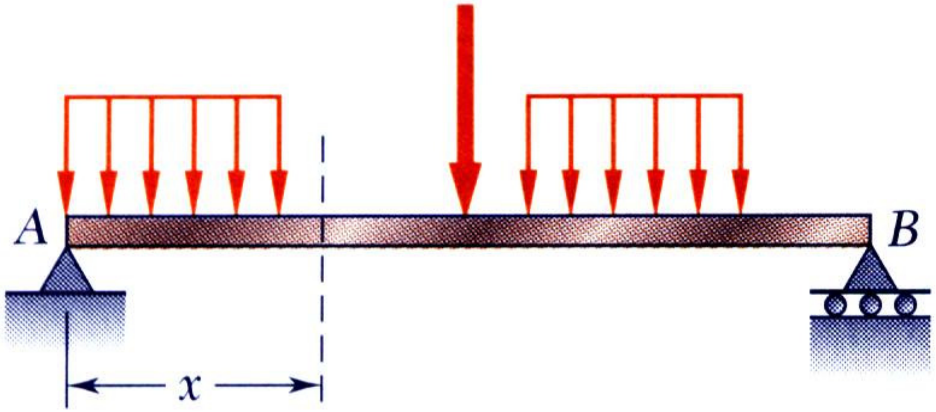
$$U = \frac{P^2 L}{2AE}$$

Energy Methods

❑ Elastic Strain Energy for Normal Stresses

- For a beam subjected to a bending load,

$$\sigma_x = \frac{M\,y}{I} \quad \Rightarrow \quad U = \int \frac{\sigma_x^2}{2E} dV = \int \frac{M^2 y^2}{2EI^2} dV$$



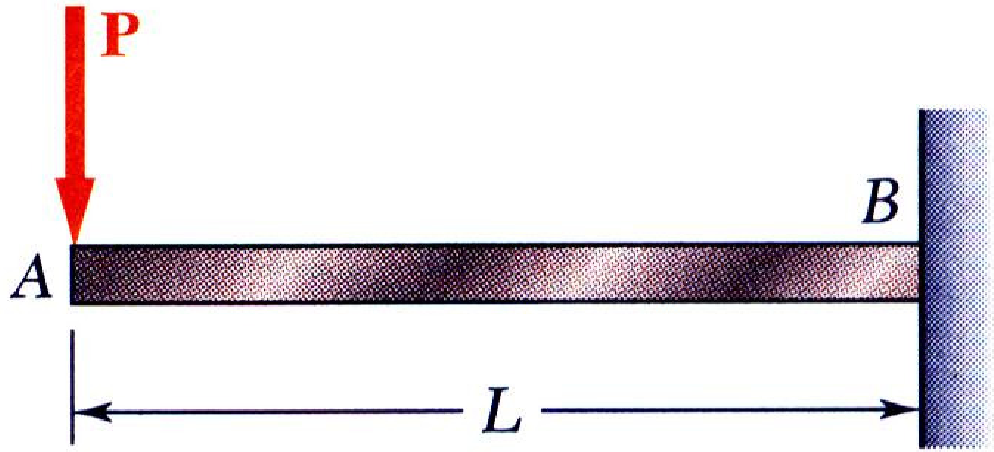
- Setting $dV = dA\,dx$,

$$U = \int_0^L \int_A \frac{M^2 y^2}{2EI^2} dA\,dx = \int_0^L \frac{M^2}{2EI^2} \left(\int_A y^2 dA \right) dx \quad \Rightarrow \quad U = \int_0^L \frac{M^2}{2EI} dx$$

❑ Elastic Strain Energy for Normal Stresses

Example 01

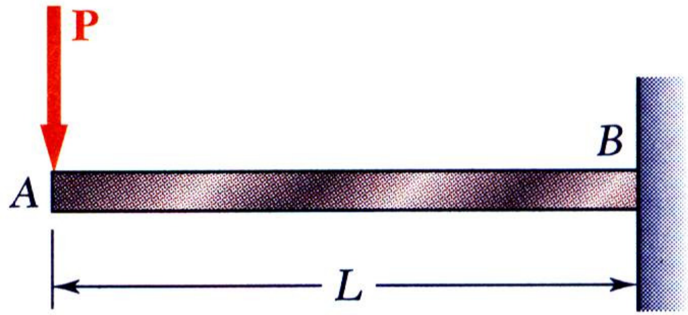
- Determine the total strain energy due to exerted force on the cantilever beam.



❑ Elastic Strain Energy for Normal Stresses

Example 01

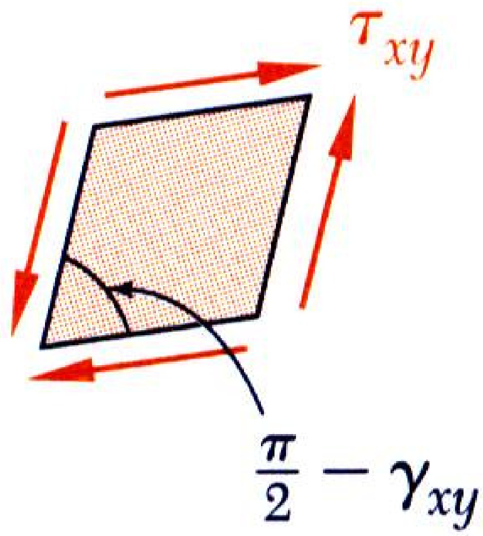
- For an end-loaded cantilever beam,



❑ Strain Energy For Shearing Stresses

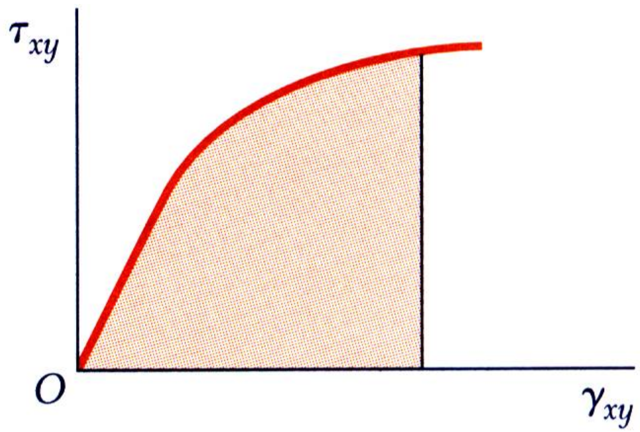
- For a material subjected to plane shearing stresses,

$$u = \int_0^{\gamma_{xy}} \tau_{xy}\, d\gamma_{xy}$$



- For values of τ_{xy} within the proportional limit,

$$u = \tfrac{1}{2} G\gamma_{xy}^2 = \tfrac{1}{2}\tau_{xy}\gamma_{xy} = \frac{\tau_{xy}^2}{2G}$$



- The total strain energy is found from

$$U = \int u\, dV \quad \Rightarrow \quad U = \int \frac{\tau_{xy}^2}{2G}\, dV$$

❑ Strain Energy For Shearing Stresses

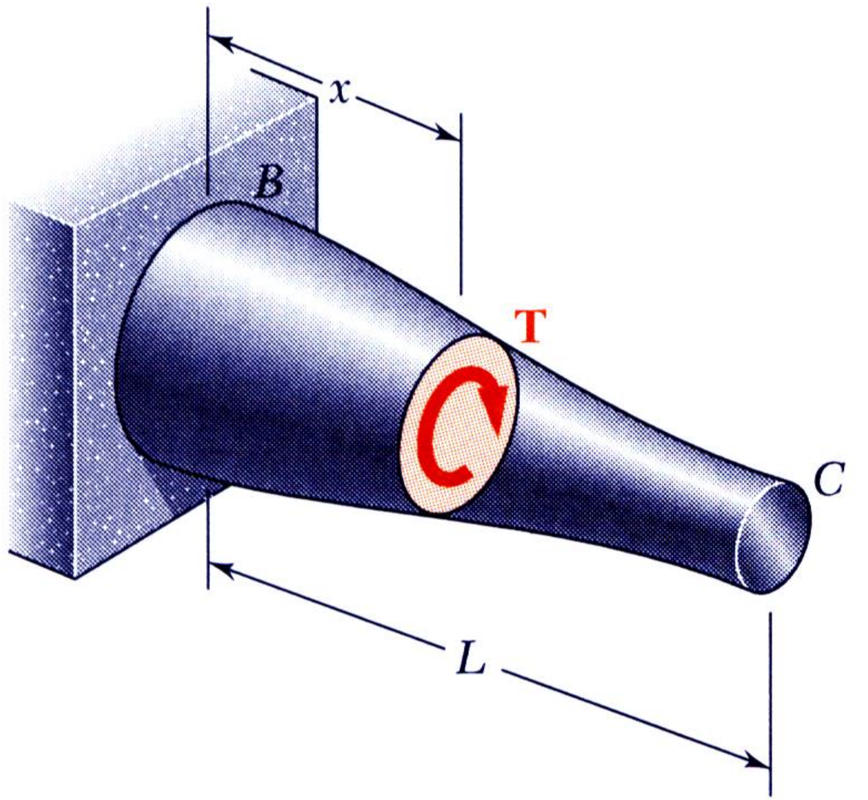
- For a shaft subjected to a torsional load,

$$\tau_{xy} = \frac{T\rho}{J} \quad \Rightarrow \quad U = \int \frac{\tau_{xy}^2}{2G} dV = \int \frac{T^2 \rho^2}{2GJ^2} dV$$

- Setting $dV = dA\, dx$,

$$U = \int_0^L \int_A \frac{T^2 \rho^2}{2GJ^2} dA\, dx = \int_0^L \frac{T^2}{2GJ^2} \left(\int_A \rho^2 dA \right) dx$$

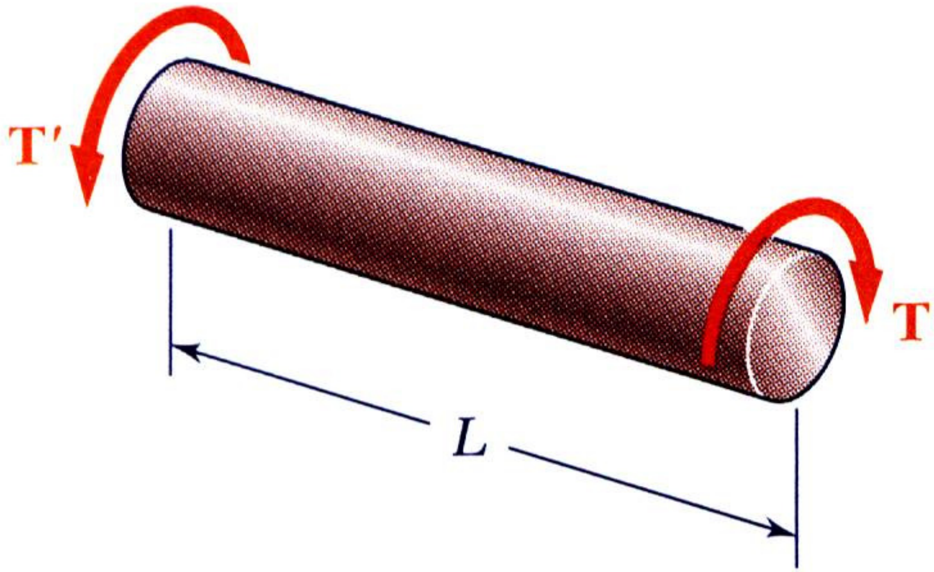
$$\Rightarrow \boxed{U = \int_0^L \frac{T^2}{2GJ} dx}$$



❑ Strain Energy For Shearing Stresses

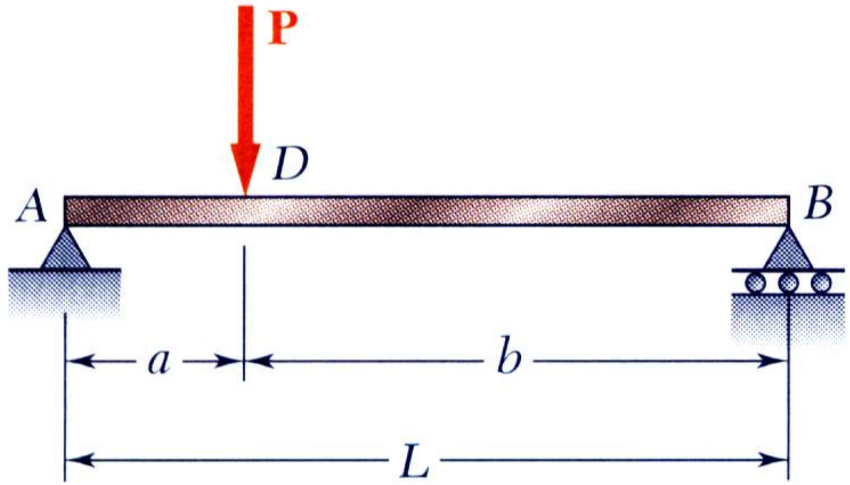
- In the case of a uniform shaft,

$$U = \int_0^L \frac{T^2}{2GJ} dx \quad \Rightarrow \quad U = \frac{T^2 L}{2GJ}$$



❑ Elastic Strain Energy for Normal Stresses

Example 02

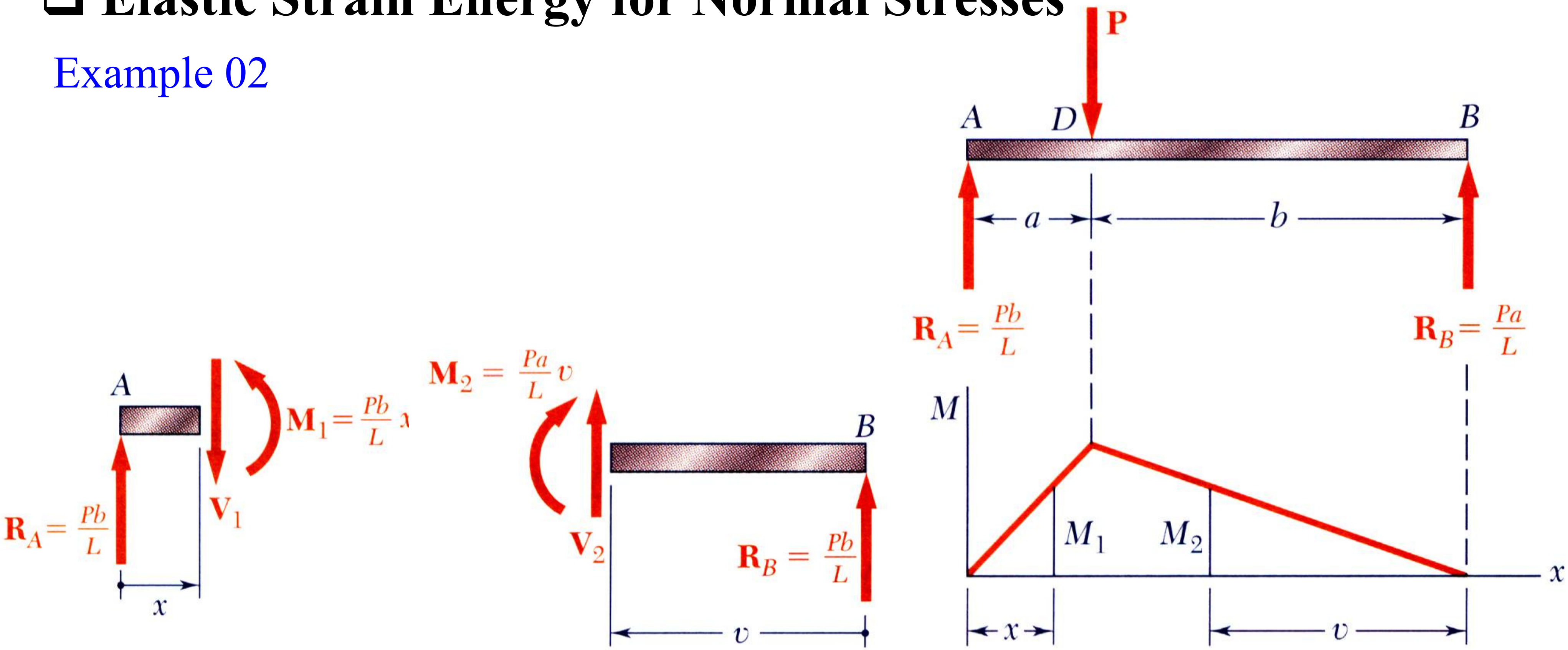


a) Taking into account only the normal stresses due to bending, determine the strain energy of the beam for the loading shown.

b) Evaluate the strain energy knowing that the beam is a W10x45, $P = 40$ kips, $L = 12$ ft, $a = 3$ ft, $b = 9$ ft, and $E = 29\text{x}10^6$ psi.

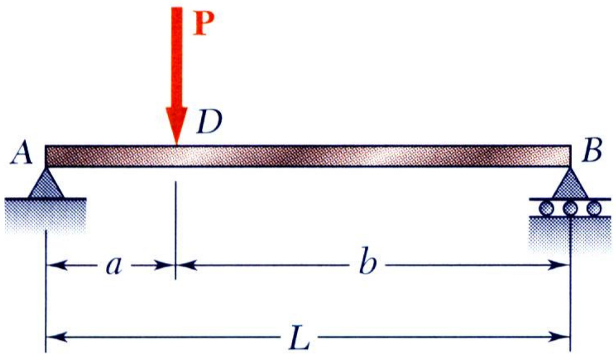
❑ Elastic Strain Energy for Normal Stresses

Example 02



❑ Elastic Strain Energy for Normal Stresses

Example 02



- Integrate over the volume of the beam to find the strain energy.

Over the portion AD,

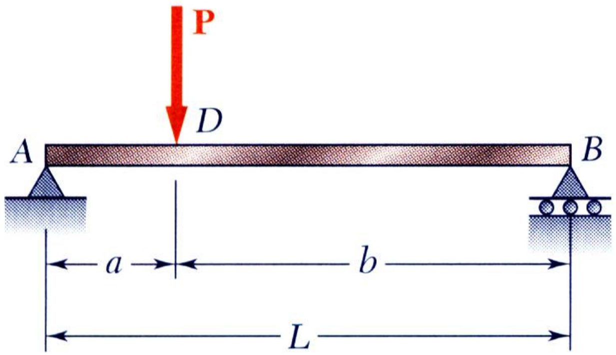
$$M_1 = \frac{Pb}{L}x$$

Over the portion BD,

$$M_2 = \frac{Pa}{L}v$$

Elastic Strain Energy for Normal Stresses

Example 02



$P = 45\,\text{kips}$ $\quad L = 144\text{ in.}$

$\text{a} = 36\text{ in.}$ $\quad b = 108\text{ in.}$

$E = 29\times10^3\,\text{ksi}$ $\quad I = 248\text{ in}^4$

❑ Strain Energy for a General State of Stress

- Previously found strain energy due to uniaxial stress and plane shearing stress. For a general state of stress,

$$u = \tfrac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}\right)$$

- With respect to the principal axes for an elastic, isotropic body,

$$u = \frac{1}{2E}\left[\sigma_a^2 + \sigma_b^2 + \sigma_c^2 - 2\nu\left(\sigma_a\sigma_b + \sigma_b\sigma_c + \sigma_c\sigma_a\right)\right] = u_v + u_d$$

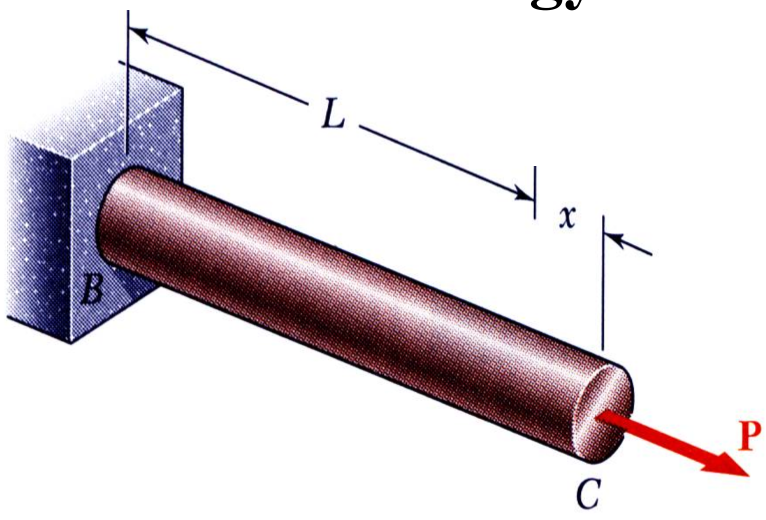
$$u_v = \frac{1-2\nu}{6E}\left(\sigma_a + \sigma_b + \sigma_c\right)^2 = \text{due to volume change}$$

$$u_d = \frac{1}{12G}\left[\left(\sigma_a - \sigma_b\right)^2 + \left(\sigma_b - \sigma_c\right)^2 + \left(\sigma_c - \sigma_a\right)^2\right] = \text{due to distortion}$$

- Basis for the *maximum distortion energy* failure criteria,

$$u_d < \left(u_d\right)_Y = \frac{\sigma_Y^2}{6G} \quad \text{for a tensile test specimen}$$

❑ Work and Energy Under a Single Load



- Previously, we found the strain energy by integrating the energy density over the volume.
  For a uniform rod,

$$U = \int u\,dV = \int \frac{\sigma^2}{2E}\,dV$$

$$= \int_0^L \frac{(P_1/A)^2}{2E}\,A\,dx = \frac{P_1^2 L}{2AE}$$

- Strain energy may also be found from the work of the single load P_1,

$$U = \int_0^{x_1} P\,dx$$

- For an elastic deformation,

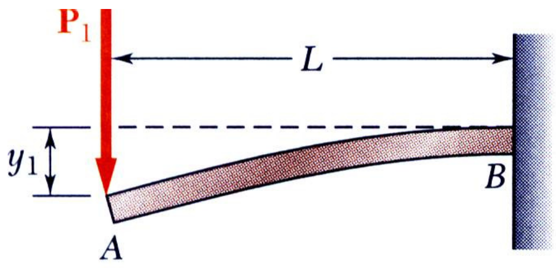
$$U = \int_0^{x_1} P\,dx = \int_0^{x_1} kx\,dx = \tfrac{1}{2}kx_1^2 = \tfrac{1}{2}P_1 x_1$$

- Knowing the relationship between force and displacement,

$$x_1 = \frac{P_1 L}{AE} \quad \Rightarrow \quad U = \tfrac{1}{2}P_1\left(\frac{P_1 L}{AE}\right) = \frac{P_1^2 L}{2AE}$$

❑ Work and Energy Under a Single Load

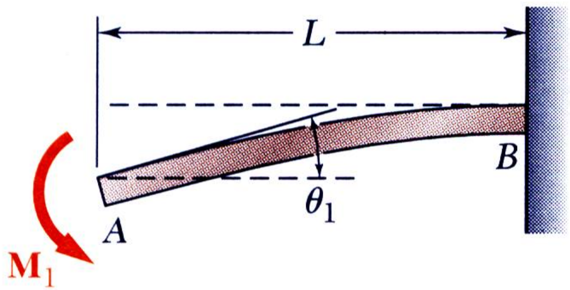
- Transverse load



$$U = \int_0^{y_1} P\,dy = \tfrac{1}{2} P_1 y_1$$

$$= \tfrac{1}{2} P_1 \left(\frac{P_1 L^3}{3EI} \right) = \frac{P_1^2 L^3}{6EI}$$

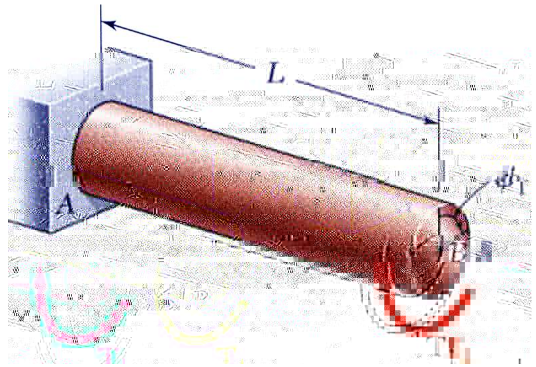
- Bending couple



$$U = \int_0^{\theta_1} M\,d\theta = \tfrac{1}{2} M_1 \theta_1$$

$$= \tfrac{1}{2} M_1 \left(\frac{M_1 L}{EI} \right) = \frac{M_1^2 L}{2EI}$$

- Torsional couple



$$U = \int_0^{\phi_1} T\,d\phi = \tfrac{1}{2} T_1 \phi_1$$

$$= \tfrac{1}{2} T_1 \left(\frac{T_1 L}{JG} \right) = \frac{T_1^2 L}{2JG}$$

❑ Deflection Under a Single Load

- If the strain energy of a structure due to a single concentrated load is known, then the equality between the work of the load and energy may be used to find the deflection.

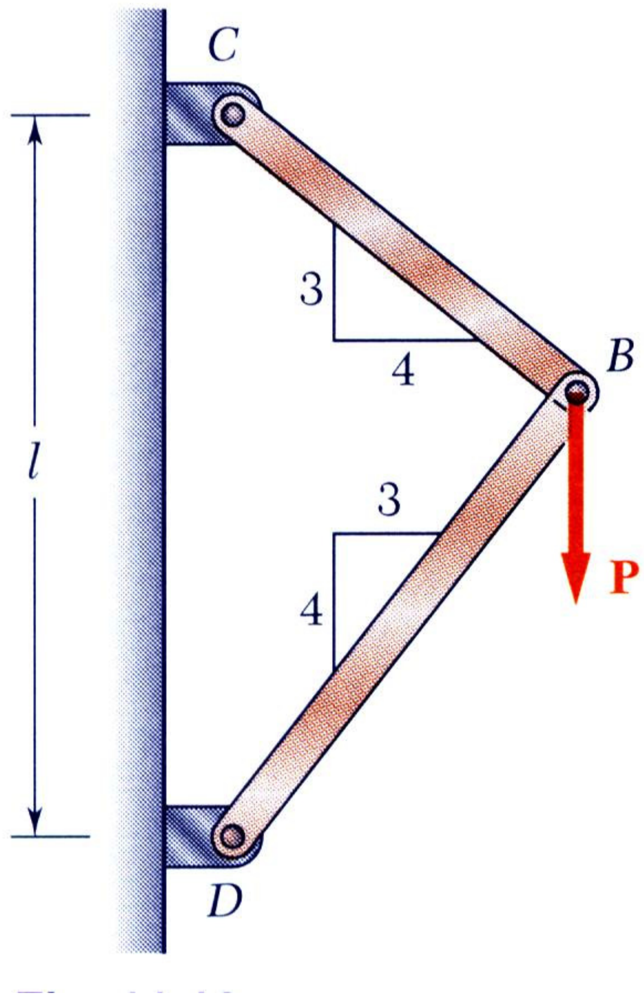
Work = Energy ⇒

Deflection

❑ Deflection Under a Single Load

Example 03

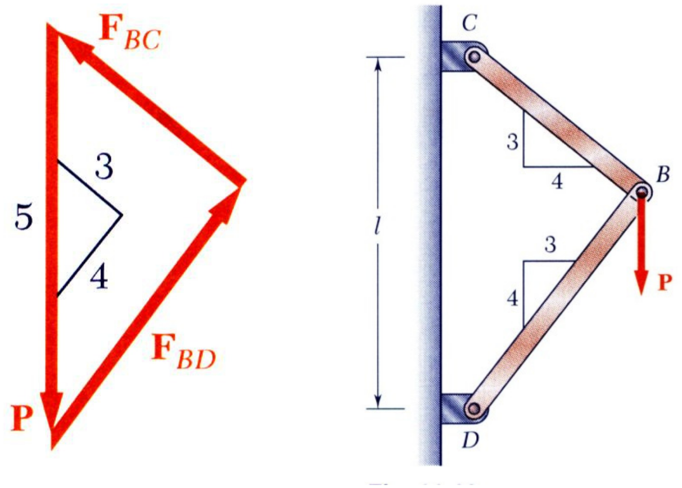
Determine vertical deflection at point B.



Energy Methods

❑ Deflection Under a Single Load

Example 03



From the given geometry,

From statics,

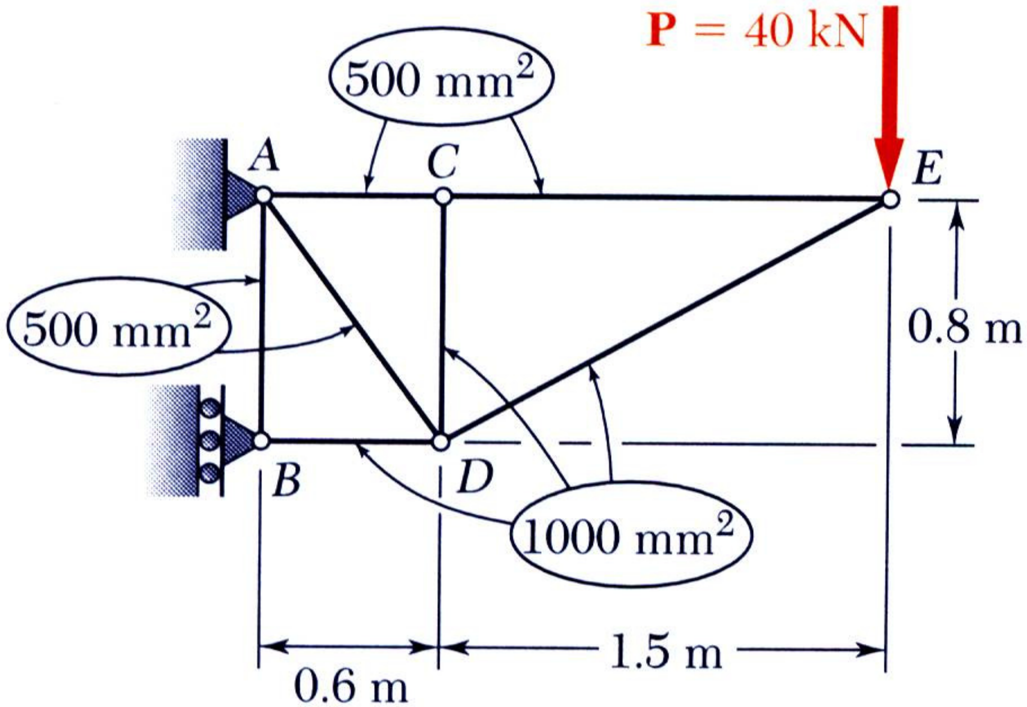
- Strain energy of the structure,

- Equating work and strain energy,

❑ Deflection Under a Single Load

Example 04

Members of the truss shown consist of sections of aluminum pipe with the cross-sectional areas indicated. Using $E = 73$ GPa, determine the vertical deflection of the point E caused by the load P.

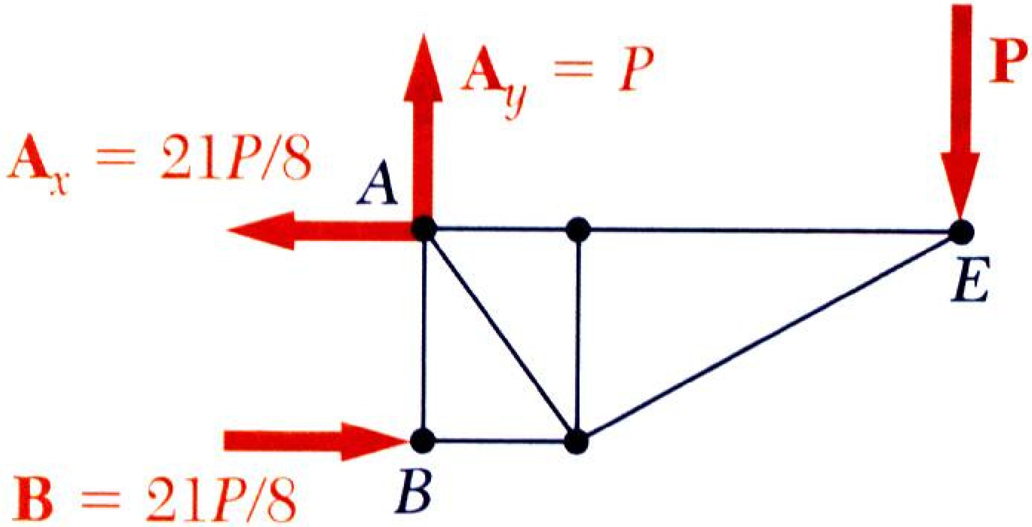


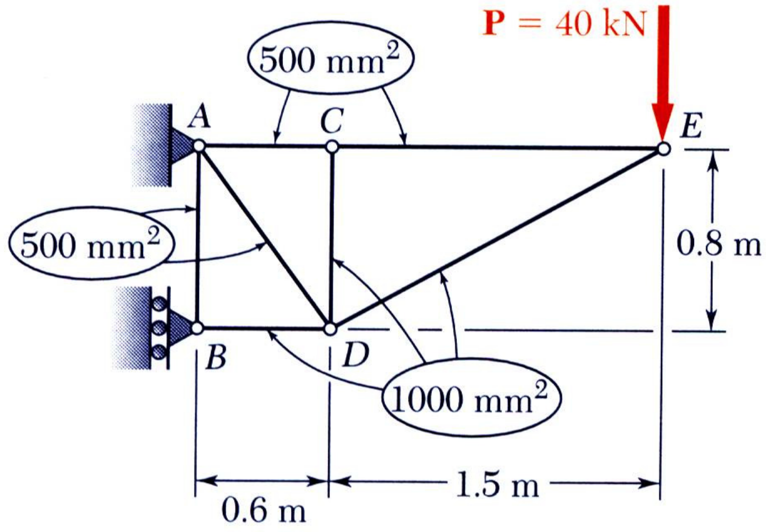
❑ Deflection Under a Single Load

Example 04

SOLUTION:

- Find the reactions at A and B from a free-body diagram of the entire truss.

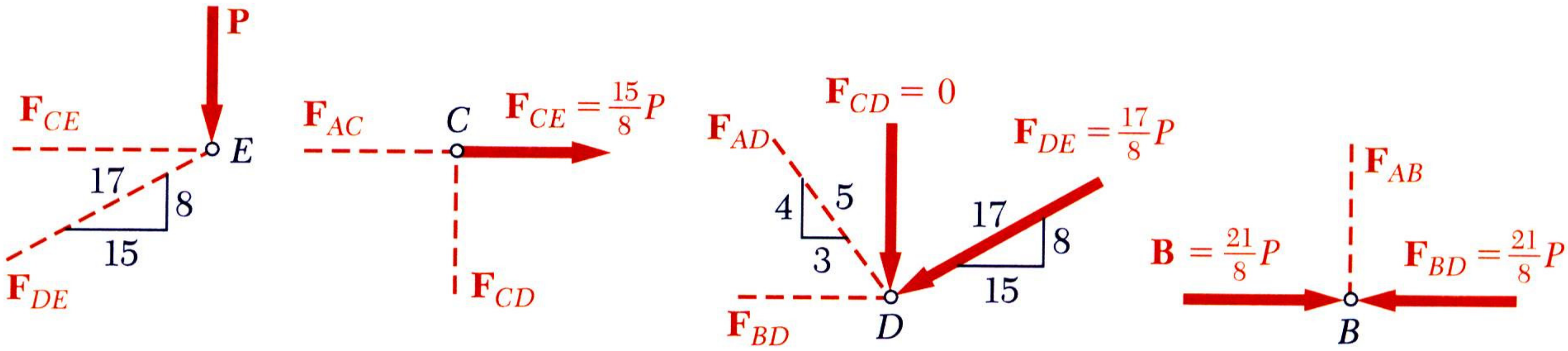




❑ Deflection Under a Single Load

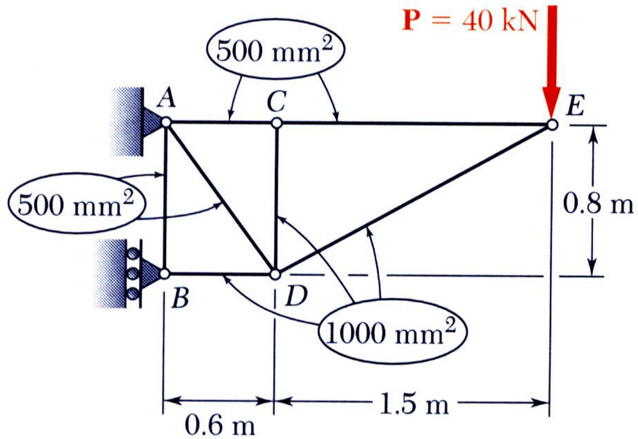
Example 04

- Apply the method of joints to determine the axial force in each member.



❑ Deflection Under a Single Load

Example 04

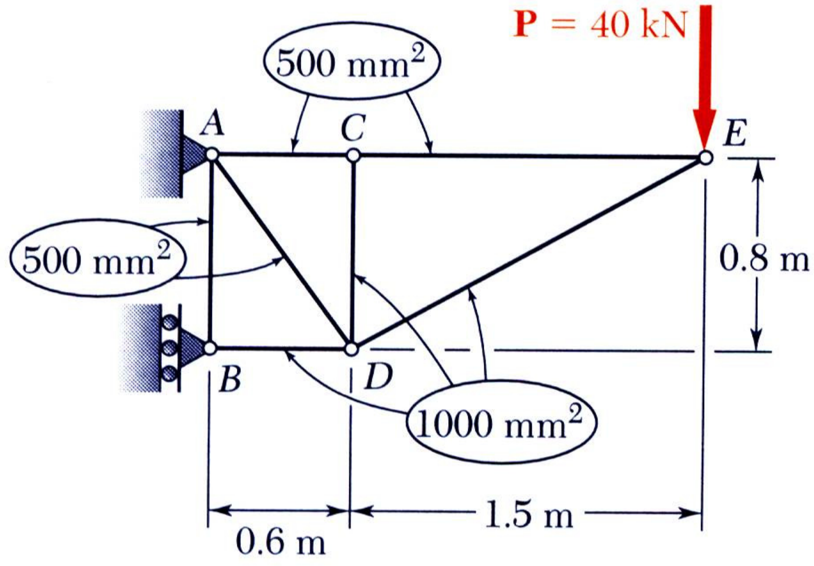


- Evaluate the strain energy of the truss due to the load *P*.

Member	F_i	L_i, m	A_i, m²	$\frac{F_i^2 L_i}{A_i}$
AB	0	0.8	500×10^{-6}	0
AC	$+15P/8$	0.6	500×10^{-6}	$4\,219P^2$
AD	$+5P/4$	1.0	500×10^{-6}	$3\,125P^2$
BD	$-21P/8$	0.6	1000×10^{-6}	$4\,134P^2$
CD	0	0.8	1000×10^{-6}	0
CE	$+15P/8$	1.5	500×10^{-6}	$10\,547P^2$
DE	$-17P/8$	1.7	1000×10^{-6}	$7\,677P^2$

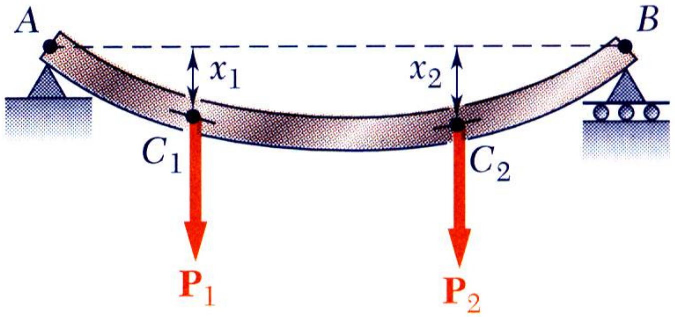
❑ Deflection Under a Single Load

Example 04



- Equate the strain energy to the work by *P* and solve for the displacement.

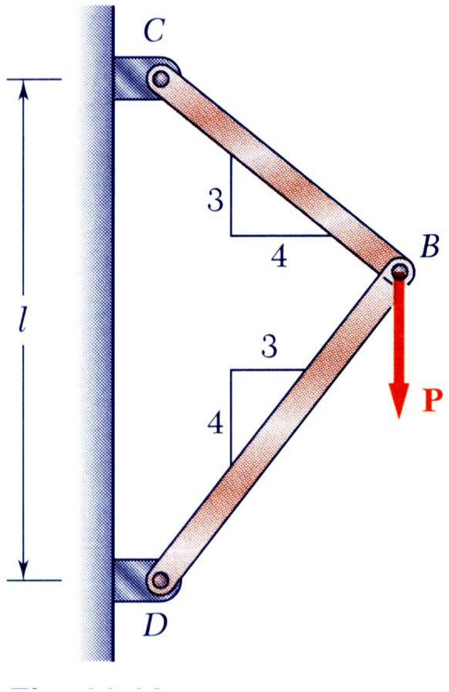
❑ Castigliano's Theorem



- In the case of a beam,

$$U = \int_0^L \frac{M^2}{2EI} dx$$

$$x_j = \frac{\partial U}{\partial P_j} = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial P_j} dx$$



- For a truss,

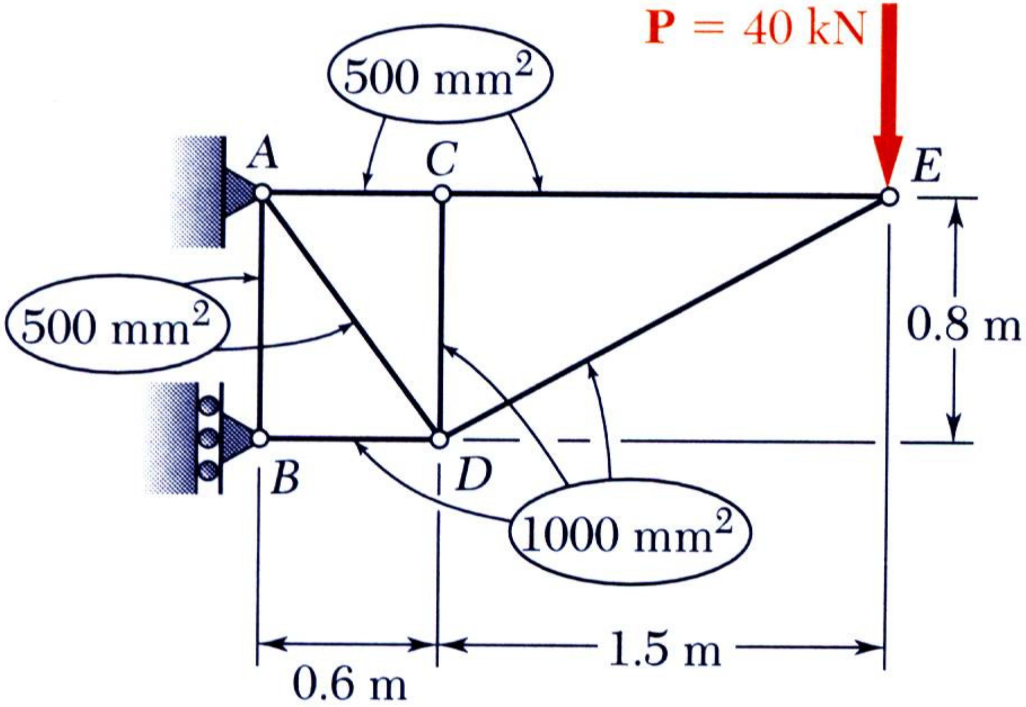
$$U = \sum_{i=1}^{n} \frac{F_i^2 L_i}{2 A_i E}$$

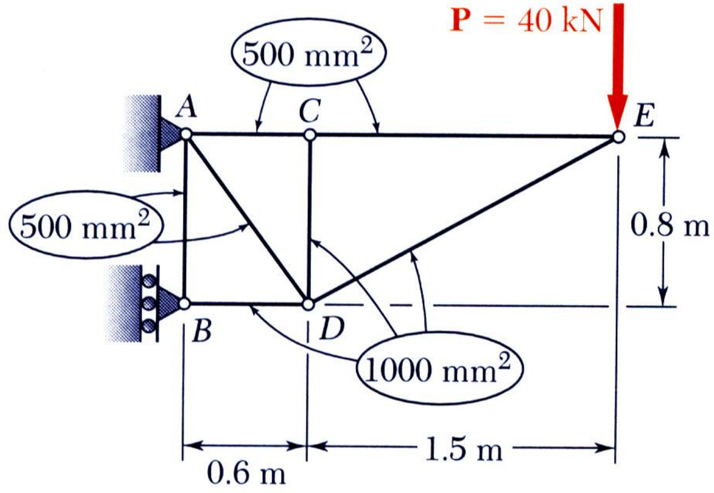
$$x_j = \frac{\partial U}{\partial P_j} = \sum_{i=1}^{n} \frac{F_i L_i}{A_i E} \frac{\partial F_i}{\partial P_j}$$

❑ Deflection Under a Single Load

Example 05

Members of the truss shown consist of sections of aluminum pipe with the cross-sectional areas indicated. Using $E = 73$ GPa, determine the vertical deflection of the joint C caused by the load P.



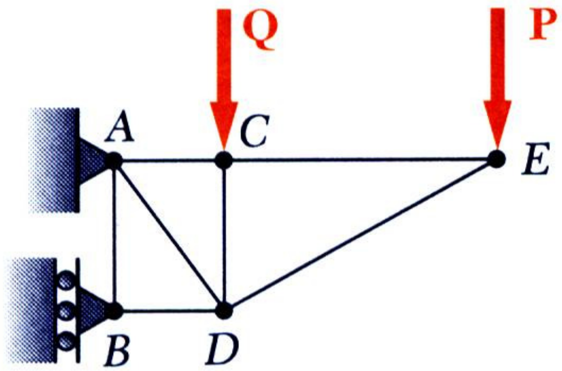


❑ Deflection Under a Single Load

Example 05

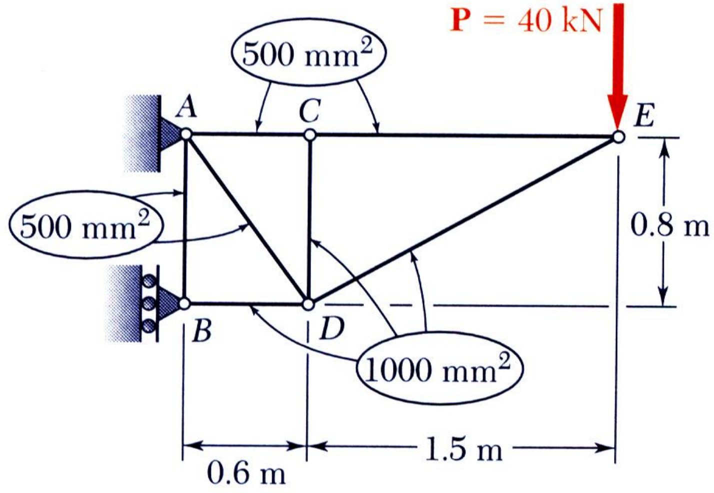
SOLUTION:

- Find the reactions at A and B due to a dummy load Q at C from a free-body diagram of the entire truss.

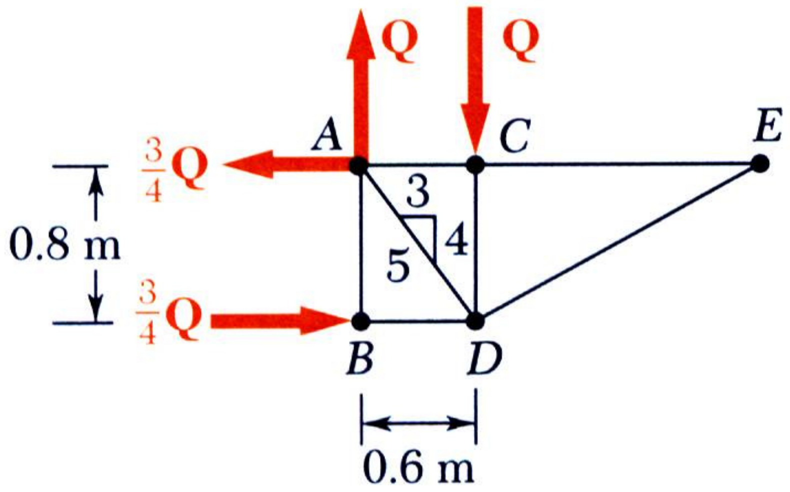


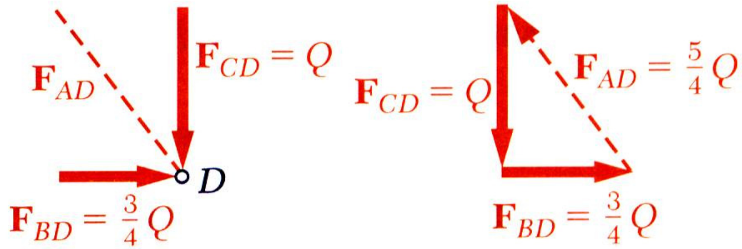
❑ Deflection Under a Single Load

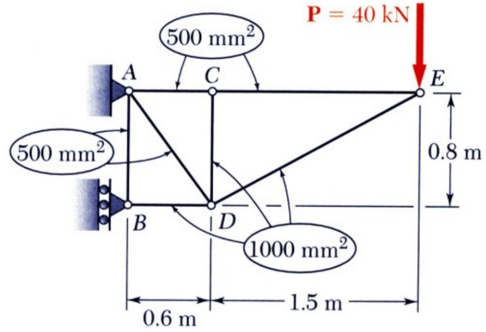
Example 05



- Apply the method of joints to determine the axial force in each member due to Q.







❑ Deflection Under a Single Load

Example 05

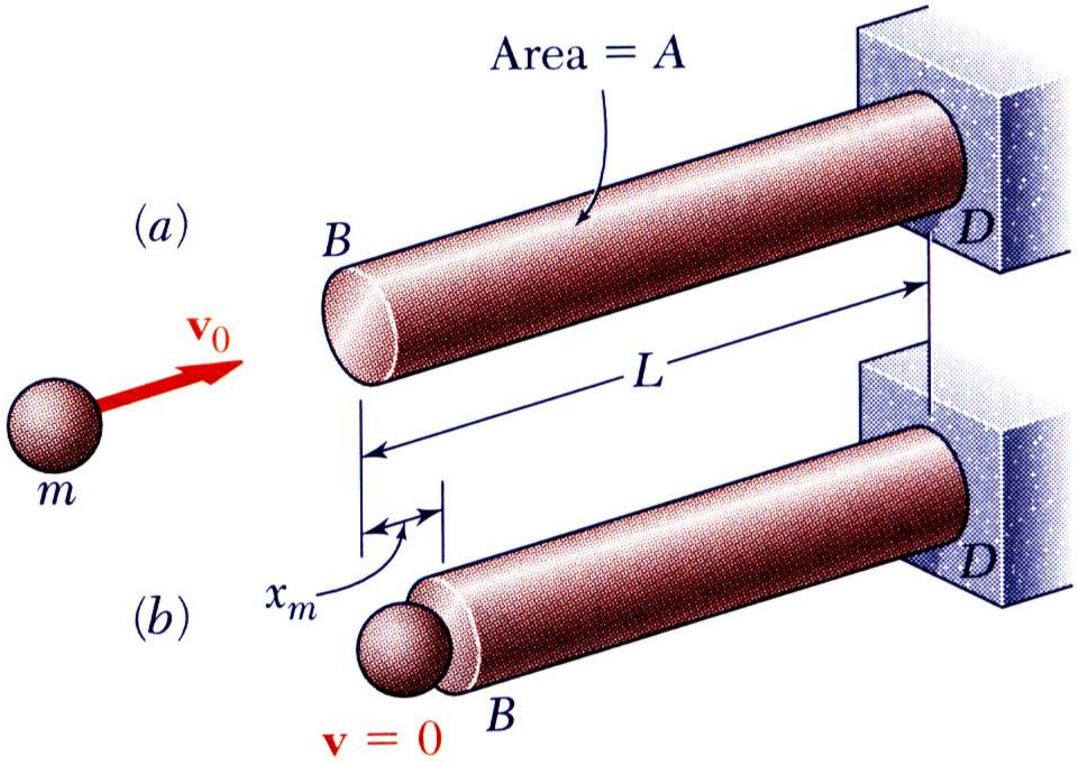
Member	F_i	$\partial F_i/\partial Q$	L_i, m	A_i, m²	$\left(\frac{F_i L_i}{A_i}\right)\frac{\partial F_i}{\partial Q}$
AB	0	0	0.8	500×10^{-6}	0
AC	$+15P/8$	0	0.6	500×10^{-6}	0
AD	$+5P/4 + 5Q/4$	$\frac{5}{4}$	1.0	500×10^{-6}	$+3125P + 3125Q$
BD	$-21P/8 - 3Q/4$	$-\frac{3}{4}$	0.6	1000×10^{-6}	$+1181P + 338Q$
CD	$-Q$	-1	0.8	1000×10^{-6}	$+ 800Q$
CE	$+15P/8$	0	1.5	500×10^{-6}	0
DE	$-17P/8$	0	1.7	1000×10^{-6}	0

Energy Methods

❑ Impact Loading

- Consider a rod which is hit at its end with a body of mass *m* moving with a velocity v_0.

- Rod deforms under impact. Stresses reach a maximum value σ_m and then disappear.



Energy Methods

❑ Impact Loading

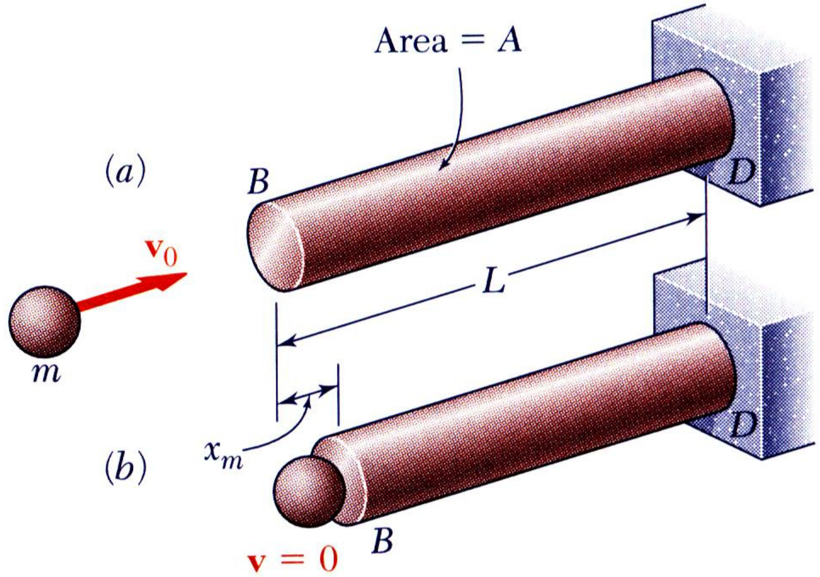
- To determine the maximum stress σ_m
  - Assume that the kinetic energy is transferred entirely to the structure,

$$U_m = \frac{1}{2} m v_0^2$$

  - Assume that the stress-strain diagram obtained from a static test is also valid under impact loading.

- Maximum value of the strain energy,

$$U_m = \int \frac{\sigma_m^2}{2E} dV$$



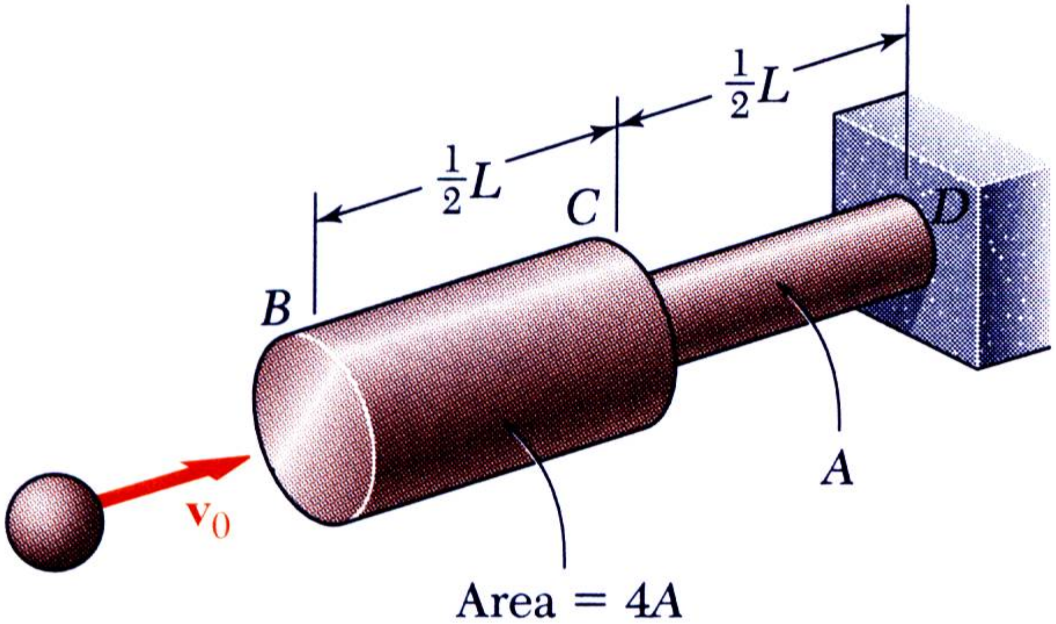
- For the case of a uniform rod,

$$\sigma_m = \sqrt{\frac{2U_m E}{V}} = \sqrt{\frac{m v_0^2 E}{V}}$$

❑ Impact Loading

Example 06

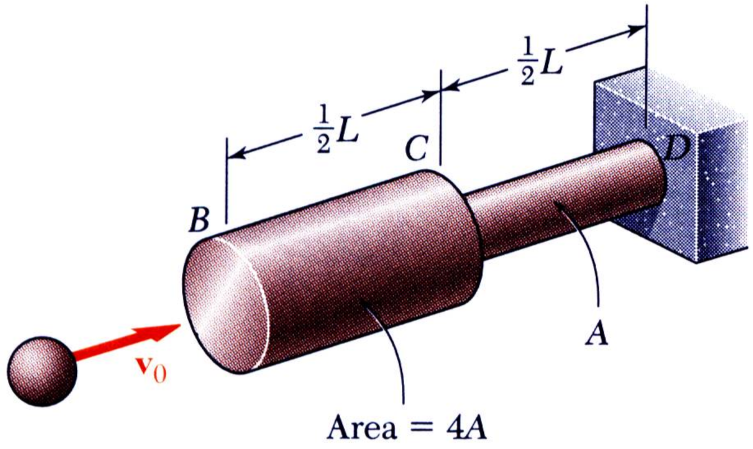
Body of mass *m* with velocity v_0 hits the end of the Nonuniform rod *BCD*. Knowing that the diameter of the portion *BC* is twice the diameter of portion *CD*, determine the maximum value of the normal stress in the rod.



Energy Methods

❑ Impact Loading

Example 06

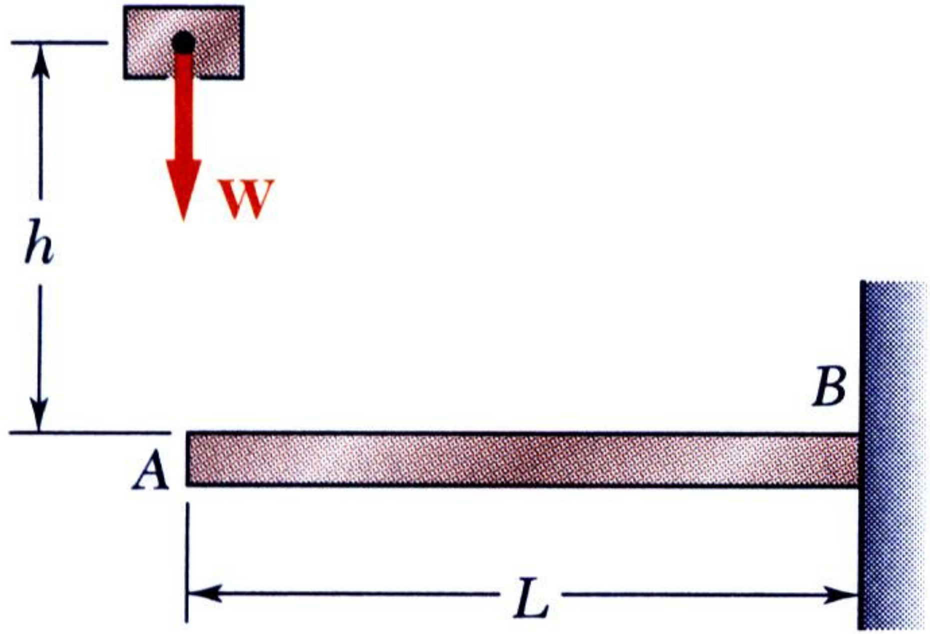


- Find the static load P_m which produces the same strain energy as the impact.

- Evaluate the maximum stress resulting from the static load P_m

❑ Impact Loading

Example 07

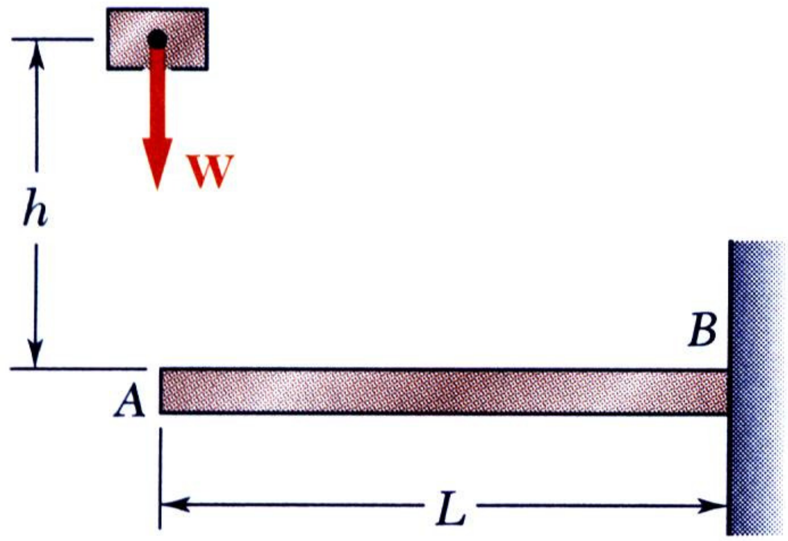


A block of weight W is dropped from a height h onto the free end of the cantilever beam. Determine the maximum value of the stresses in the beam.

Energy Methods

❑ Impact Loading

Example 07



- Find the static load P_m which produces the same strain energy as the impact.

For an end-loaded cantilever beam,

- Evaluate the maximum stress resulting from the static load P_m