Mechanics of Materials

دانشگاه کردستان
University of Kurdistan
زانکۆی کوردستان

Ferdinand P.Beer, E.Russel Johnston, Jr., John T.Dewolf

Other Reference:
J.Wat Oler "Lectures notes on Mechanics od Materials"
Ibrahim A.Assakkaf "Lectures notes on Mechanics od Materials"

Deflection of Beams

By: Kaveh Karami
Associate Prof. of Structural Engineering
https://prof.uok.ac.ir/Ka.Karami

Introductions

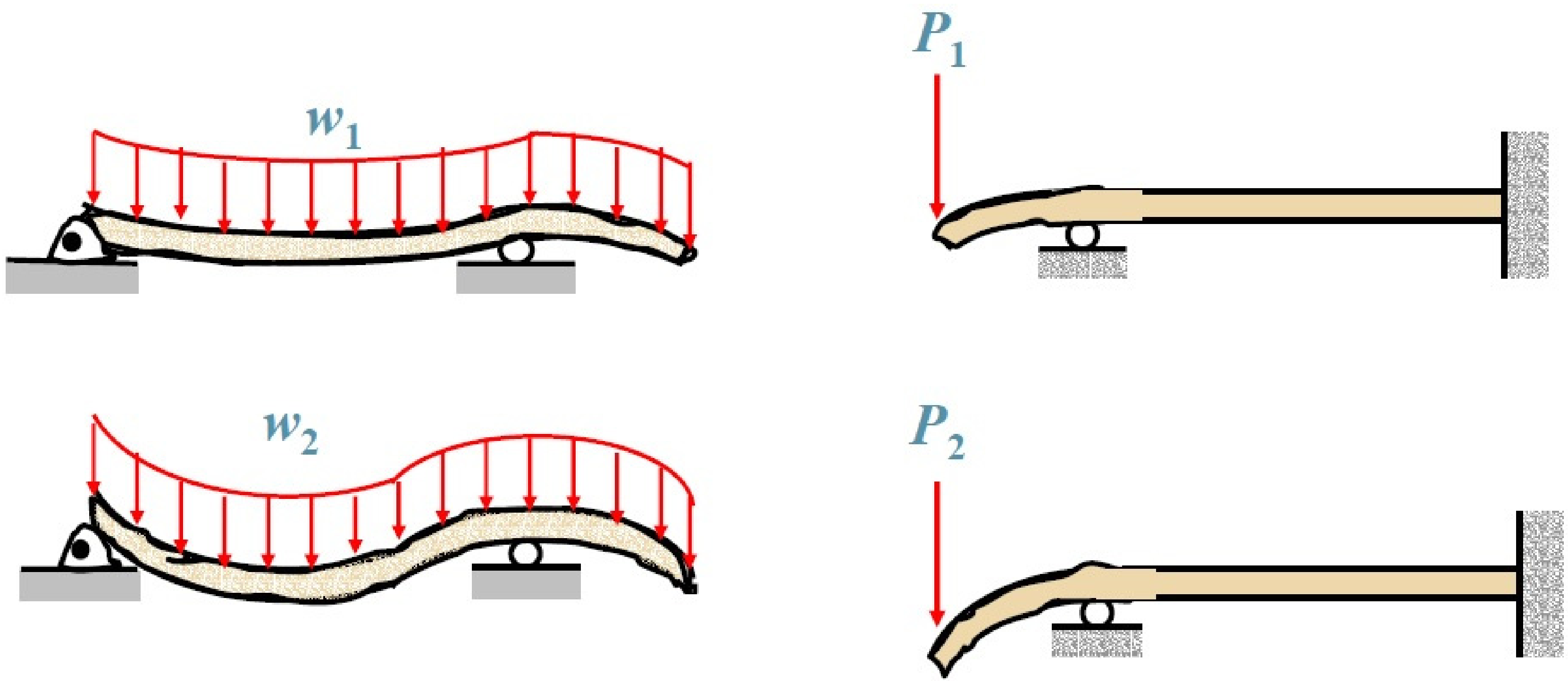
- In this chapter our concern will be beam ***deformation (or deflection)***.
- There are ***important relations*** between applied load and stress (flexural and shear) and the amount of deformation or deflection that a beam can exhibit.
- In design of beams, it is important sometimes to ***limit the deflection*** for specified load.
- So, in these situations, it is not enough only to ***design for the strength*** (flexural normal and shearing stresses), ***but also for excessive deflections*** of beams.

Deflection of Beams

❑ Introductions

– Figure shows generally two examples of how **the amount of deflections increase with the applied loads**.

– Failure to control beam deflections within proper limits in building construction is frequently reflected by the development of cracks in plastered walls and ceilings.



(a) $w_2 >> w_1$ (b) $P_2 >> P_1$

Deflection of Beams

❑ Introductions

The deflection of a beam depends on four general factors:

- ***Stiffness of the materials that the beam is made of,***
- ***Dimensions of the beam,***
- ***Applied loads, and***
- ***Supports***

Three methods are commonly used to find beam deflections:

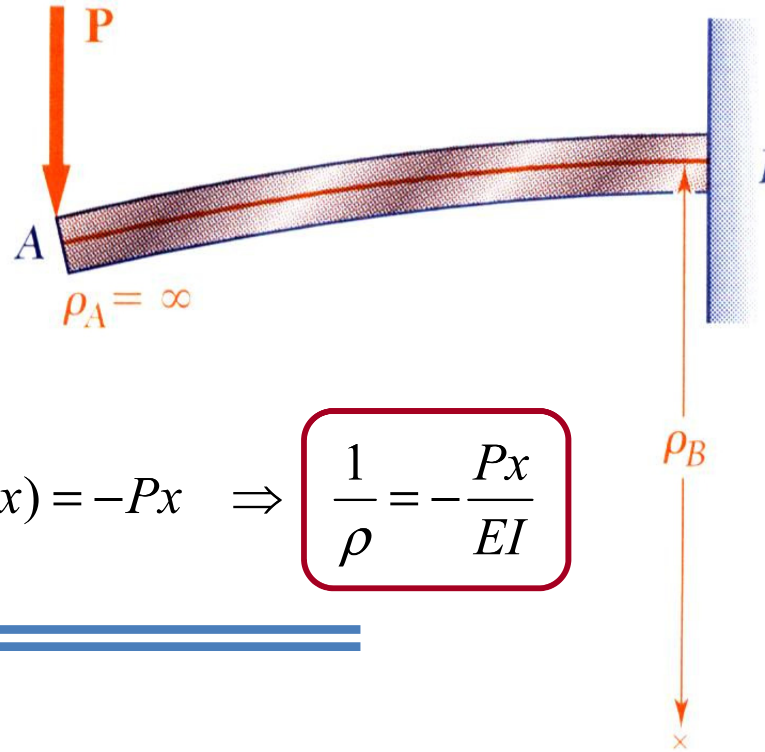
- ***The double integration method,***
- ***The singularity function method, and***
- ***The superposition method***

Deflection of Beams

❑ Deformation of a Beam Under Transverse Loading

- Relationship between bending moment and curvature for pure bending ***remains valid for general transverse loadings.***

$$\frac{1}{\rho} = \frac{M(x)}{EI}$$



- Cantilever beam subjected to concentrated load at the free end,

$$M(x) = -Px \quad \Rightarrow \quad \frac{1}{\rho} = -\frac{Px}{EI}$$

- Curvature varies linearly with x

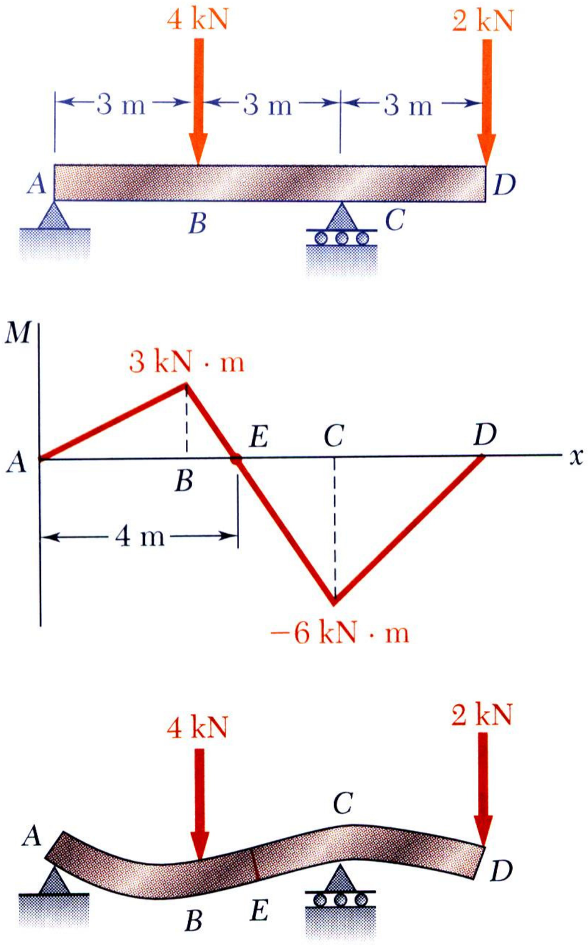
At the free end A, $\quad \frac{1}{\rho_A} = 0, \quad \rho_A = \infty$

At the support B, $\quad \frac{1}{\rho_B} \neq 0, \quad |\rho_B| = \frac{EI}{PL}$

Deflection of Beams

❑ Deformation of a Beam Under Transverse Loading

- Overhanging beam



- Curvature is zero at points where the bending moment is zero, i.e., at each end and at *E*.

$$\frac{1}{\rho} = \frac{M(x)}{EI}$$

- Maximum curvature occurs where the moment magnitude is a maximum.

- Beam is ***concave upwards*** where the bending moment is ***positive*** and ***concave downwards*** where it is ***negative***.

Deflection of Beams

❑ Deformation of a Beam Under Transverse Loading

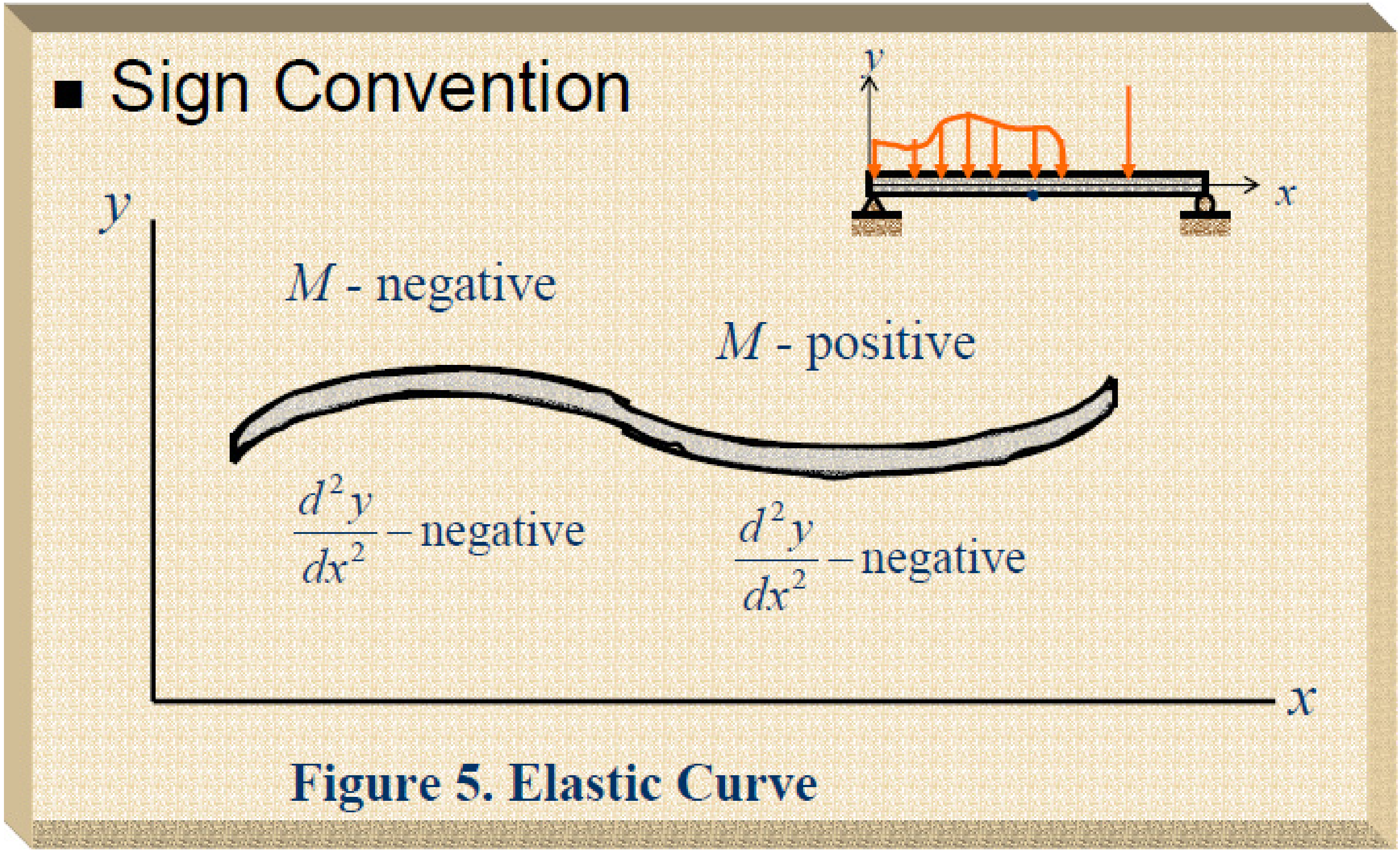
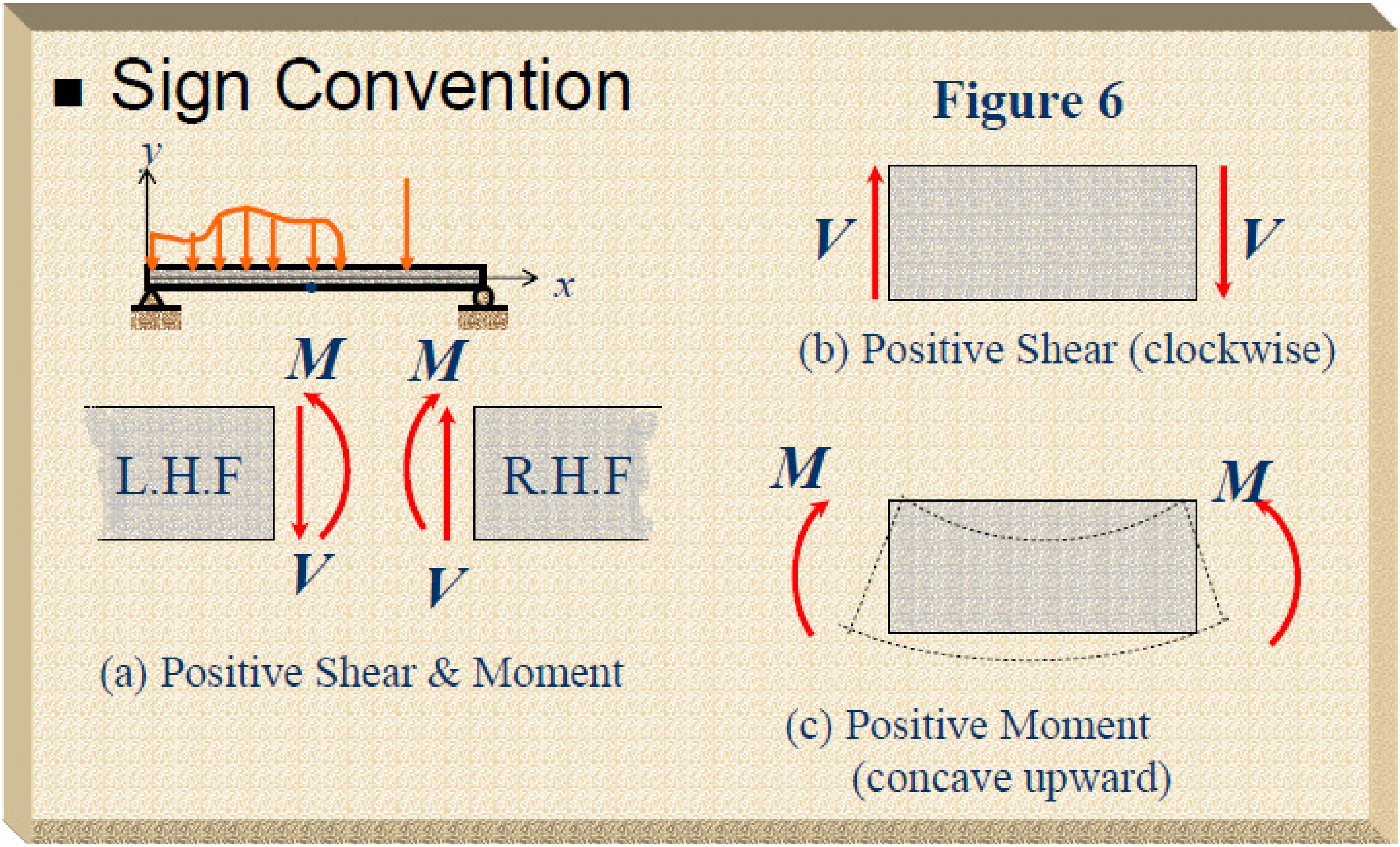


Figure 5. Elastic Curve

Deflection of Beams

❑ Deformation of a Beam Under Transverse Loading



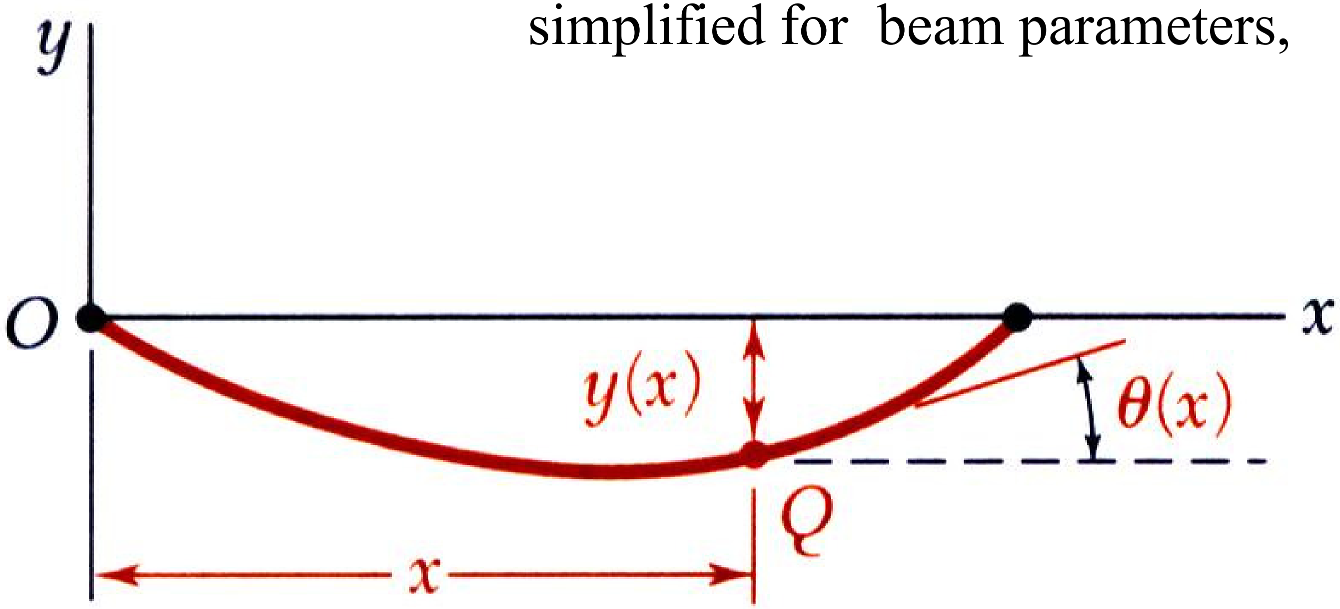
Deflection of Beams

❑ Deformation of a Beam Under Transverse Loading

An equation for the beam shape or *elastic curve* is required to determine maximum deflection and slope.

From elementary calculus, simplified for beam parameters,

$$\frac{1}{\rho} = \frac{\dfrac{d^2 y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}$$

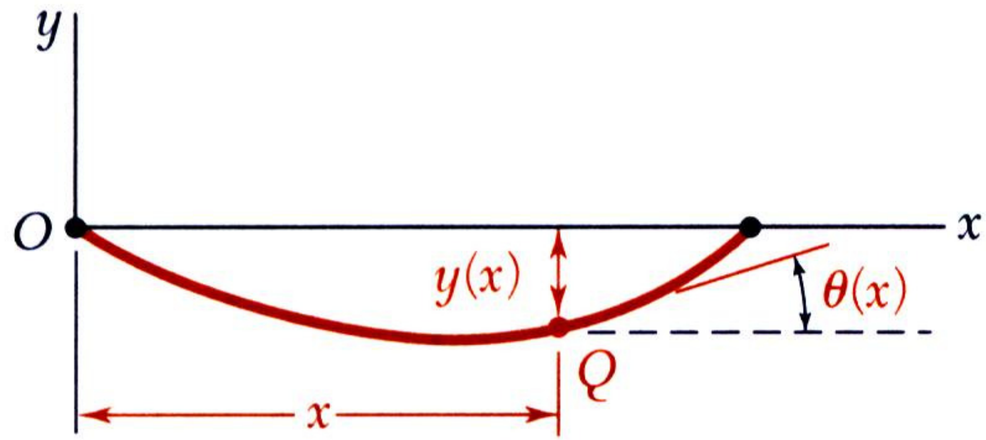


Deflection of Beams

❑ Deformation of a Beam Under Transverse Loading

However, if we realize that for most beams the slope is very small,

$$\left(\frac{dy}{dx}\right) = verys\ mall$$



and its square is much smaller, then term in Eq. 5 can be neglected as compared to unity.

$$\left(\frac{dy}{dx}\right)^2 \approx 0$$

$$\Rightarrow \frac{1}{\rho} \approx \frac{d^2y}{dx^2}$$

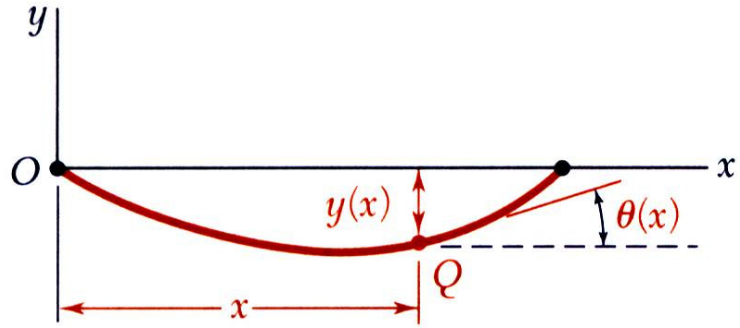
Deflection of Beams

Deformation of a Beam Under Transverse Loading

$$\frac{1}{\rho} \approx \frac{d^2 y}{dx^2}$$

$$\frac{1}{\rho} = \frac{M(x)}{EI}$$

$$\Rightarrow \quad EI\frac{d^2 y}{dx^2} = M(x)$$



- By integrating $\quad EI\int \frac{d^2 y}{dx^2} = \int M(x)$

$$\Rightarrow \quad EI\frac{dy}{dx} = EI\,\theta = \int_0^x M(x)dx + C_1$$

$$\Rightarrow \quad EI\,y = \int_0^x \left(\int_0^x M(x)\,dx \right) dx + C_1 x + C_2$$

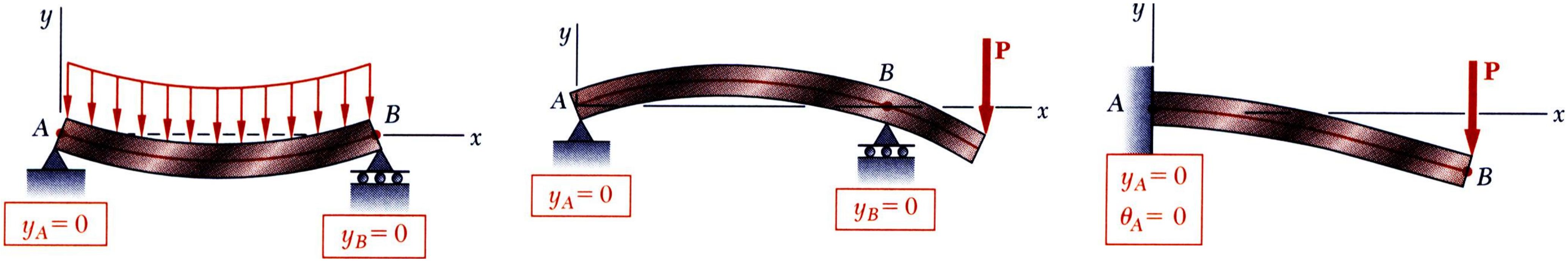
Deflection of Beams

❑ Deformation of a Beam Under Transverse Loading

- Constants are determined from boundary conditions

$$EI\,y = \int_0^x \left(\int_0^x M(x)\,dx \right) dx + C_1 x + C_2$$

- Three cases for statically determinant beams,

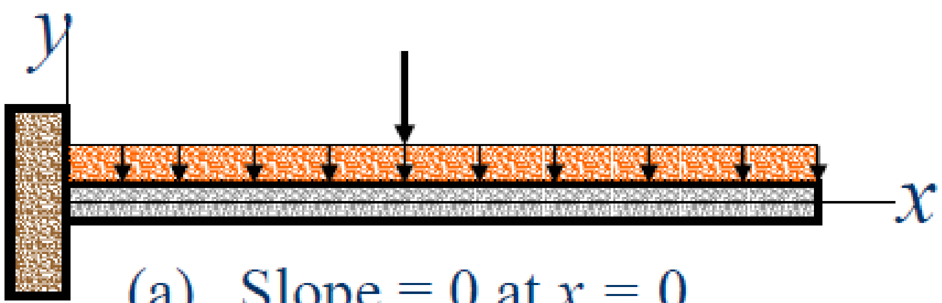


Simply supported beam **Overhanging beam** **Cantilever beam**

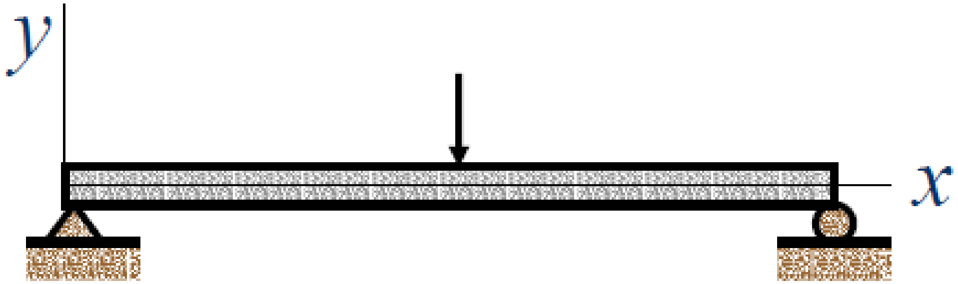
- More complicated loadings require multiple integrals and application of requirement for continuity of displacement and slope.

Deflection of Beams

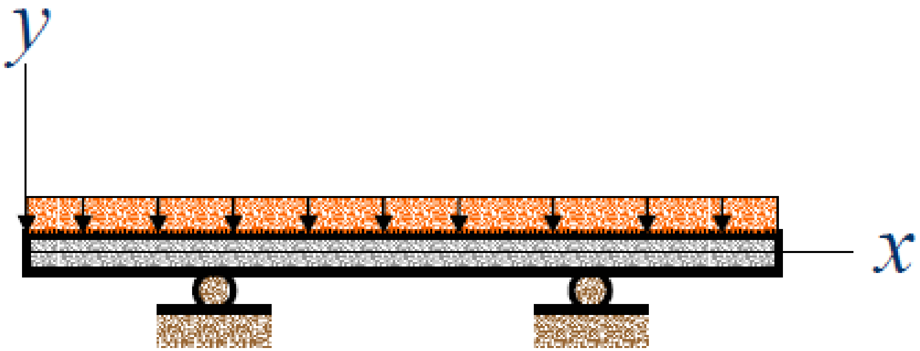
❑ Deformation of a Beam Under Transverse Loading



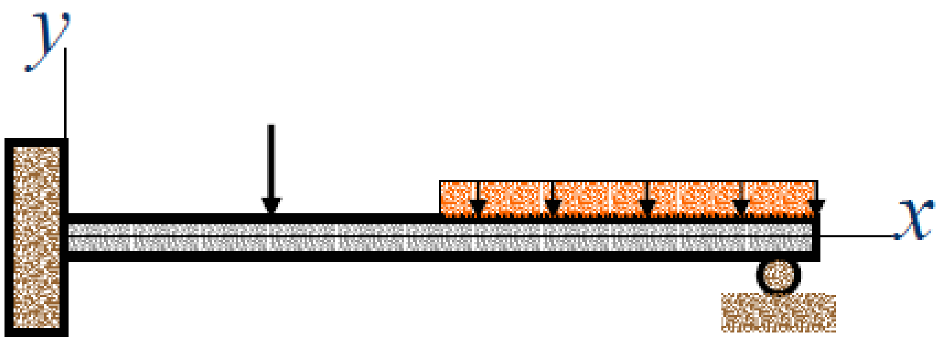
(a) Slope = 0 at $x = 0$
Deflection = 0 at $x = 0$



(b) Slope at $L/2 = 0$
Deflection = 0 at $x = 0$, and L



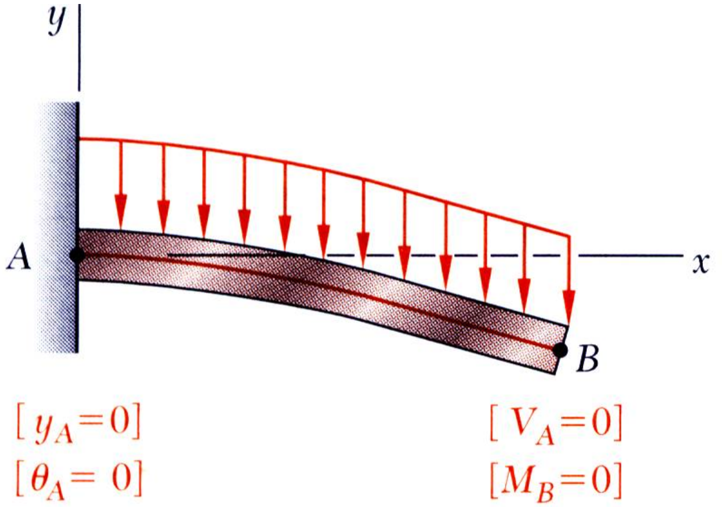
(c) Slope at rollers ?
Deflection at rollers = 0



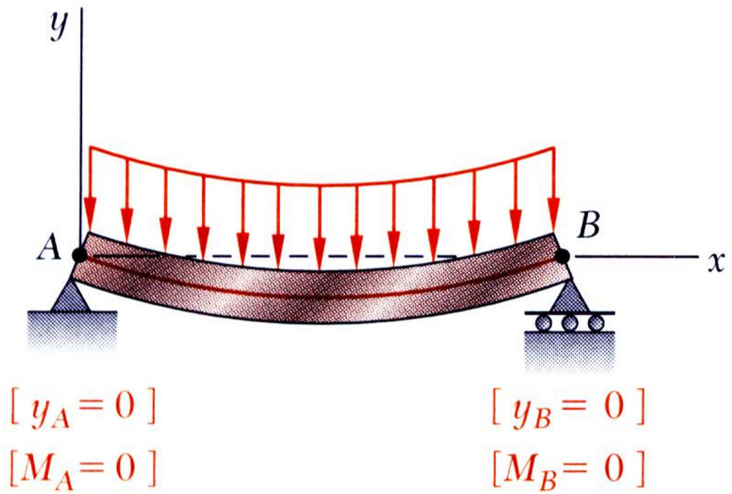
(d) Slope = 0 at $x = 0$
Deflection = 0 at $x = 0$ and $x = L$

Deflection of Beams

❑ Deformation of a Beam Under Transverse Loading



(*a*) Cantilever beam



(*b*) Simply supported beam

- For a beam subjected to a distributed load,

$$\frac{dM}{dx} = V(x) \qquad \frac{d^2M}{dx^2} = \frac{dV}{dx} = -w(x)$$

- Equation for beam displacement becomes

$$\frac{d^2M}{dx^2} = EI\,\frac{d^4y}{dx^4} = -w(x)$$

- Integrating four times yields

$$EI\,y(x) = -\int\left(\int\left(\int\left(\int w(x)\,dx\right)dx\right)dx\right)dx + \tfrac{1}{6}C_1x^3 + \tfrac{1}{2}C_2x^2 + C_3x + C_4$$

- Constants are determined from boundary conditions.

Deflection of Beams

❑ Deformation of a Beam Under Transverse Loading

- **Relation of the Deflection *y* with Physical Quantities such as *V* and *M***

$$deflection = y$$

$$slope = \frac{dy}{dx}$$

$$moment(M) = EI\frac{d^2y}{dx^2}$$

$$shear(V) = \frac{dM}{dx} = EI\frac{d^3y}{dx^3} \quad (for\ EI = cte)$$

$$load(w) = \frac{dv}{dx} = EI\frac{d^4y}{dx^4} \quad (for\ EI = cte)$$

Deflection of Beams

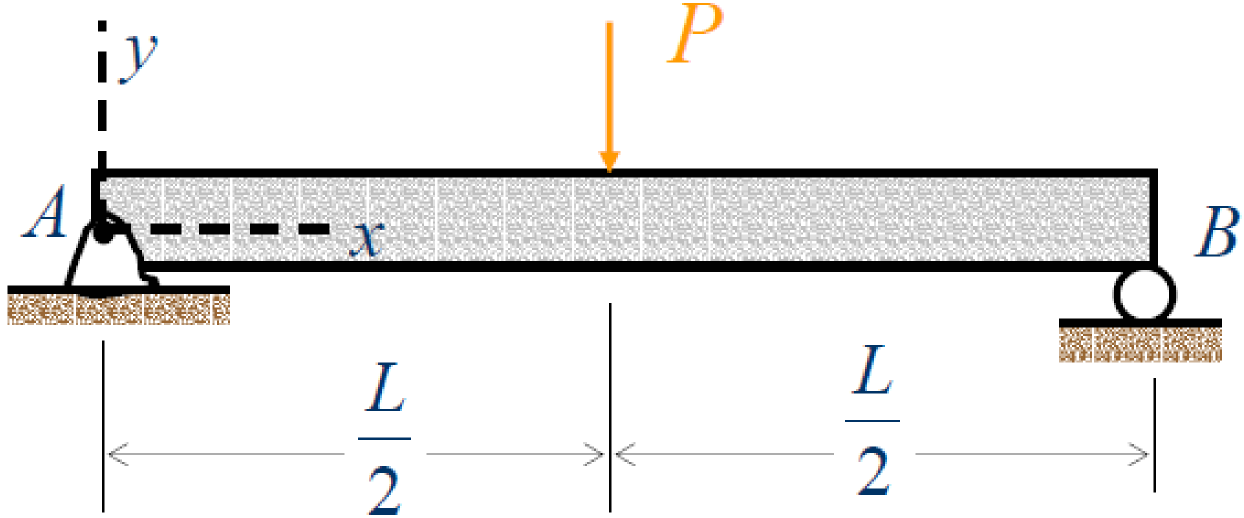
Example 01

A beam is loaded and supported as shown in the figure.

a) Derive the equation of the elastic curve in terms of P, L, x, E, and I.

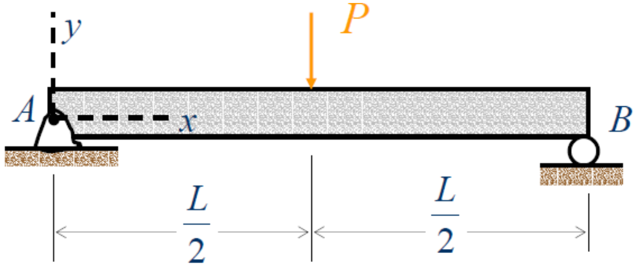
b) Determine the slope at the left end of the beam.

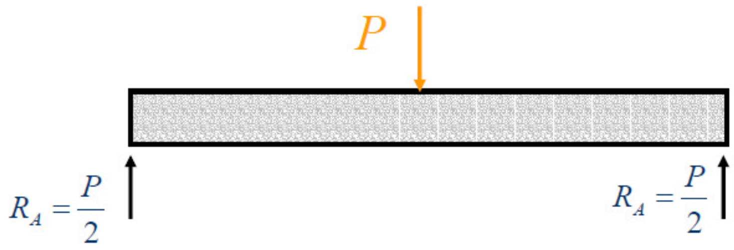
c) Determine the deflection at $x = L/2$.

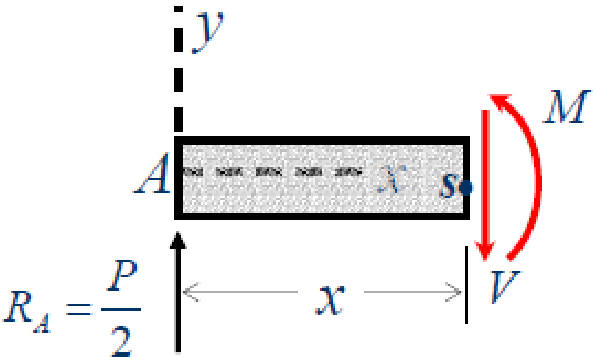


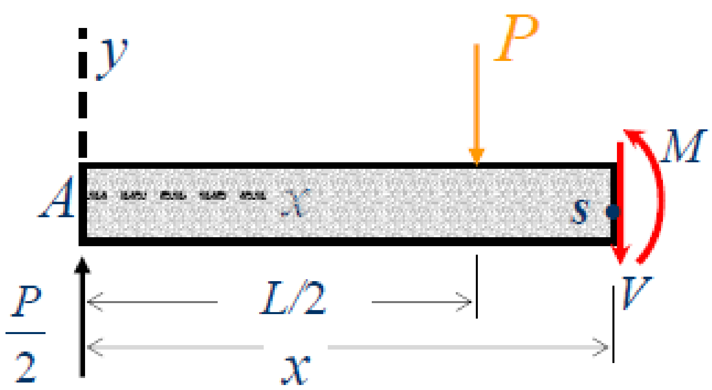
Deflection of Beams

Example 01

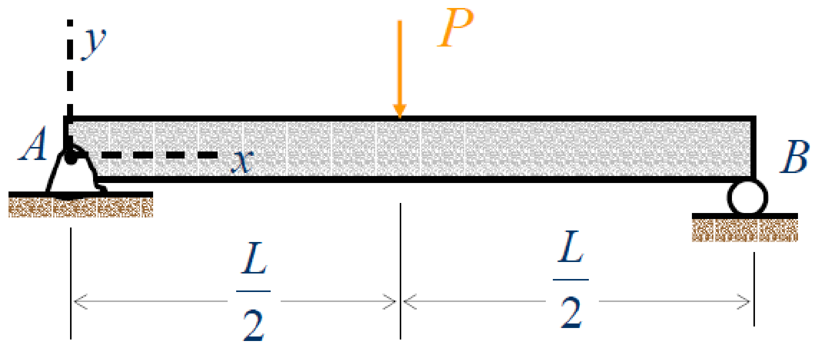








Deflection of Beams



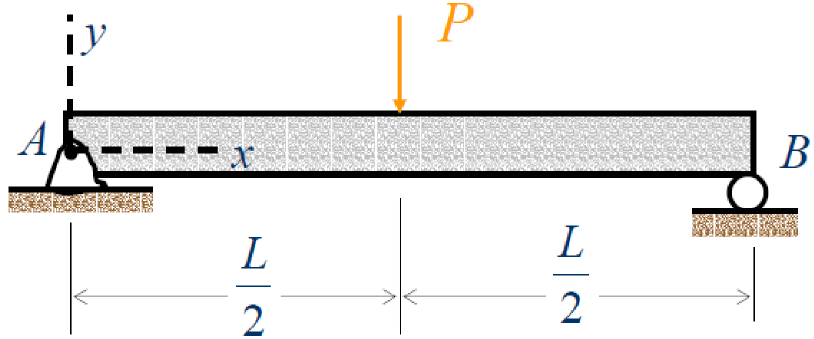
Example 01

Boundary conditions:

- $\theta = 0$ at $x = L/2$ (from symmetry)
- $y = 0$ at $x = 0$ and $x = L$

$$for\ 0 \le x \le \frac{L}{2} \quad M = \frac{P}{2}x$$

Deflection of Beams



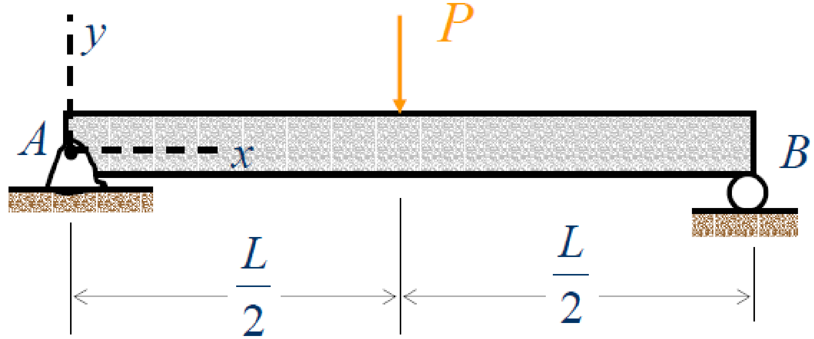
Example 01

Boundary conditions:

- $\theta = 0$ at $x = L/2$ (from symmetry)
- $y = 0$ at $x = 0$ and $x = L$

$$for\ 0 \le x \le \frac{L}{2} \quad M = \frac{P}{2}x$$

Deflection of Beams

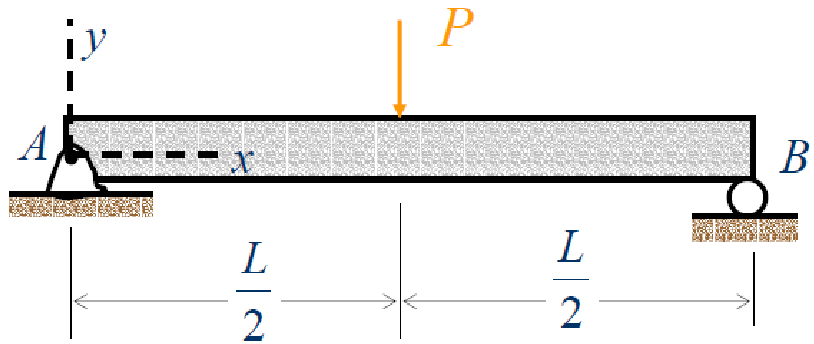


Example 01

- $\theta = 0$ at $x = L/2$ (from symmetry)

- $y = 0$ at $x = 0$

Deflection of Beams



Example 01

(a)The equation of elastic curve

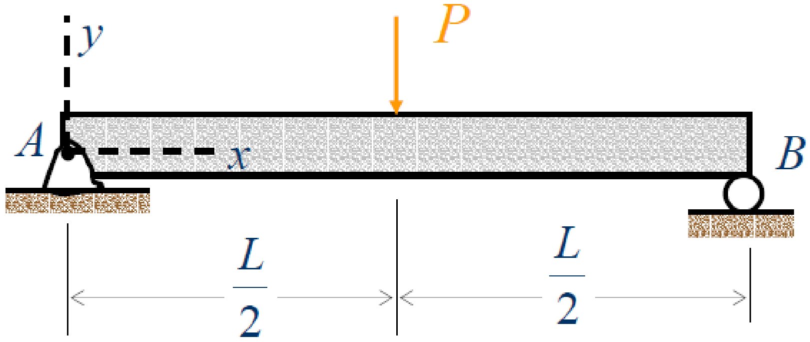
(b) Slope at the left end of the beam:

Therefore,

Deflection of Beams

Example 01

(c) Deflection at x = L/2

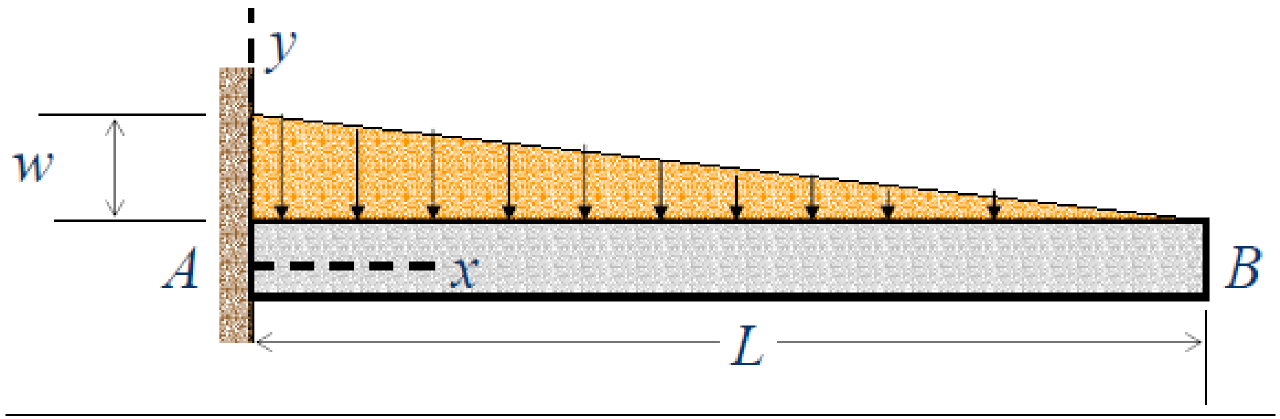


Deflection of Beams

Example 02

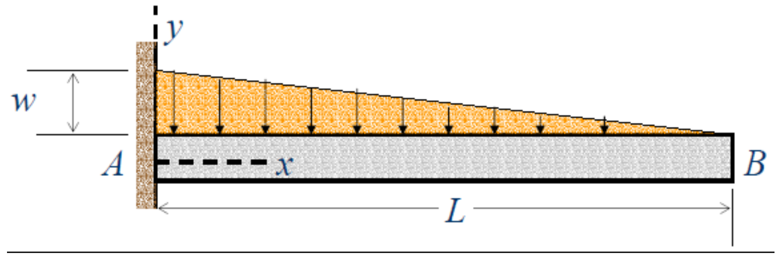
A beam is loaded and supported as shown in the figure.

a) Determine the slope at the right end of the beam.
b) Derive the equation for the elastic curve in terms of w, L, x, E, and I.
c) Find the deflection at x = L.



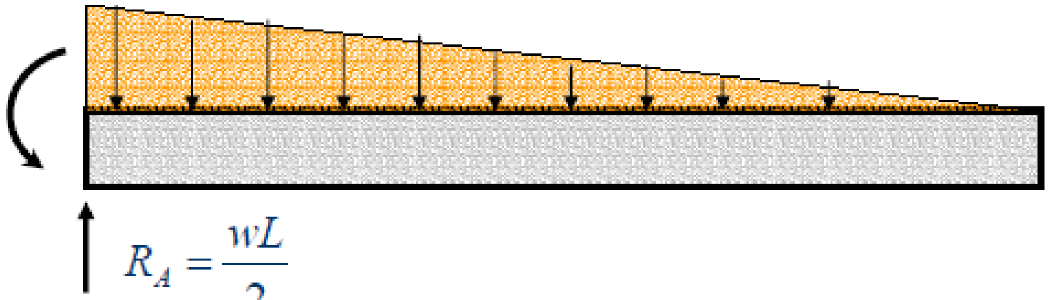
Deflection of Beams

Example 02



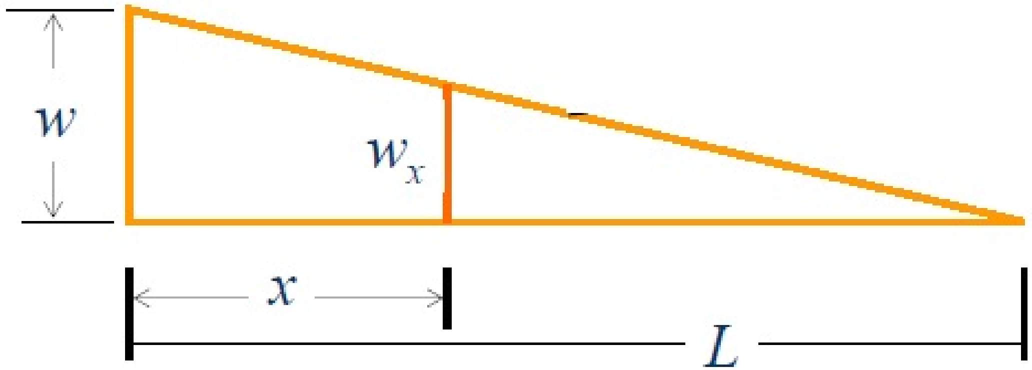
Using equilibrium equation the support reactions are determ

$$M_A = \frac{wL}{2}\left(\frac{L}{3}\right) = \frac{wL^2}{6}$$



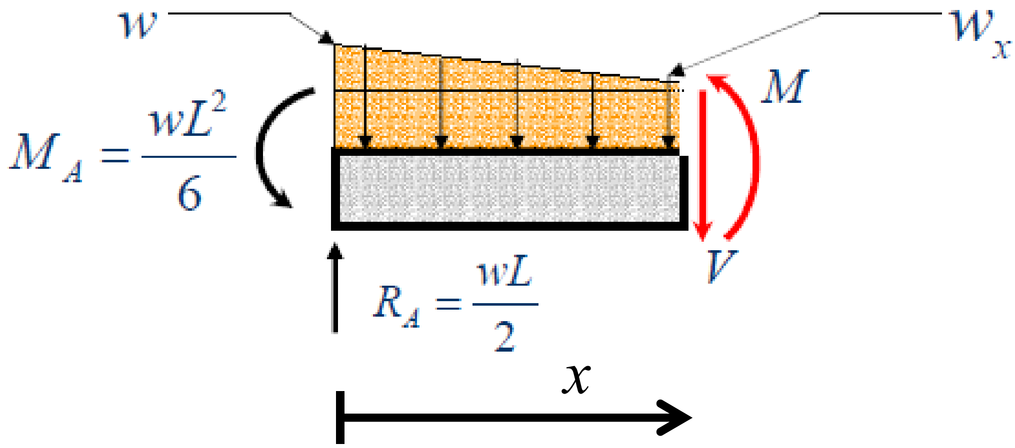
$$R_A = \frac{wL}{2}$$

Find an expression for a segment of the distributed load:



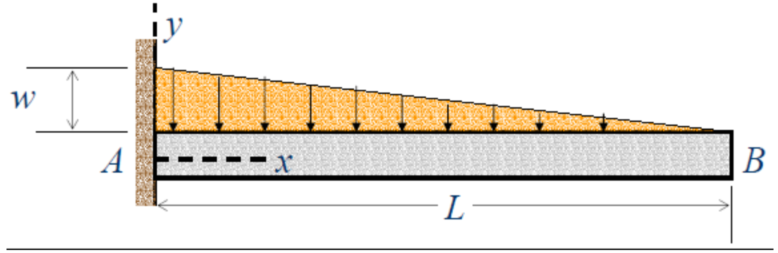
Deflection of Beams

Example 02



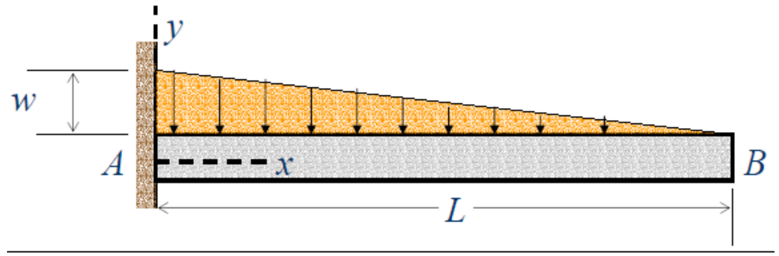
Deflection of Beams

Example 02



Deflection of Beams

Example 02

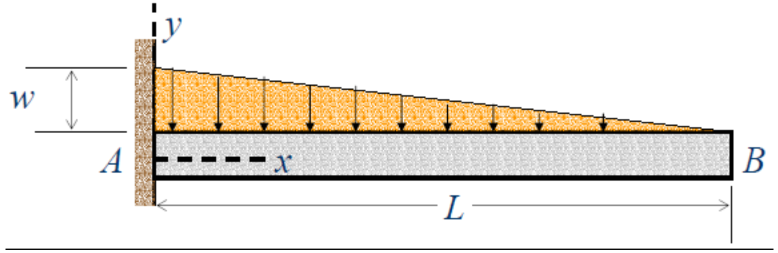


Boundary conditions: $\theta = 0$ at $x = 0$

a) Determine the slope at the right end of the beam.

Deflection of Beams

Example 02

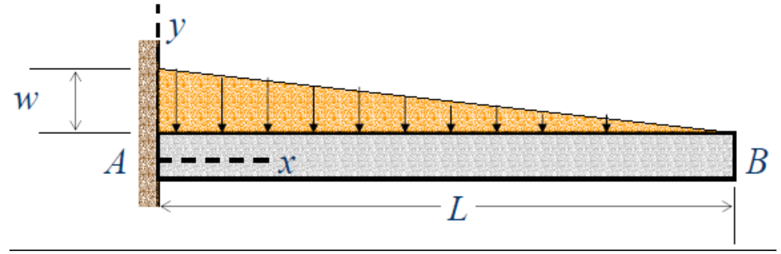


b) Derive the equation for the elastic curve in terms of w, L, x, E, and I.

Boundary conditions: $y = 0$ at $x = 0$

Deflection of Beams

Example 02



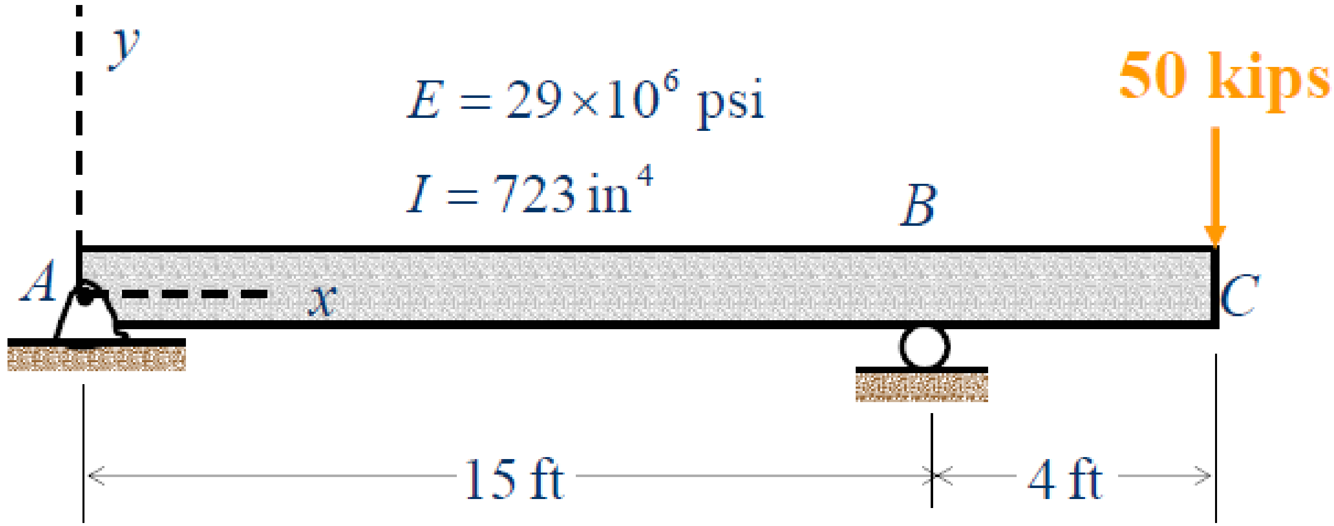
c) Find the deflection at x = L.

Deflection of Beams

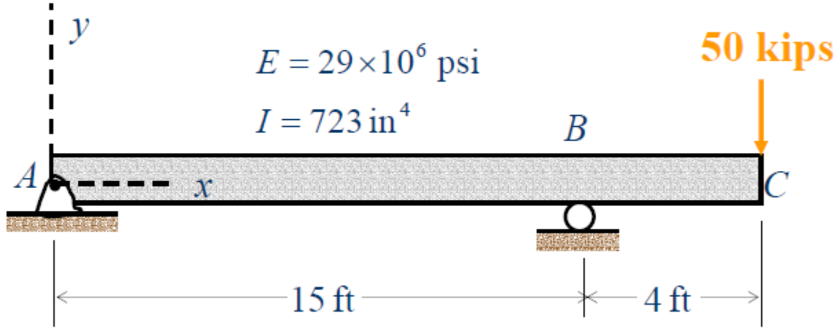
Example 03

For the overhanging steel beam ABC that subjected to concentrated load of 50 kips as shown,

(a) derive an expression for the elastic curve,
(b) determine the maximum deflection, and
(c) find the slope at point A.

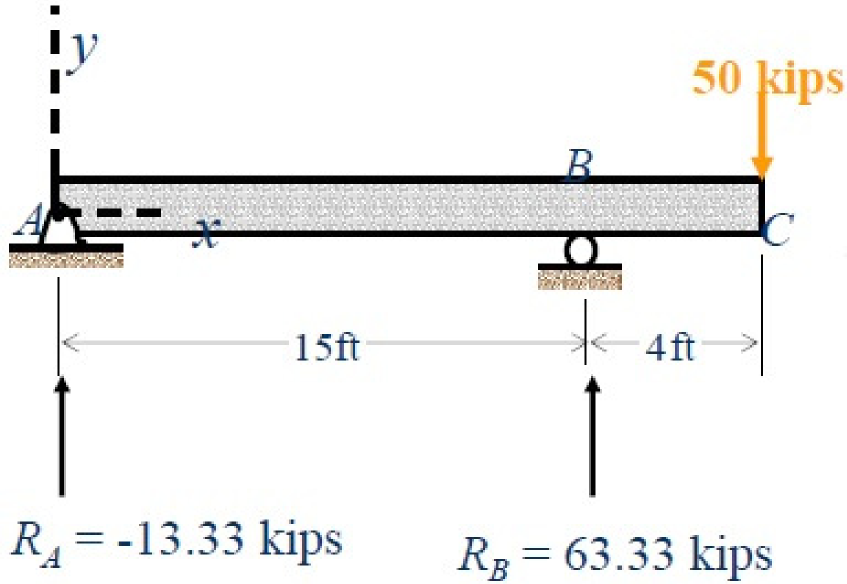


Deflection of Beams



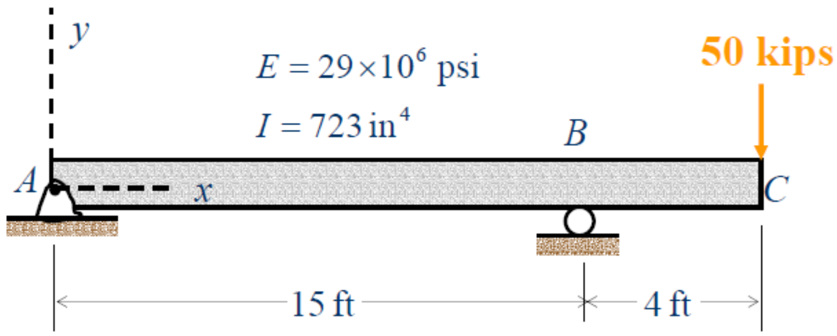
Example 03

We first find the reactions as follows:

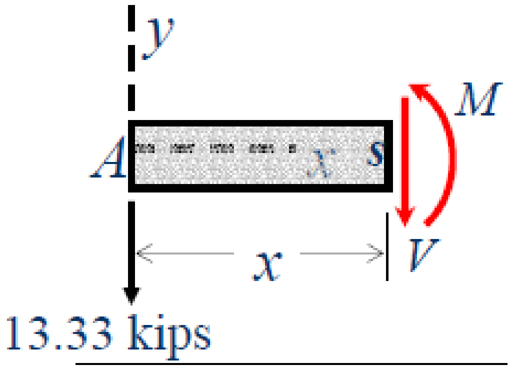


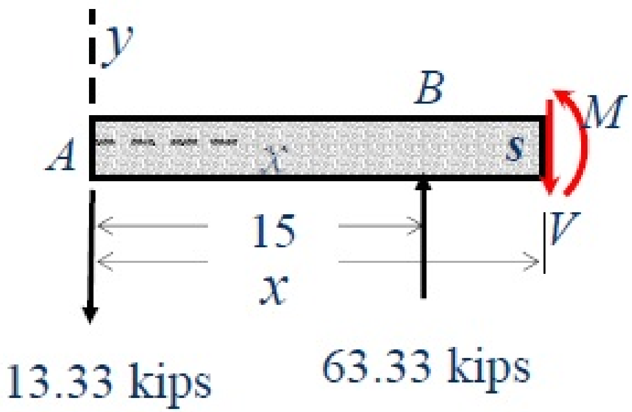
Deflection of Beams

Example 03



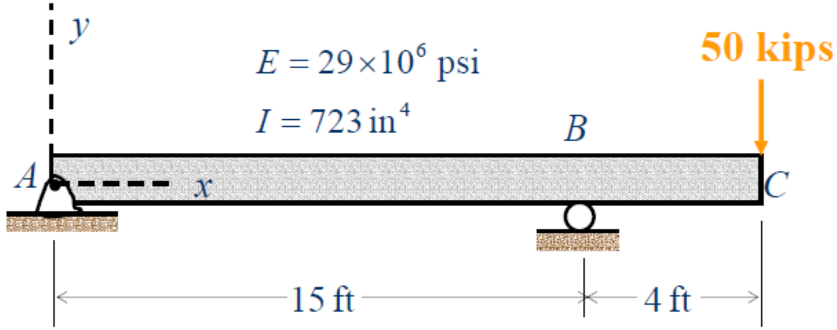
Determine moment equation as follows:





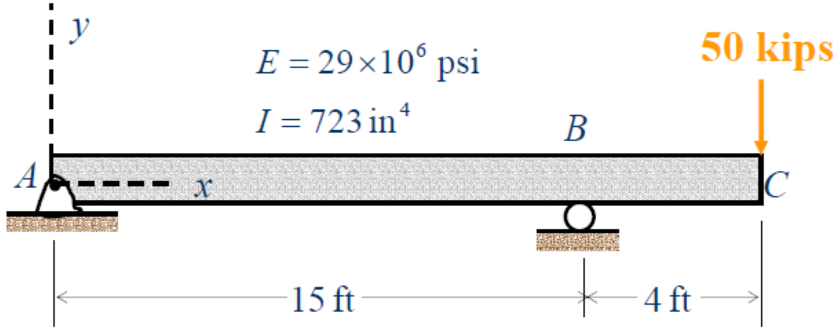
Deflection of Beams

Example 03



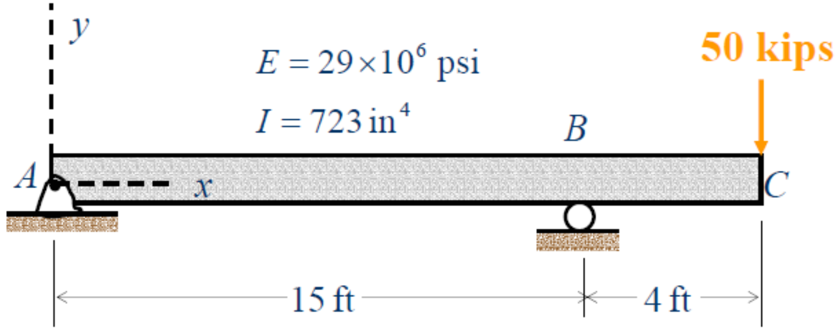
Deflection of Beams

Example 03



Deflection of Beams

Example 03



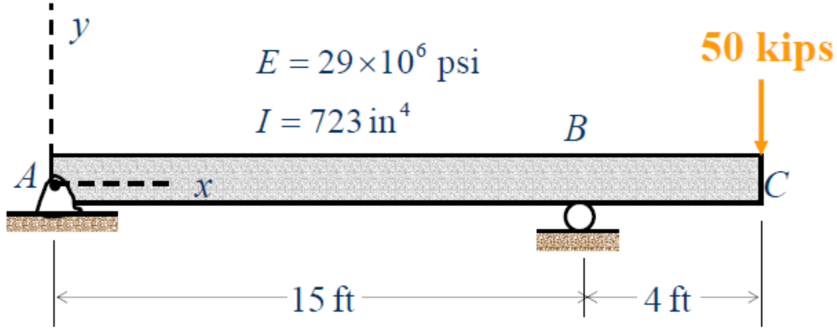
Boundary conditions:

y = 0 at $x = 0$

y = 0 at $x = 15$

Deflection of Beams

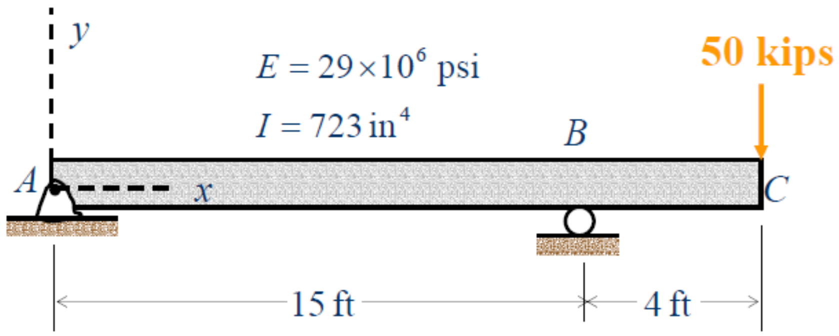
Example 03



Consistent conditions:

Deflection of Beams

Example 03

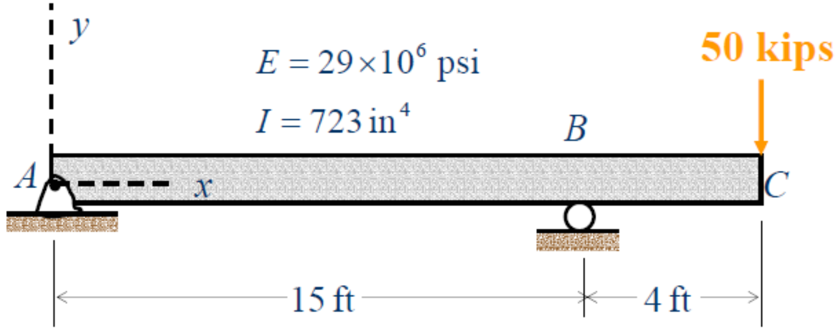


Boundary conditions:

y = 0 at $x = 15$

Deflection of Beams

Example 03

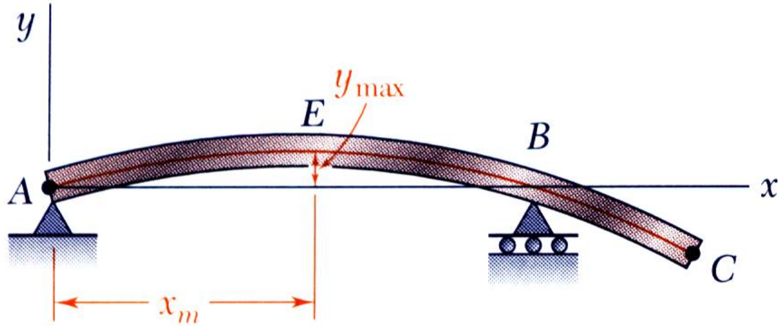


(a) The elastic curve is

$$\left(for \quad 0 \le x \le 15\right) \Rightarrow \begin{cases} EI\theta = -13.33\dfrac{x^2}{2} + 499.9 \\ EIy = -13.33\dfrac{x^3}{6} + 499.9x \end{cases}$$

$$\left(for \quad 15 \le x \le 19\right) \Rightarrow \begin{cases} EI\theta = 50\dfrac{x^2}{2} - 949.95x + 7624.5 \\ EIy = 50\dfrac{x^3}{6} - 949.95\dfrac{x^2}{2} + 7624.5x - 35623.1 \end{cases}$$

Deflection of Beams



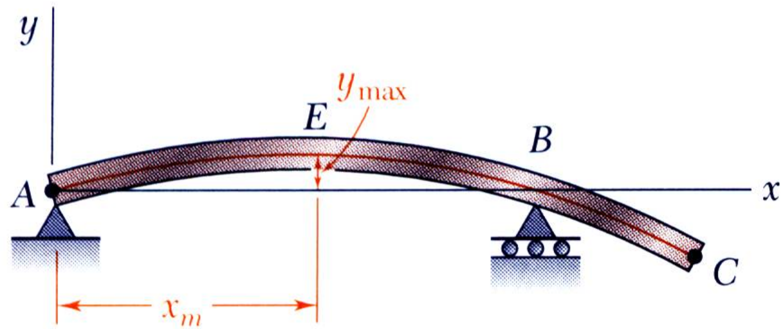
Example 03

(b) Maximum deflection occurs when the slope is zero.

Deflection of Beams

Example 03

(c) The slope at point A (x = 0) can be computed from



Deflection of Beams

- **Statically Indeterminate Beams**

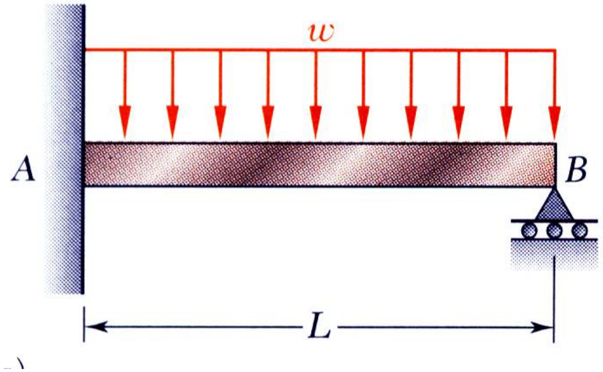
How to determine forces and stresses of transversely loaded beam that is statically indeterminate?

– In order to solve for the forces, and stresses in such beam, it becomes necessary to supplement the equilibrium equations with additional relationships based on any conditions of restraint that may exist.

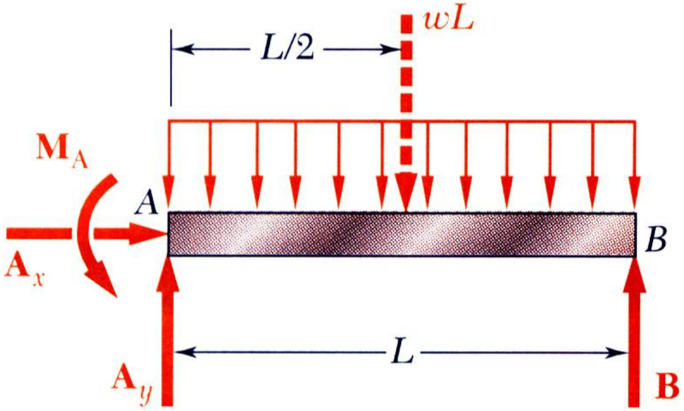
Deflection of Beams

❑ Statically Indeterminate Beams

- Consider beam with fixed support at *A* and roller support at *B*.



- From free-body diagram, note that there are ***four unknown reaction*** components.



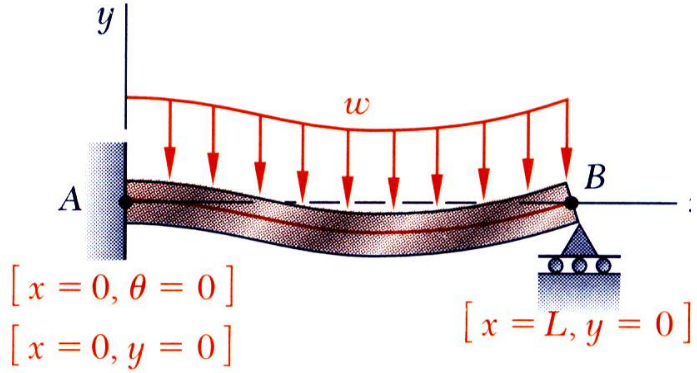
- Conditions for static equilibrium yield

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum M_A = 0$$

The beam is statically indeterminate.

Deflection of Beams

❑ Statically Indeterminate Beams



- Also have the beam deflection equation,

$$EI\,y = \int_0^x \left(\int_0^x M(x)\,dx \right) dx + C_1 x + C_2$$

which introduces ***two unknowns*** but provides three additional equations from the boundary conditions:

$$Number\ of\ unknown = 6 \quad \left(A_y, A_x, M_A, B_y \quad \& \quad C_1, C_2\right)$$

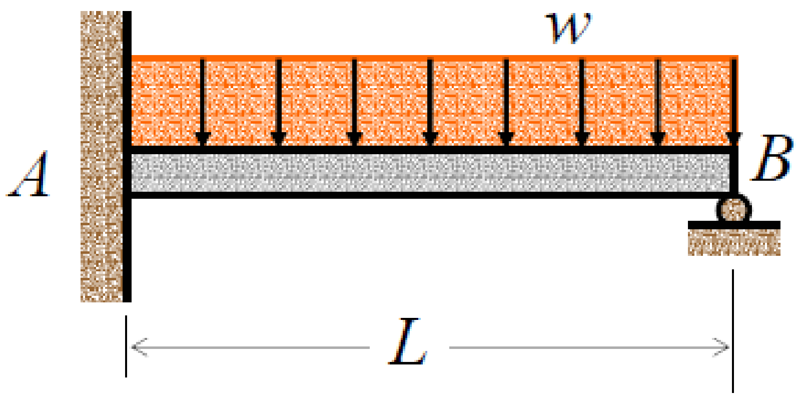
$$Number\ of\ equations = 6 \quad \left(\begin{array}{l} \sum Fx = 0, \sum Fy = 0, \sum M = 0 \\ \theta_{(x=0)} = 0, y_{(x=0)} = 0, y_{(x=L)} = 0 \end{array} \right)$$

Thus the reactions at the supports may be determined, and the equations for the elastic curve may be obtained.

Deflection of Beams

Example 04

Determine the reactions at the supports for the simply supported cantilever beam in terms of w and L.

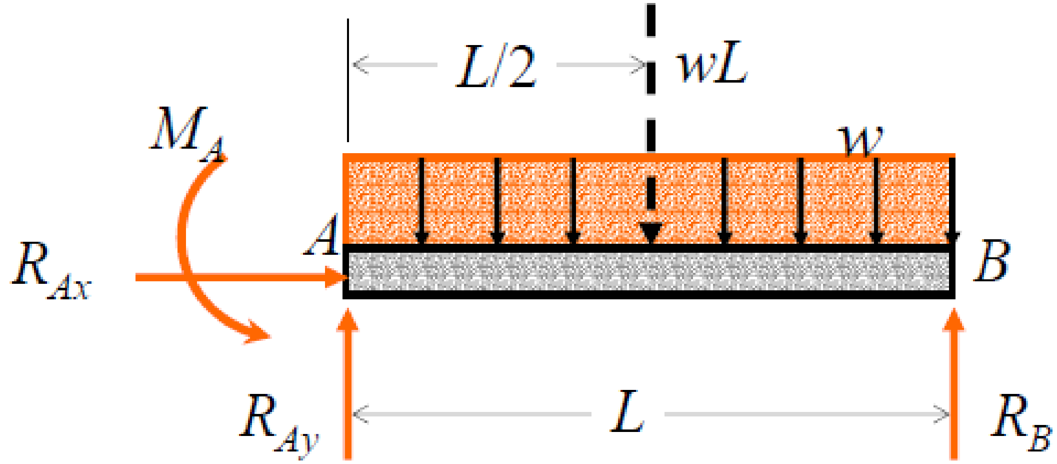


Deflection of Beams

Example 04

Equilibrium Equations:

- From the free body diagram



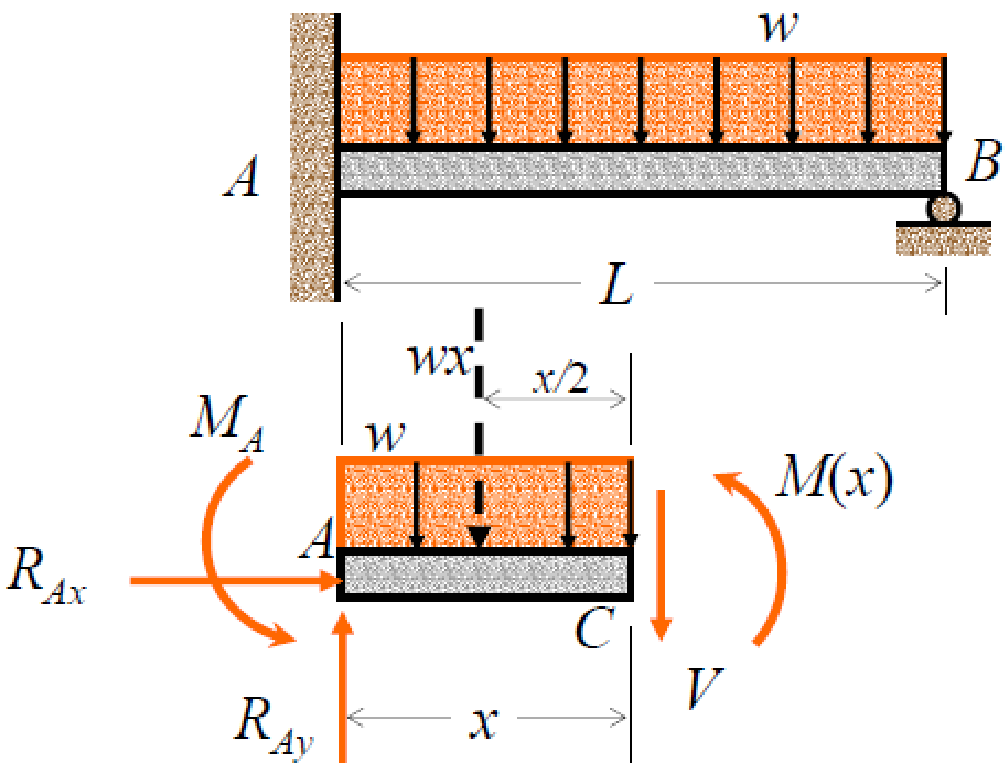
(I)

Deflection of Beams

Example 04

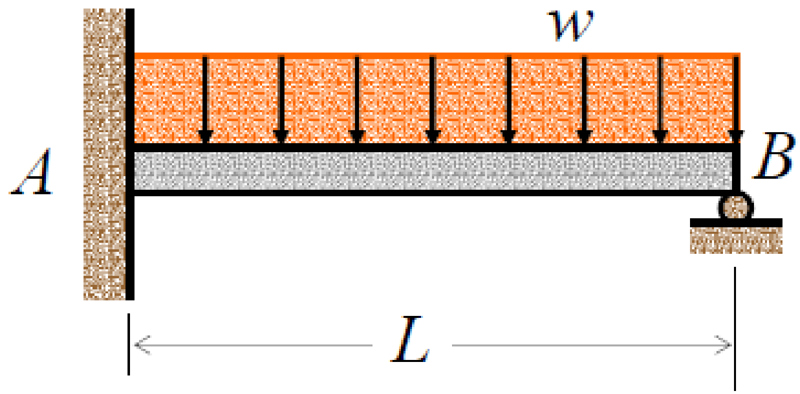
Equation of Elastic Curve:

• Drawing the free-body diagram of a portion of the beam (AC)



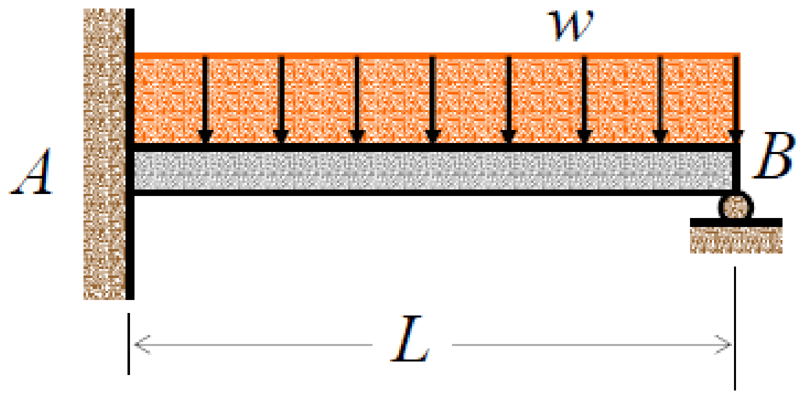
Deflection of Beams

Example 04



Equation of Elastic Curve:

Deflection of Beams



Example 04

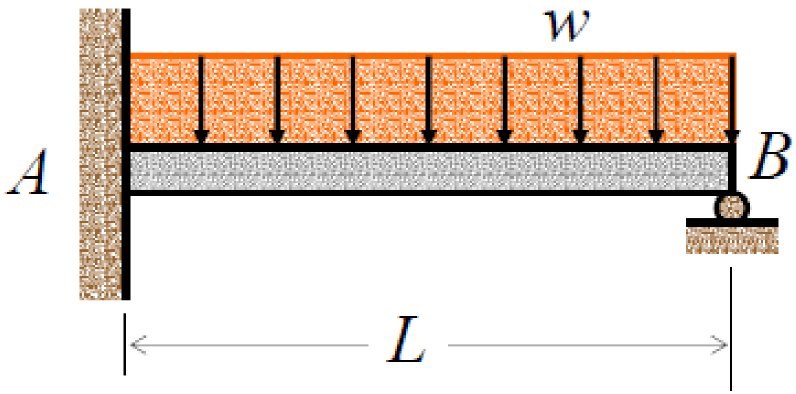
Boundary conditions:

$\Theta = 0$ at $x = 0$

$y = 0$ at $x = 0$

Deflection of Beams

Example 04



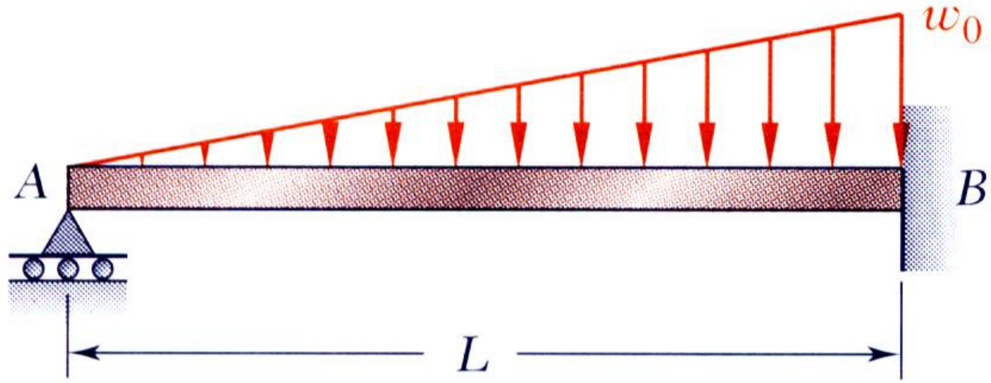
Boundary conditions:

y = 0 at $x = L$

Deflection of Beams

Example 05

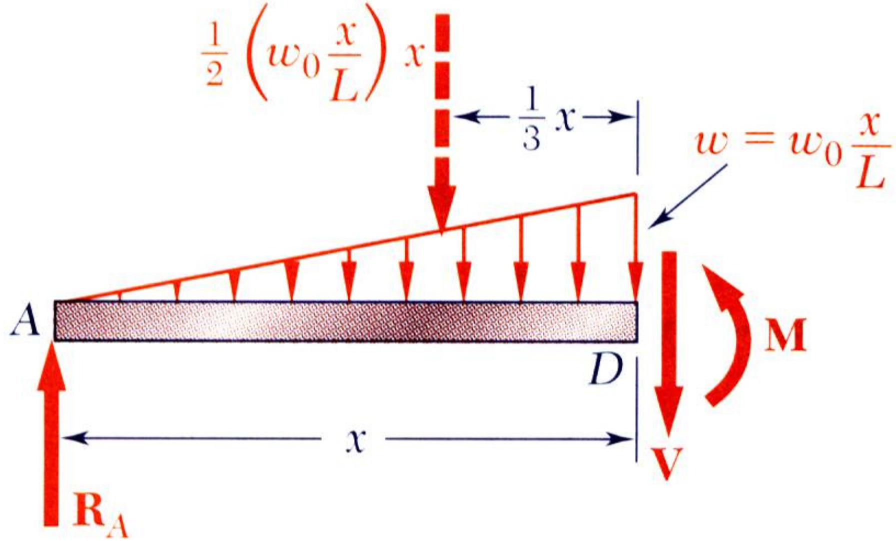
For the uniform beam, determine the reaction at A, derive the equation for the elastic curve, and determine the slope at A. (Note that the beam is statically indeterminate to the first degree)



Deflection of Beams

Example 05

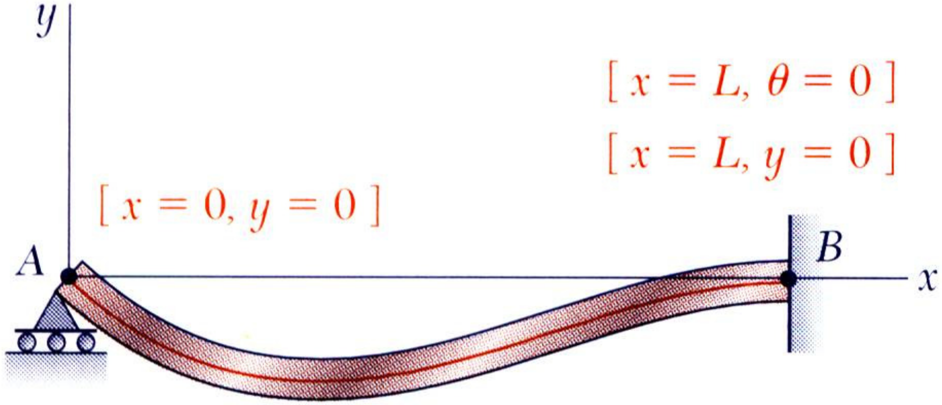
- Consider moment acting at section D,



- The differential equation for the elastic curve,

Deflection of Beams

Example 05

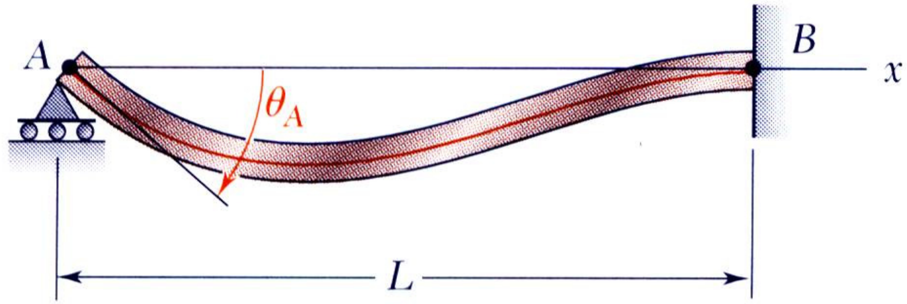


- Integrate twice

- Apply boundary conditions:

Deflection of Beams

Example 05



- Substitute for C_1, C_2, and R_A in the elastic curve equation,

- Substitute for C_1 and R_A in the slope curve equation,

Deflection of Beams

- **Method of Superposition**

– When a beam is subjected to several loads at various positions along the beam, the problem of determining the slope and the deflection usually becomes quite involved and tedious.

– This is true regardless of the method used.

– However, many complex loading conditions are merely combinations of relatively simple loading conditions

❑ Method of Superposition

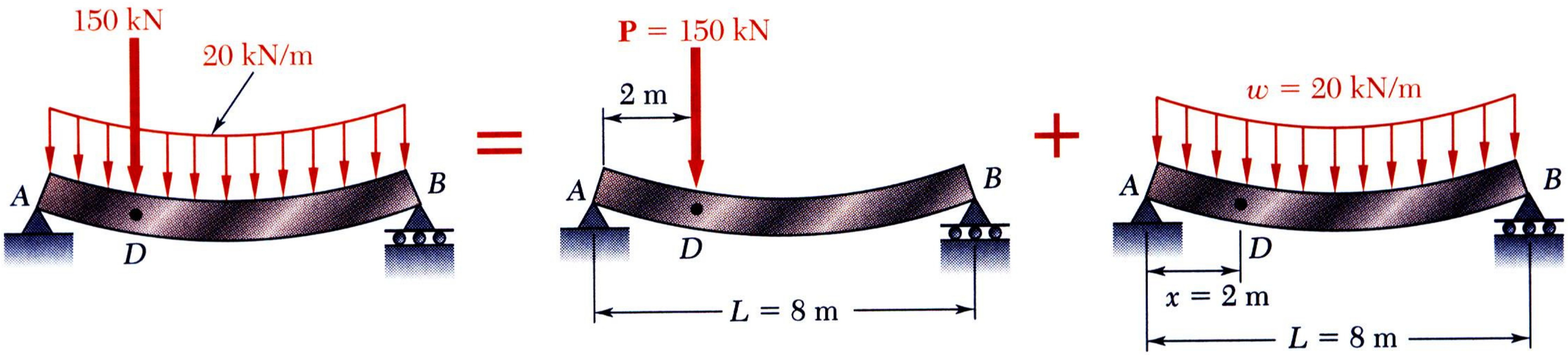
- ***Method of Superposition***

– Assumptions:

• The beam behaves elastically for the combined loading.
• The beam also behaves elastically for the each of the individual loads.
• Small deflection theory.

Deflection of Beams

❑ Method of Superposition



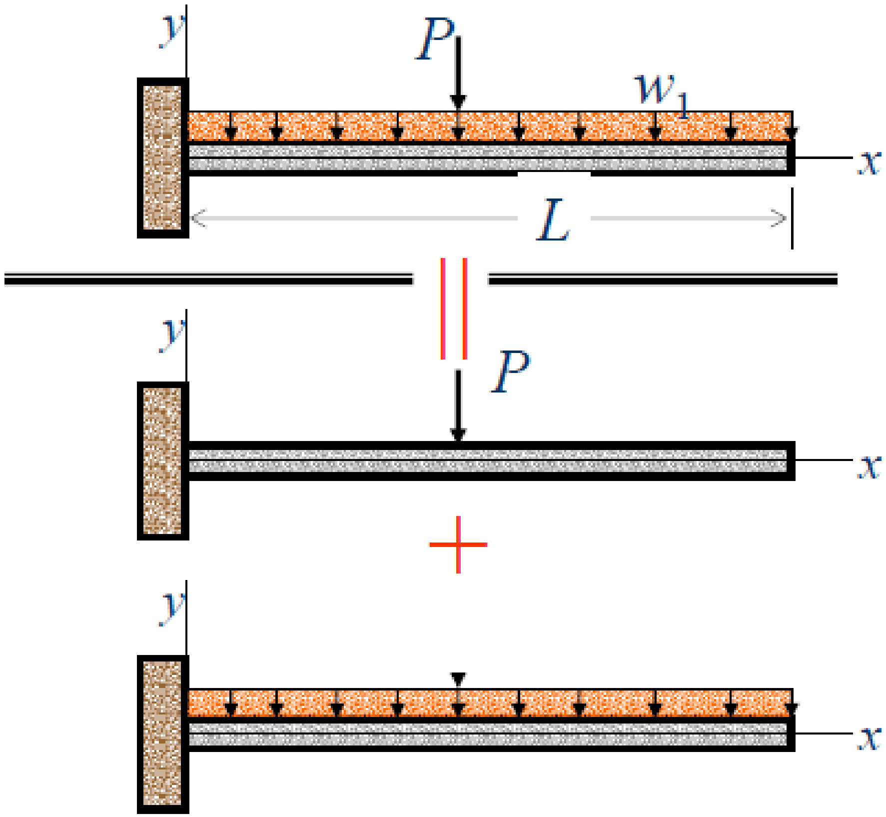
Principle of Superposition:

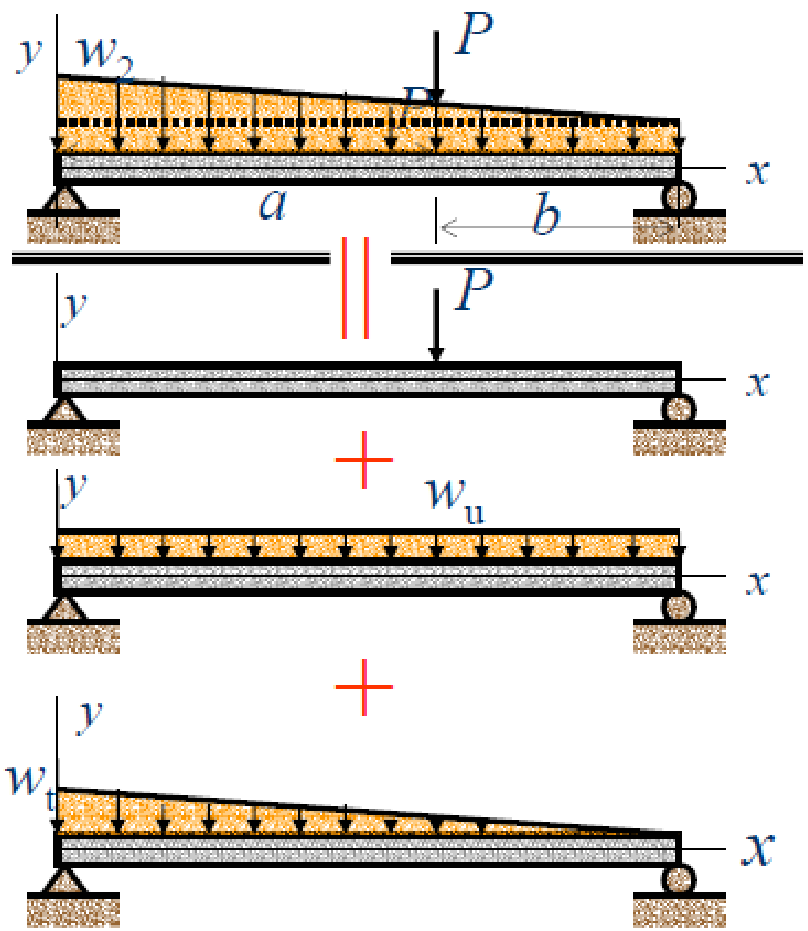
- Deformations of beams subjected to combinations of loadings may be obtained as the linear combination of the deformations from the individual loadings
- Procedure is facilitated by tables of solutions for common types of loadings and supports.

Deflection of Beams

❑ Method of Superposition

Algebraic sum

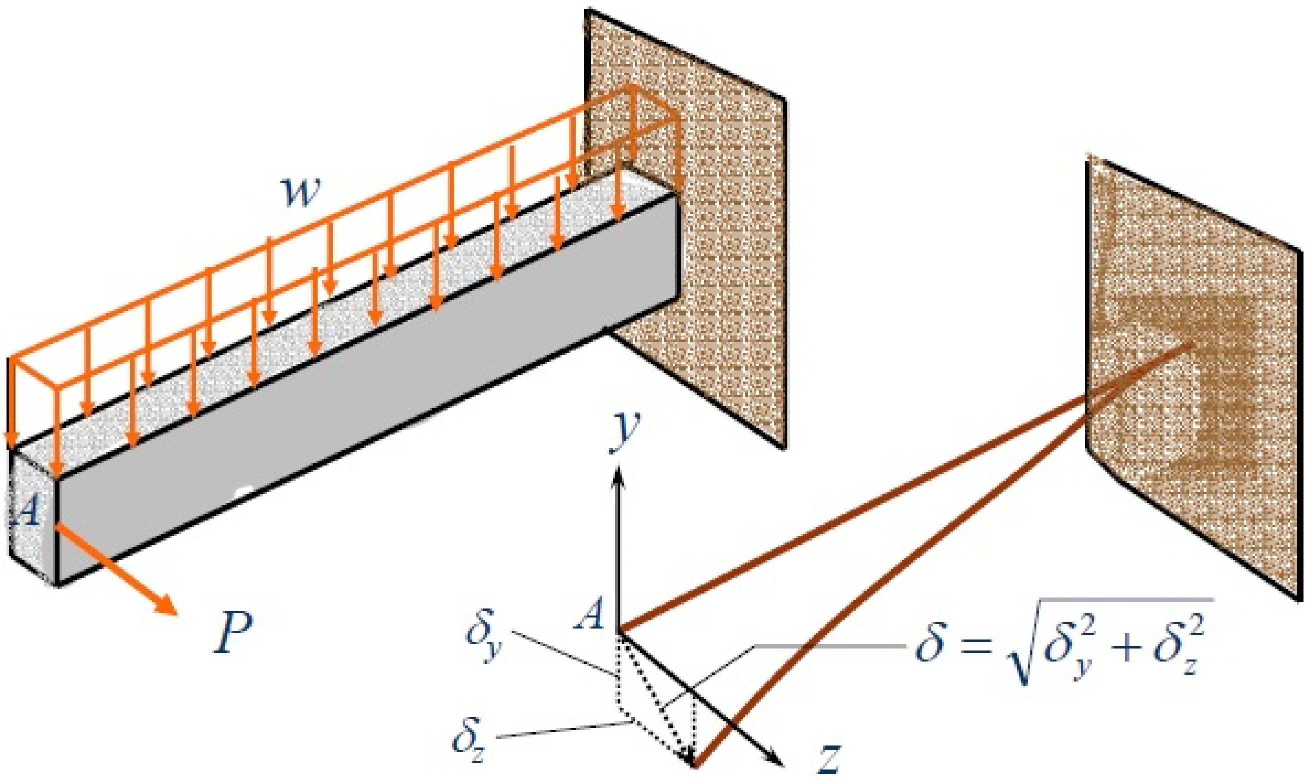




Deflection of Beams

❑ Method of Superposition

Vector sum

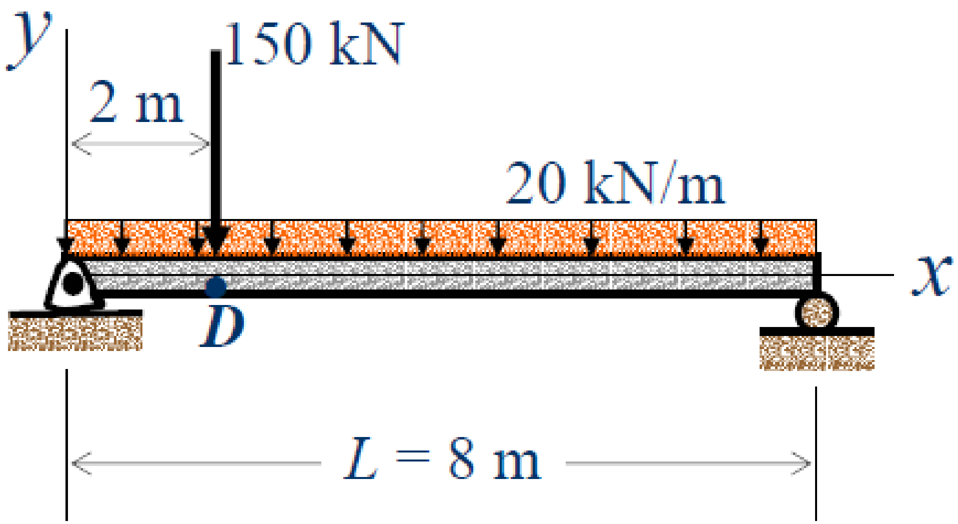


Deflection of Beams

Example 06

Determine slope and the deflection at point D in the following beam.

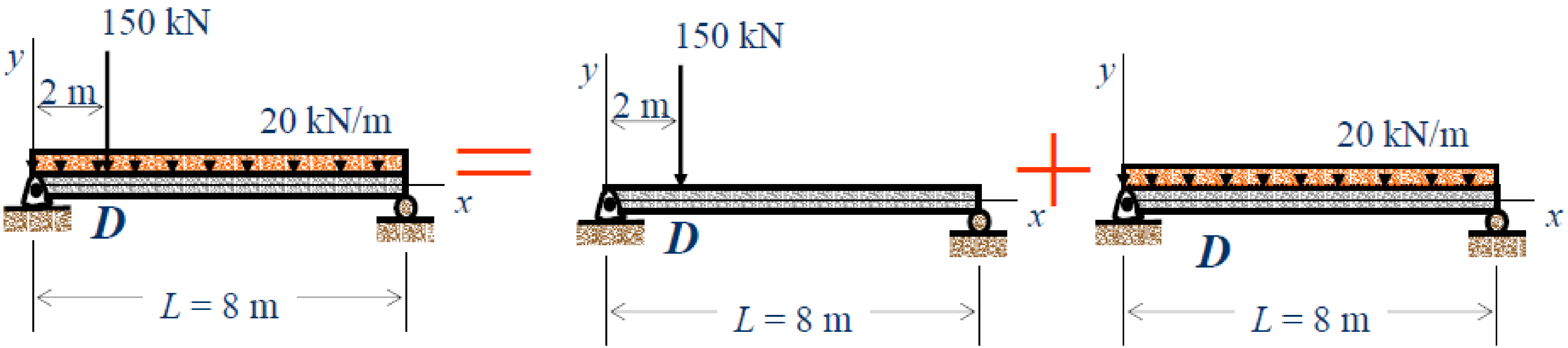
$EI = 100 \quad MN.m^2$



Deflection of Beams

Example 06

First we find the slope and deflection due the effect of each load

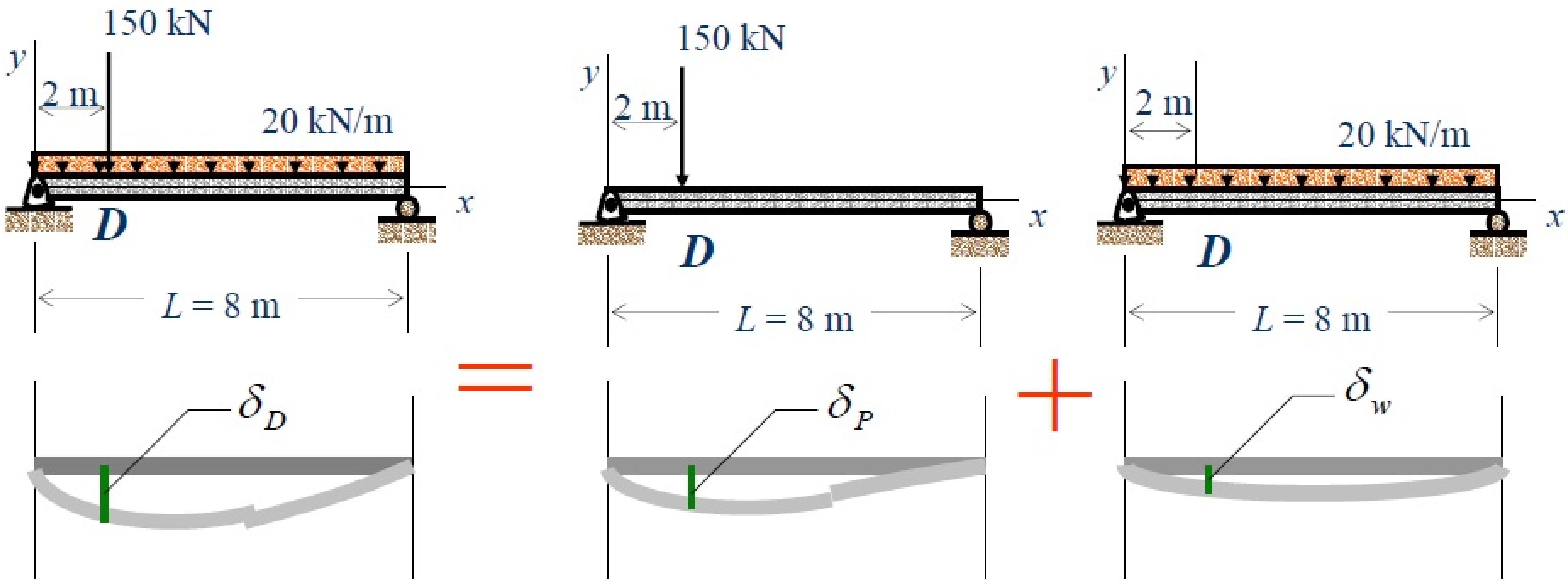


The resulting final slope and deflection of point D of the loaded beam is simply the sum of the slopes and deflections caused by each of the individual loads as shown

Deflection of Beams

Example 06

We need to find both the slope and deflection caused by the concentrated load (120 kN) and distributed load (20 kN/m)

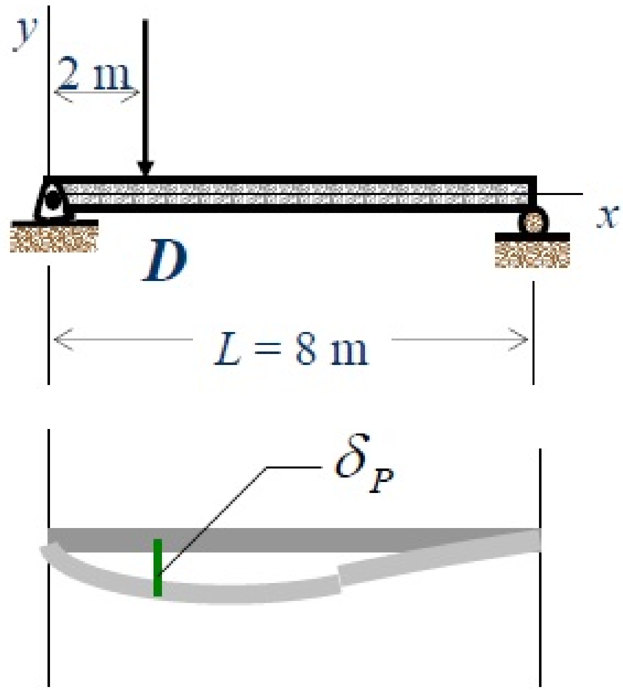


$$\delta_D = \delta_{D-\text{due to } P} + \delta_{D-\text{due to } w}$$

Deflection of Beams

Example 06

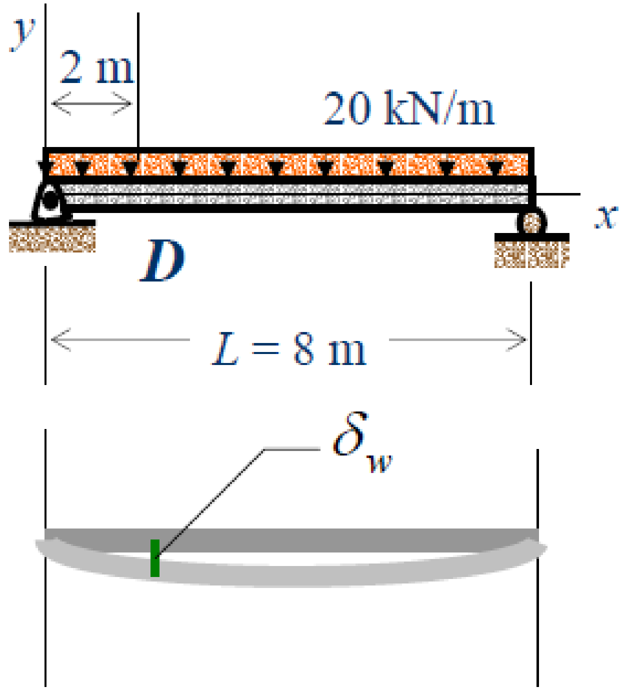
Slope and Deflection caused by P



Deflection of Beams

Example 06

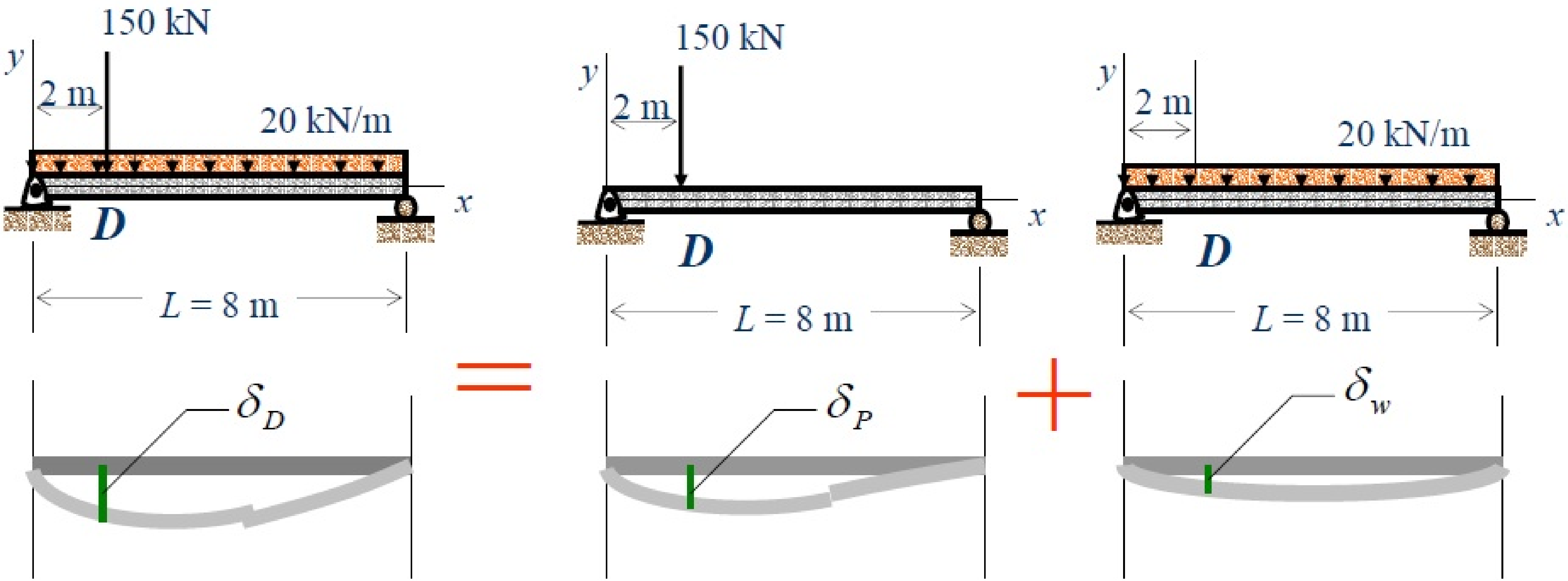
Slope and Deflection caused by w



Deflection of Beams

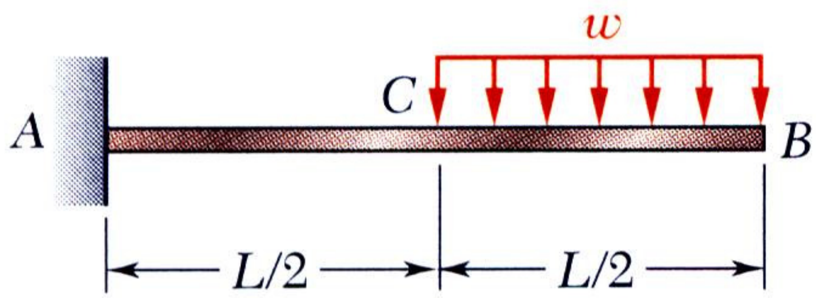
Example 06

Slope and Deflection caused by P and w



Example 07

For the beam and loading shown, determine the slope and deflection at point *B*.

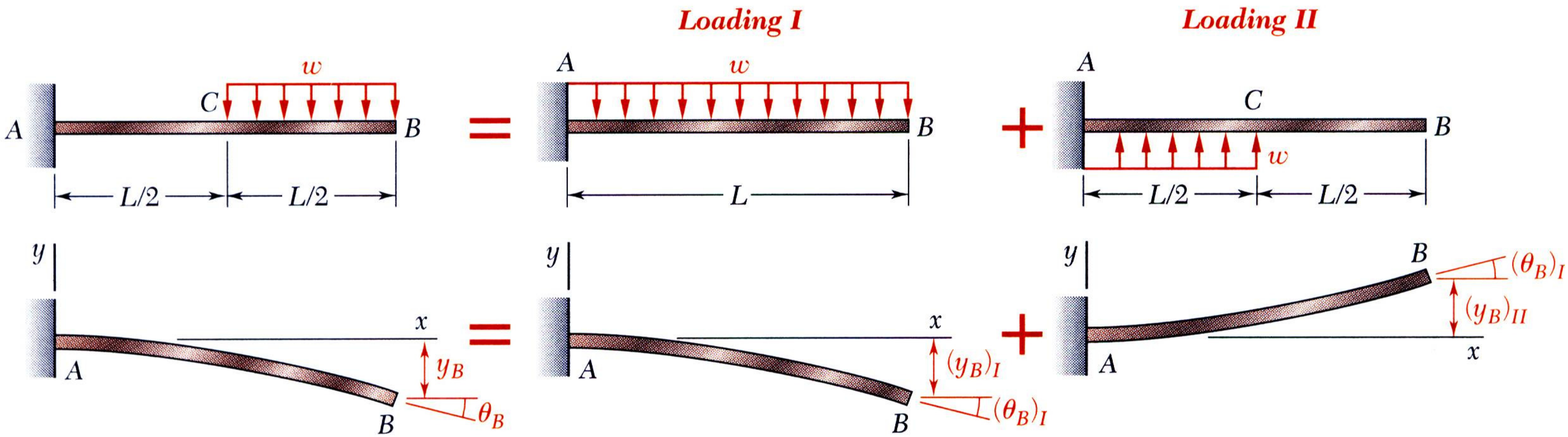


Deflection of Beams

Example 07

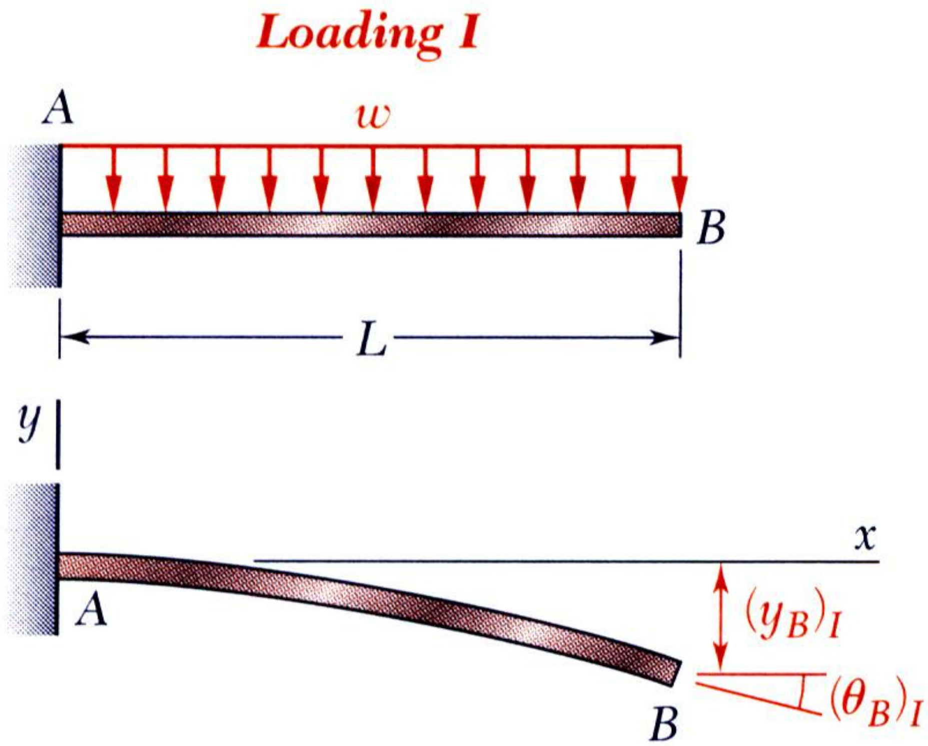
SOLUTION:

Superpose the deformations due to *Loading I* and *Loading II* as shown.



Deflection of Beams

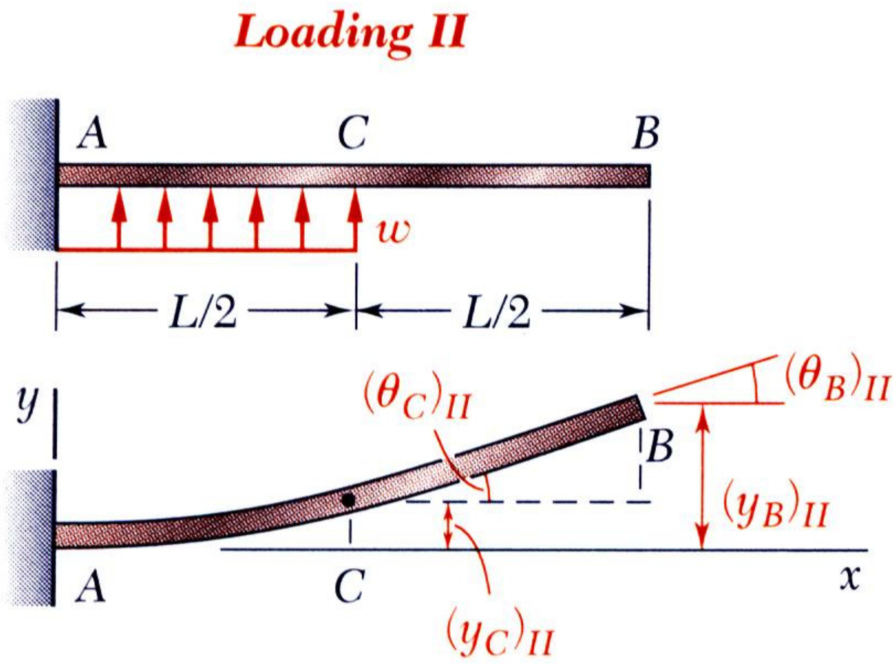
Example 07



Deflection of Beams

Example 07

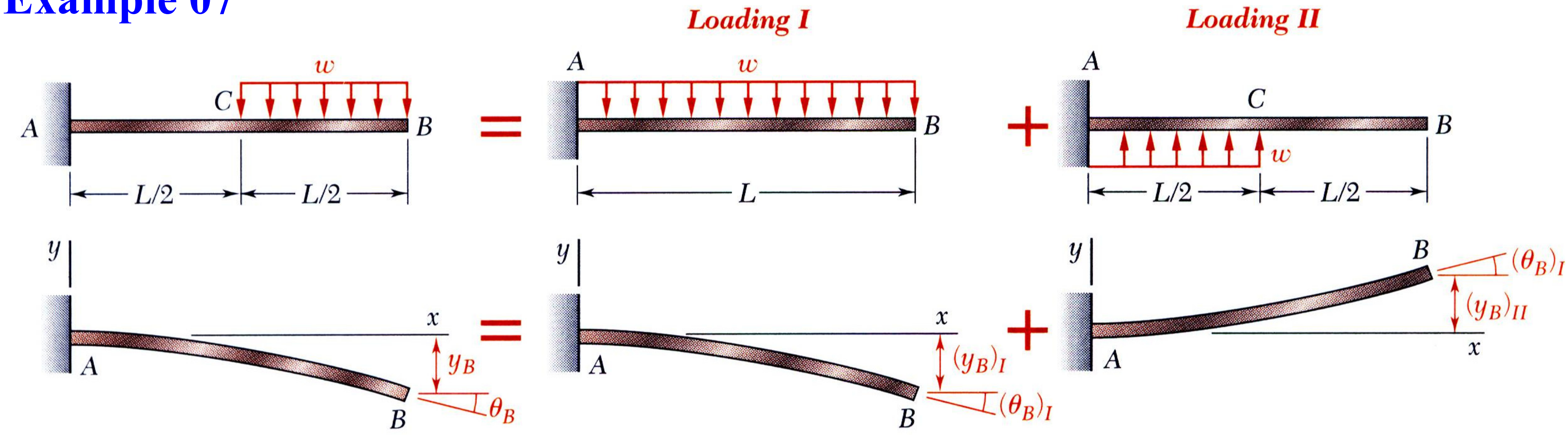
Loading II



In beam segment CB, the bending moment is zero and the elastic curve is a straight line.

Deflection of Beams

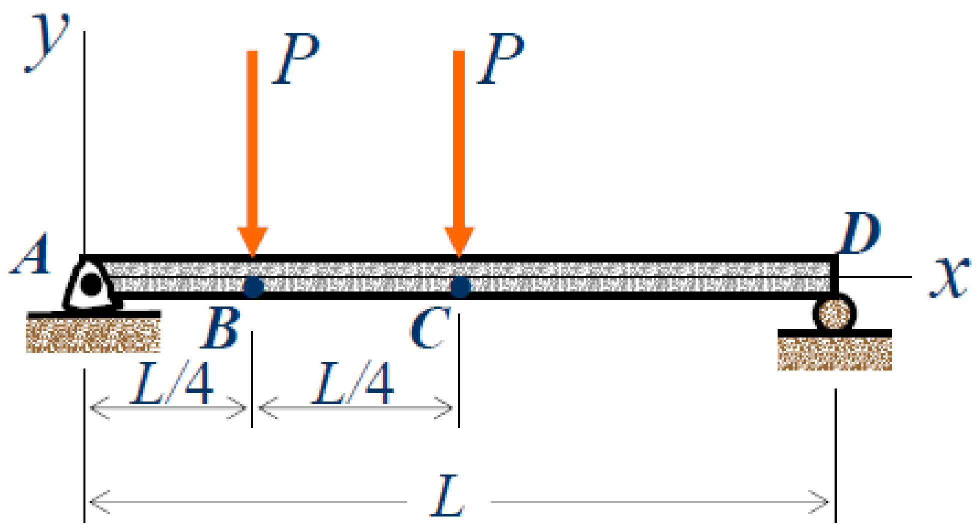
Example 07



Deflection of Beams

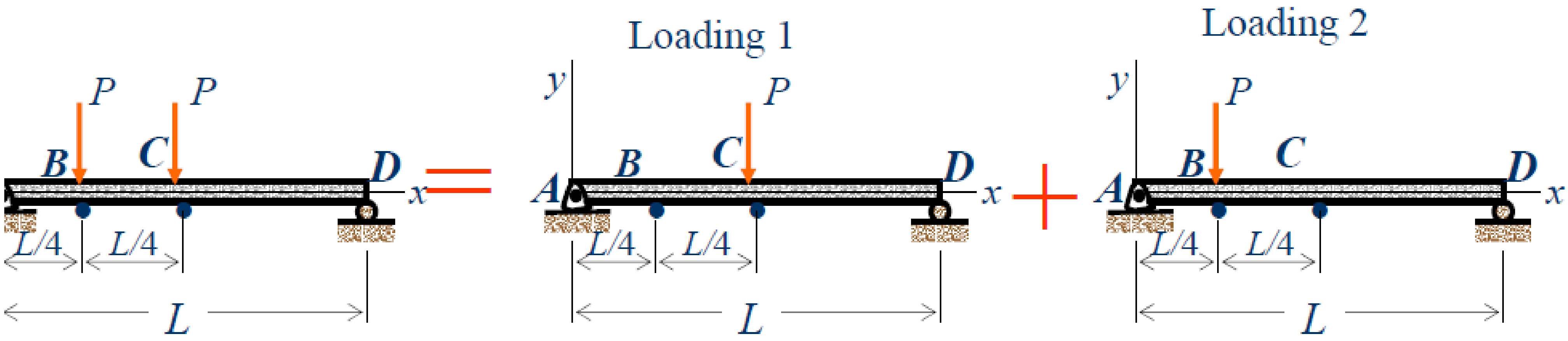
Example 08

For the simply supported beam, use the method of superposition to determine the total deflection at point *C* in terms of *P*, *L*, *E*, and *I*.



Deflection of Beams

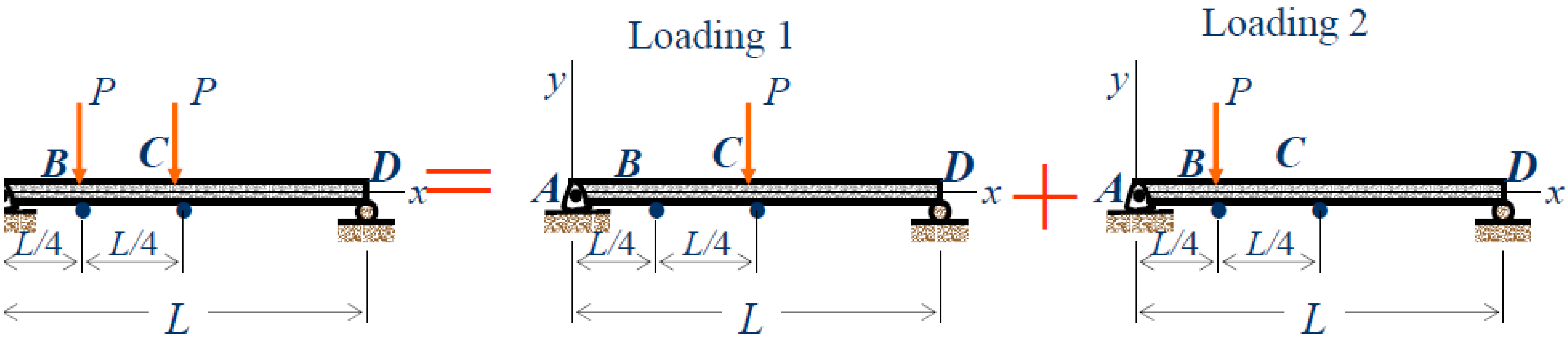
Example 08



from table $\Rightarrow$

Deflection of Beams

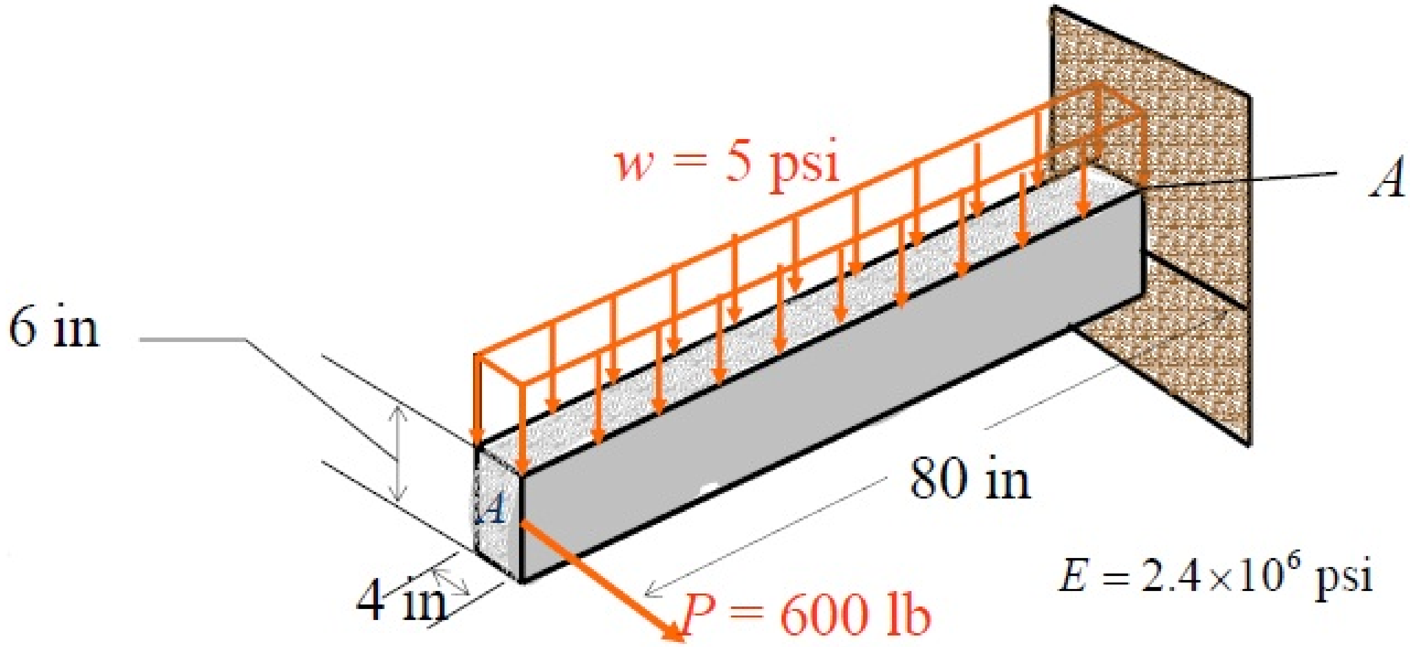
Example 08



Deflection of Beams

Example 09

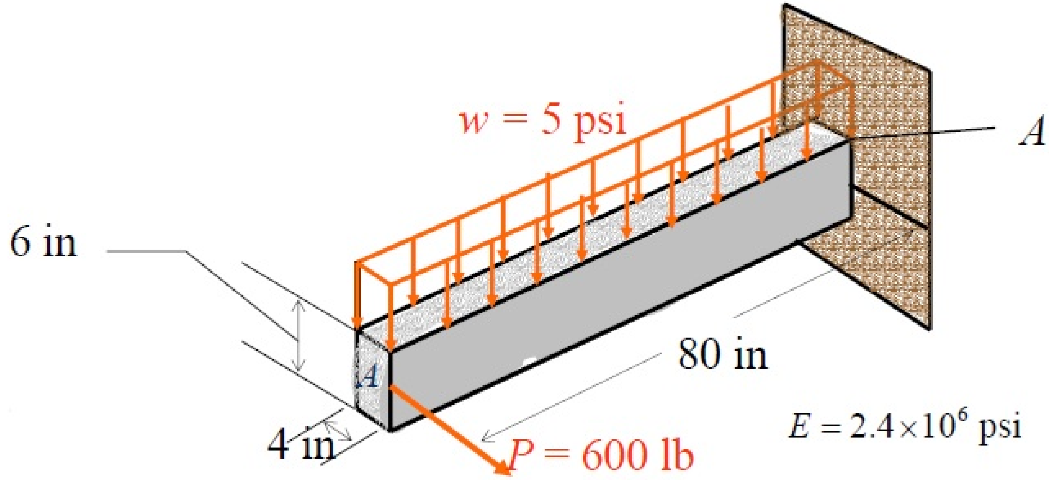
For the beam in Figure, determine the flexural stress at point A and the deflection of the left-hand end.



Deflection of Beams

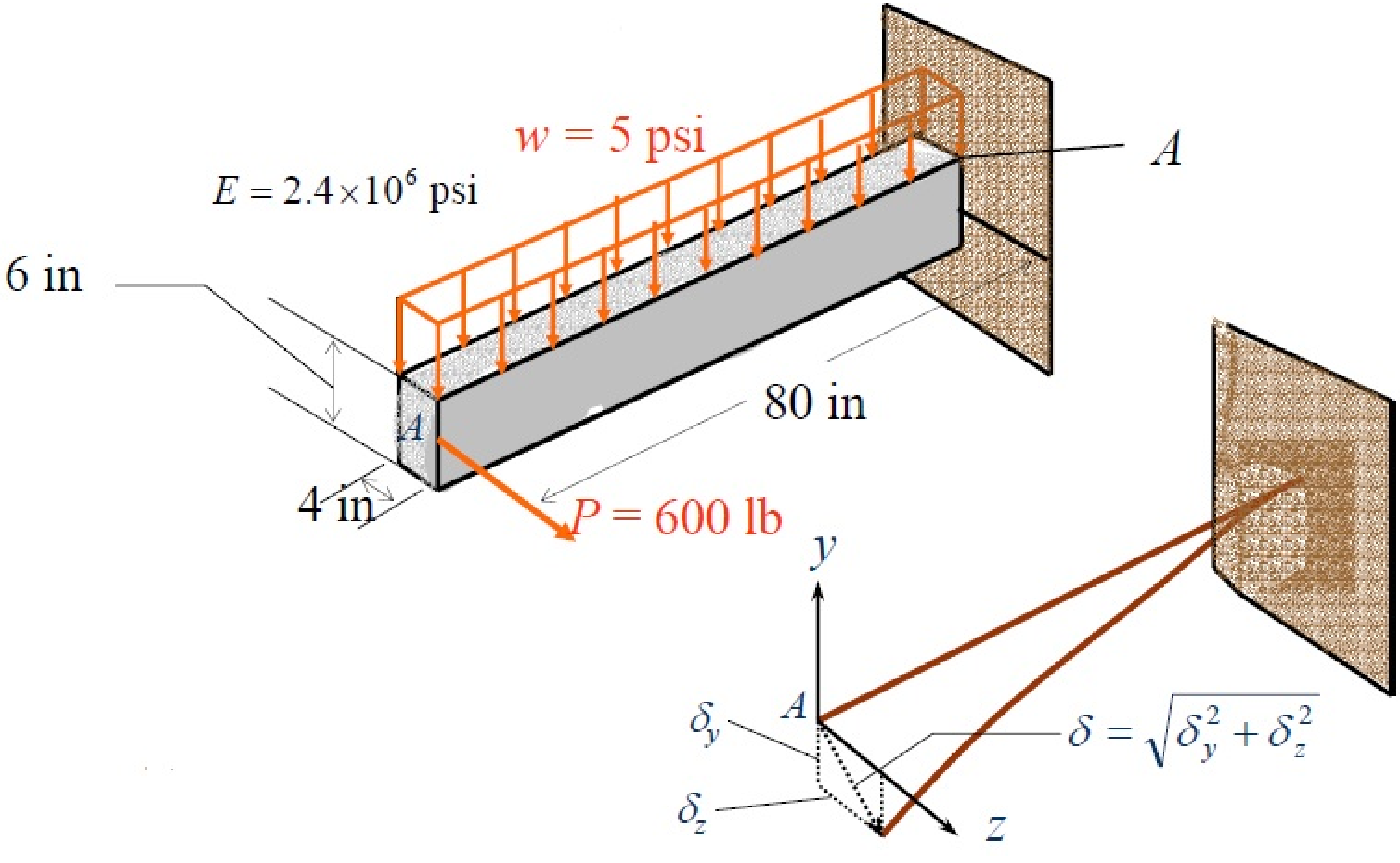
Example 09

The stress at point A is a combination of compressive flexural stress due to the concentrated load and a tensile flexural stress due to the distributed load, hence



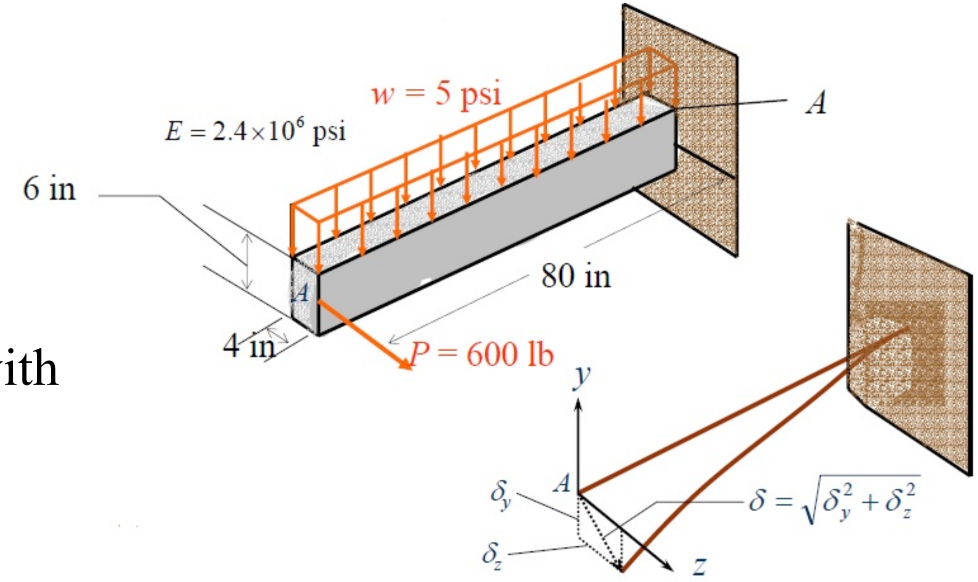
Deflection of Beams

Example 09



Deflection of Beams

Example 09

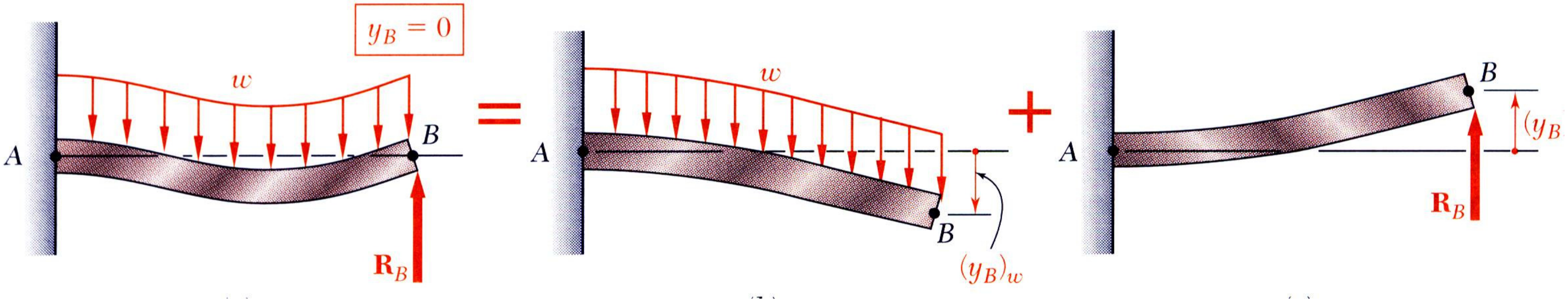


The deflection at the end of a cantilever beam with uniformly distributed load is given by

and with concentrated load at the end is given by

Superimposing the results

❑ Application of Superposition to Statically Indeterminate Beams

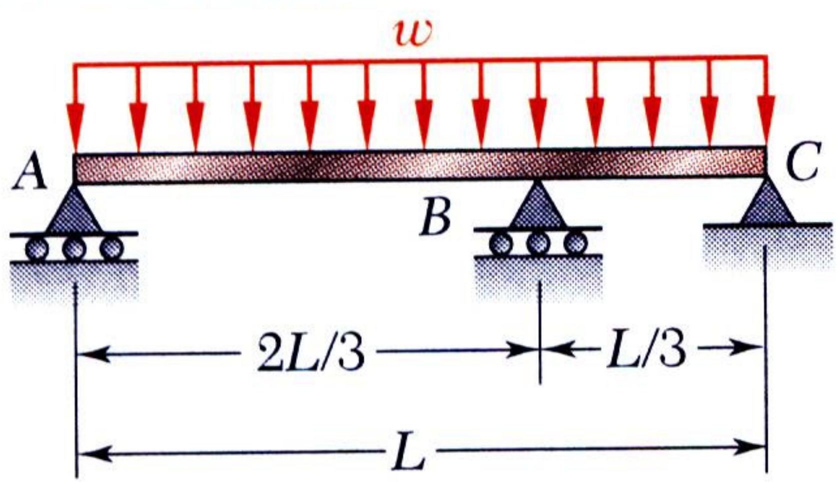


- Method of superposition may be applied to determine the reactions at the supports of statically indeterminate beams.
- Designate one of the reactions as redundant and eliminate or modify the support.
- Determine the beam deformation without the redundant support.
- Treat the redundant reaction as an unknown load which, together with the other loads, must produce deformations compatible with the original supports.

Deflection of Beams

Example 10

For the uniform beam and loading shown, determine the reaction at each support and the slope at end A.

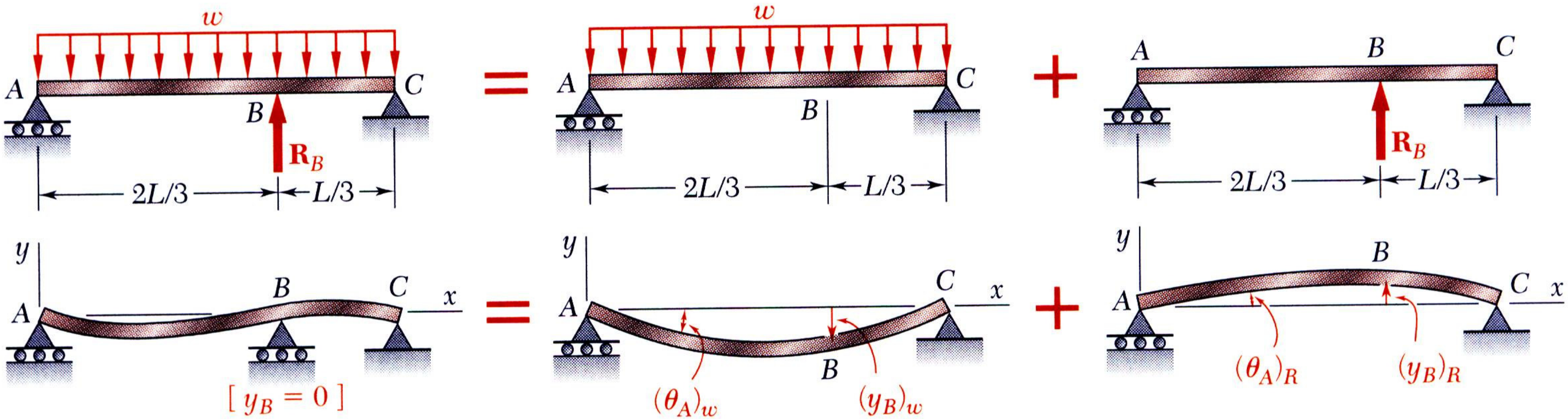


Deflection of Beams

Example 10

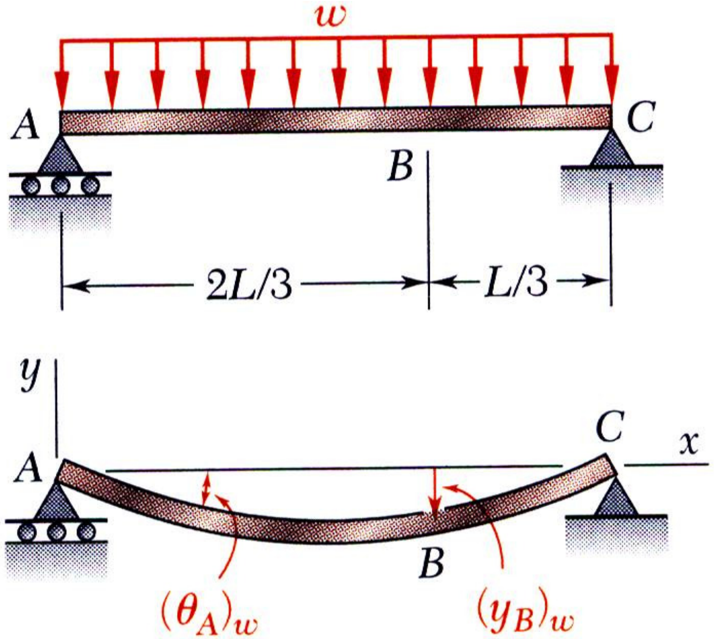
SOLUTION:

- Release the "redundant" support at B, and find deformation.
- Apply reaction at *B* as an unknown load to force zero displacement at *B*.

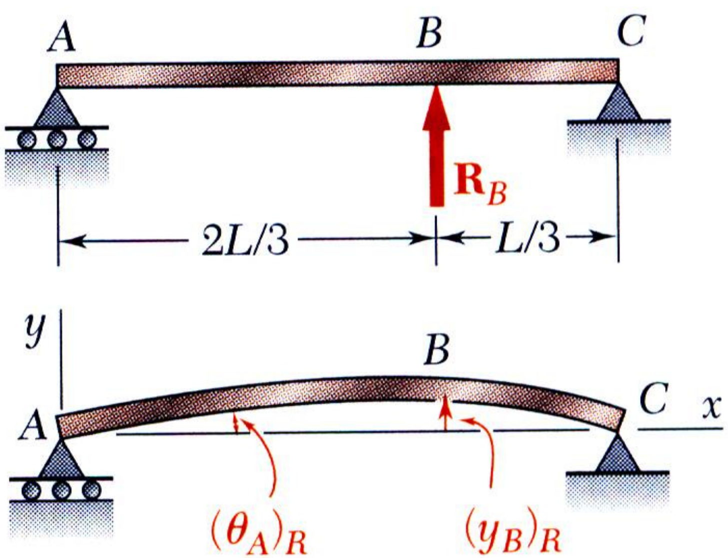


Deflection of Beams

Example 10



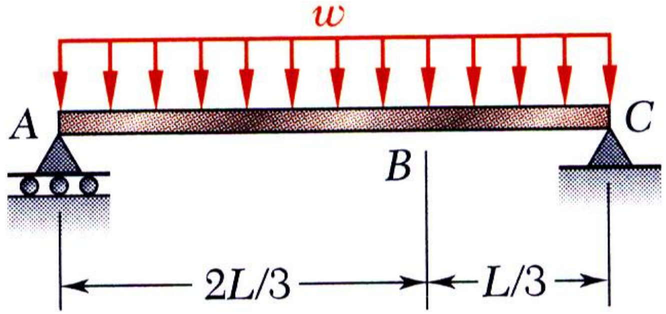
- Distributed Loading:



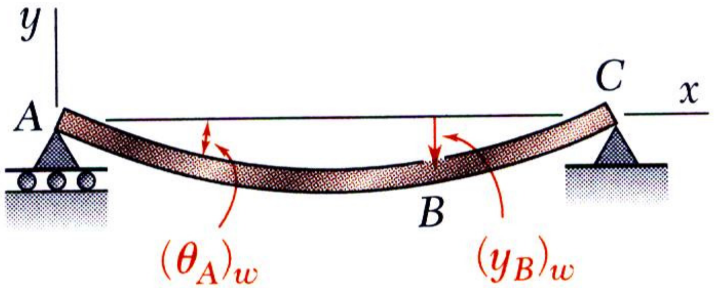
- Redundant Reaction Loading:

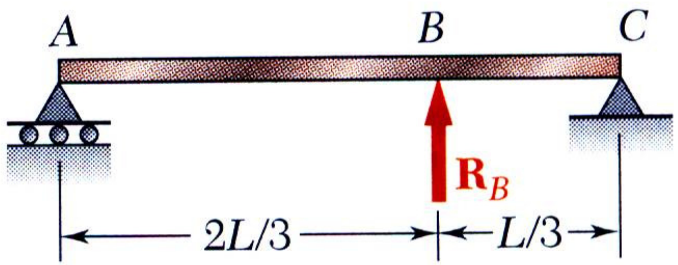
Deflection of Beams

Example 10

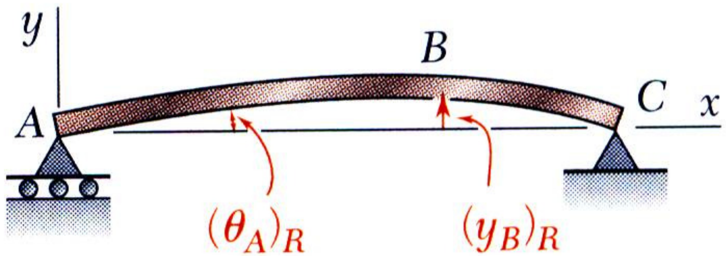


- For compatibility with original supports, $y_B = 0$



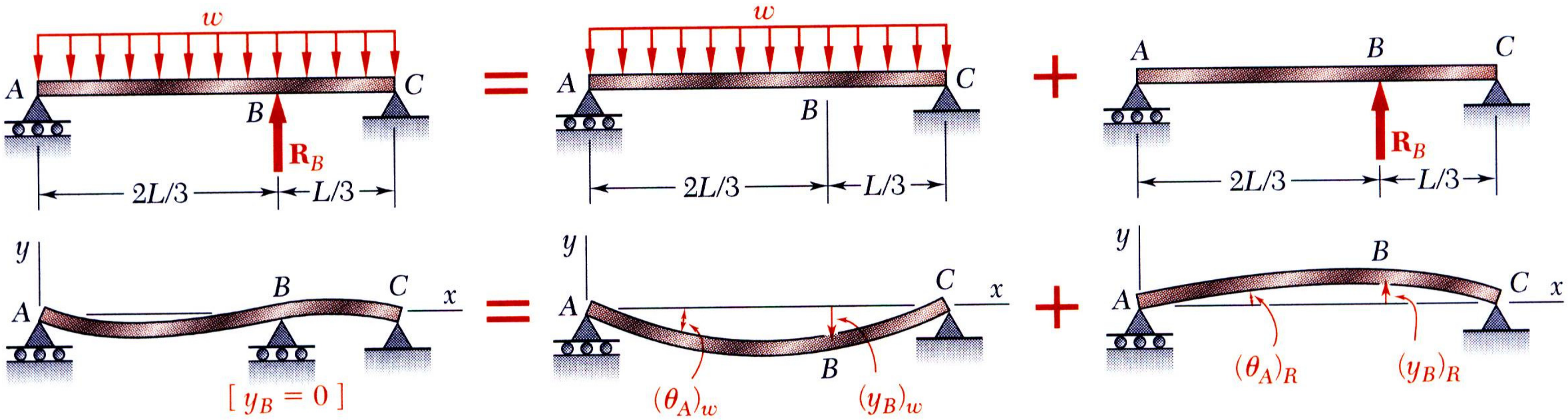


- From statics,



Deflection of Beams

Example 10



Slope at end *A*,