Mechanics of Materials

دانشگاه کردستان
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Ferdinand P.Beer, E.Russel Johnston, Jr., John T.Dewolf

Other Reference:
J.Wat Oler "Lectures notes on Mechanics od Materials"
Ibrahim A.Assakkaf "Lectures notes on Mechanics od Materials"

Transformations of Stress and Strain

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❑ Introductions

– Formulas for determining normal and shearing stresses on a specific planes are:

- Axially loaded bars
- Circular shafts
- Beams

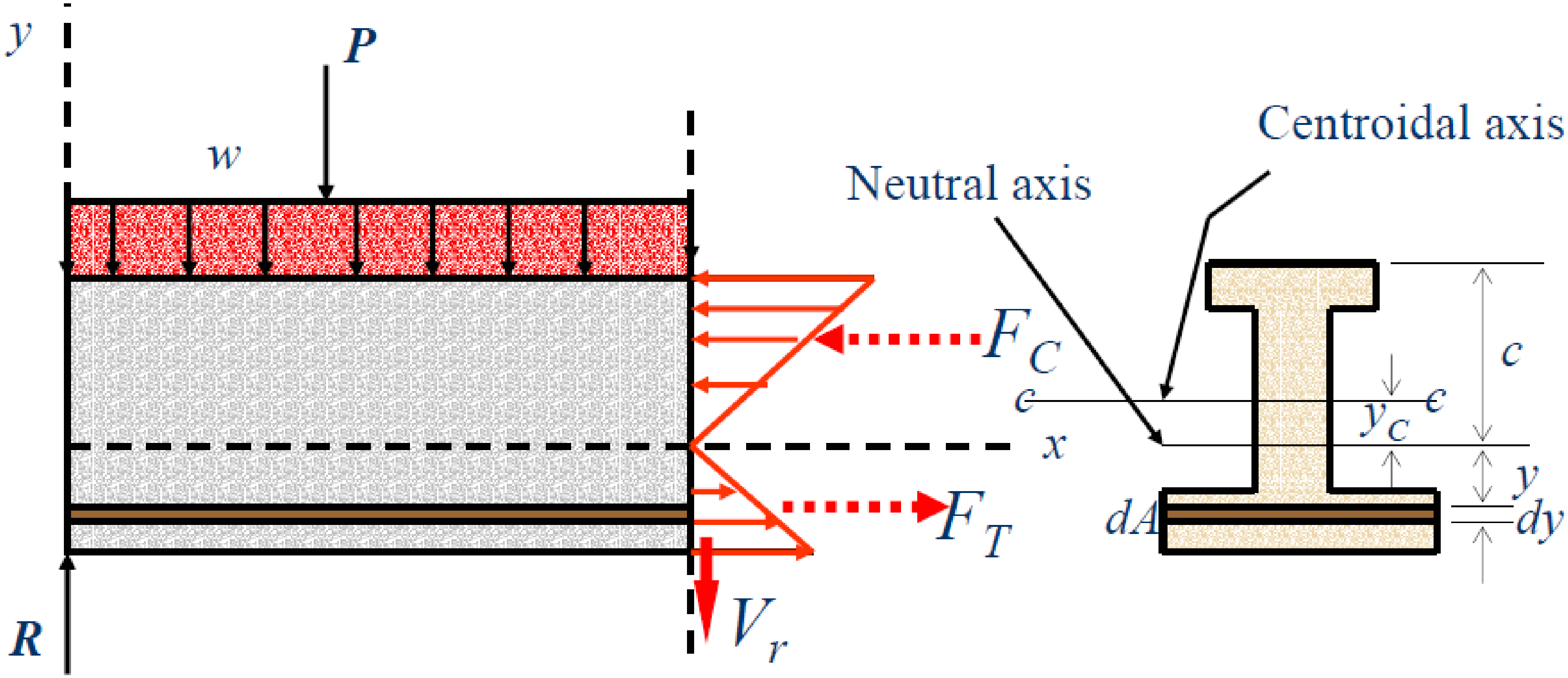
❑ Introductions

The elastic flexural formula for normal stress is given by:

$$\sigma_x = \frac{M \cdot y}{I}$$

$$\sigma_{max} = \frac{M \cdot c}{I}$$

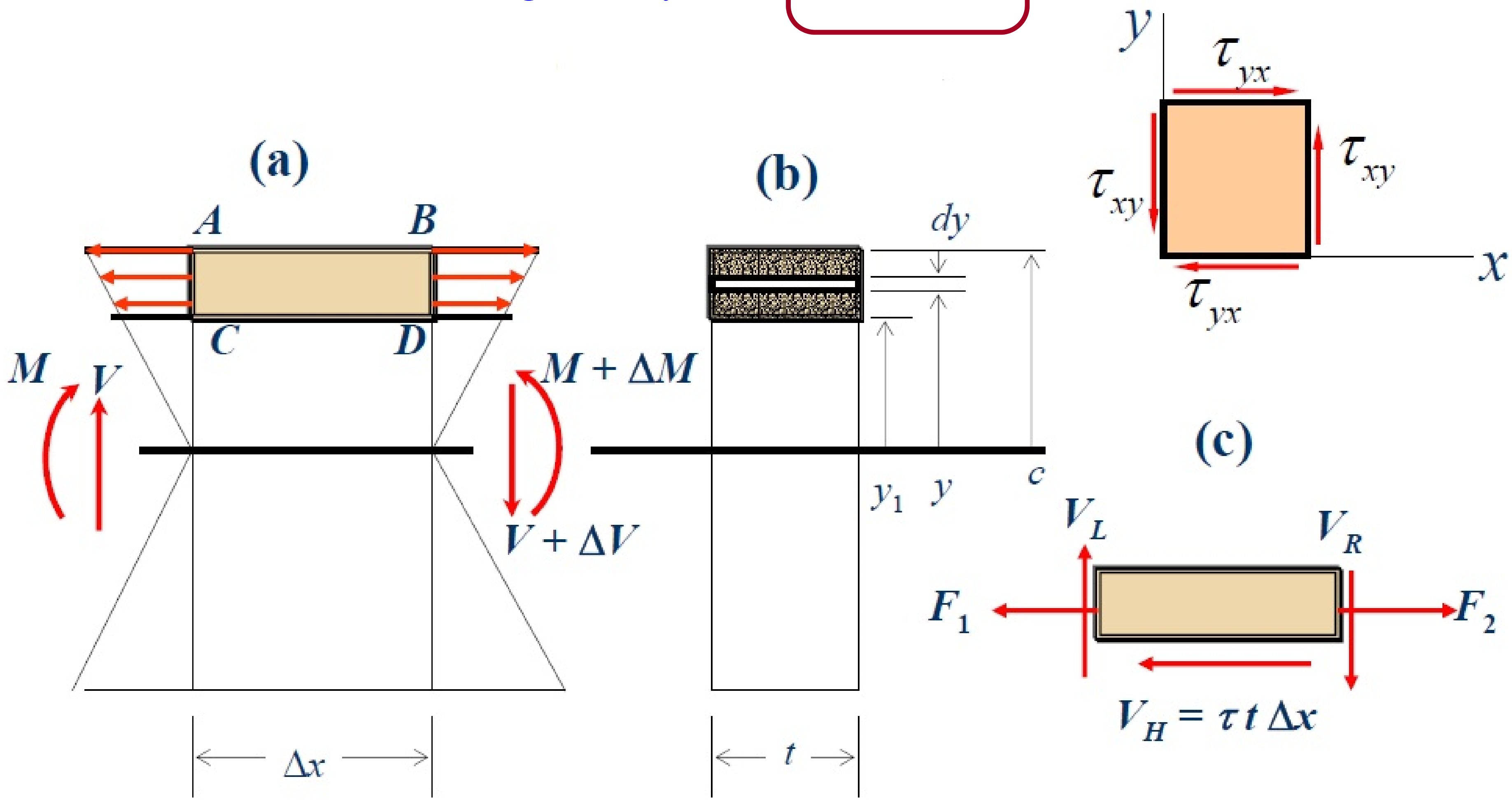
Distribution of Normal Stress in a Beam Cross Section



❑ Introductions

The shearing stress at the same point on the cross section of the beam is given by:

$$\tau = \frac{V \cdot Q}{I \cdot t}$$



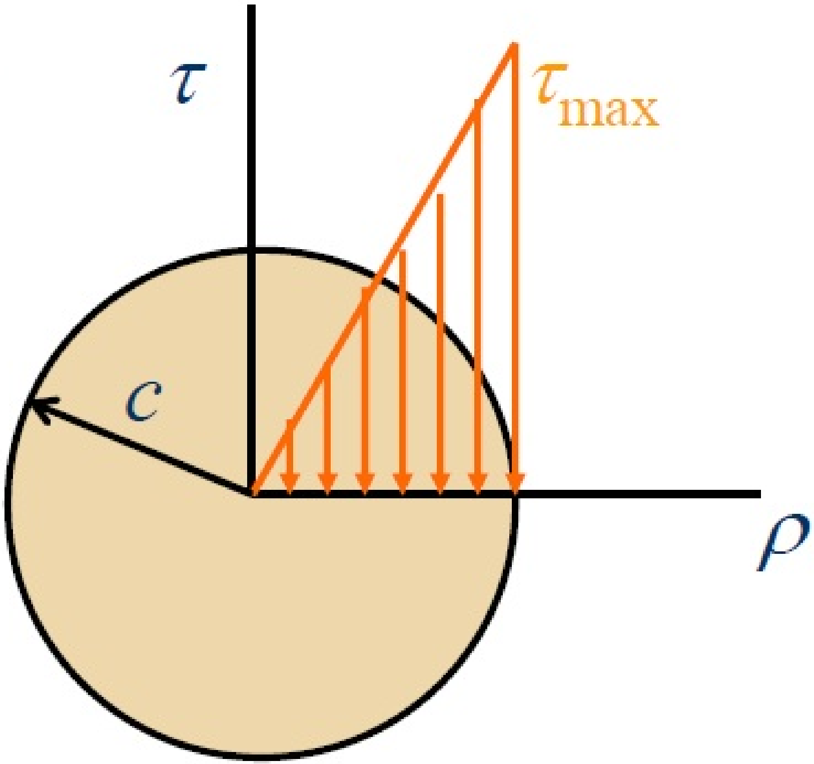
❑ Introductions

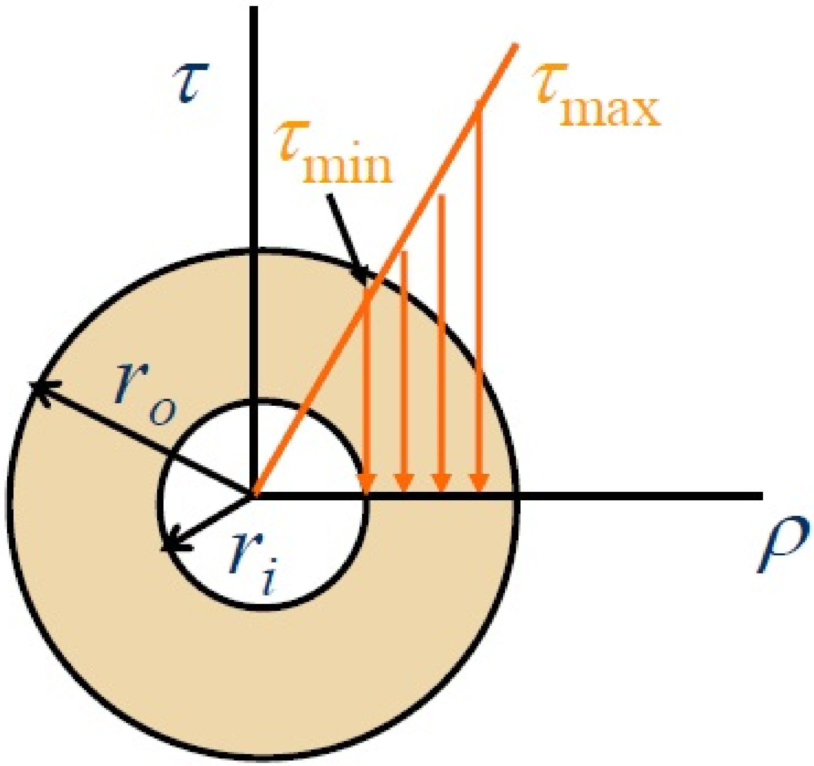
the stress on circular shafts due to torsion is given by:

$$\tau_\rho = \frac{T \cdot \rho}{J}$$

$$\tau_{\max} = \frac{T \cdot c}{J}$$

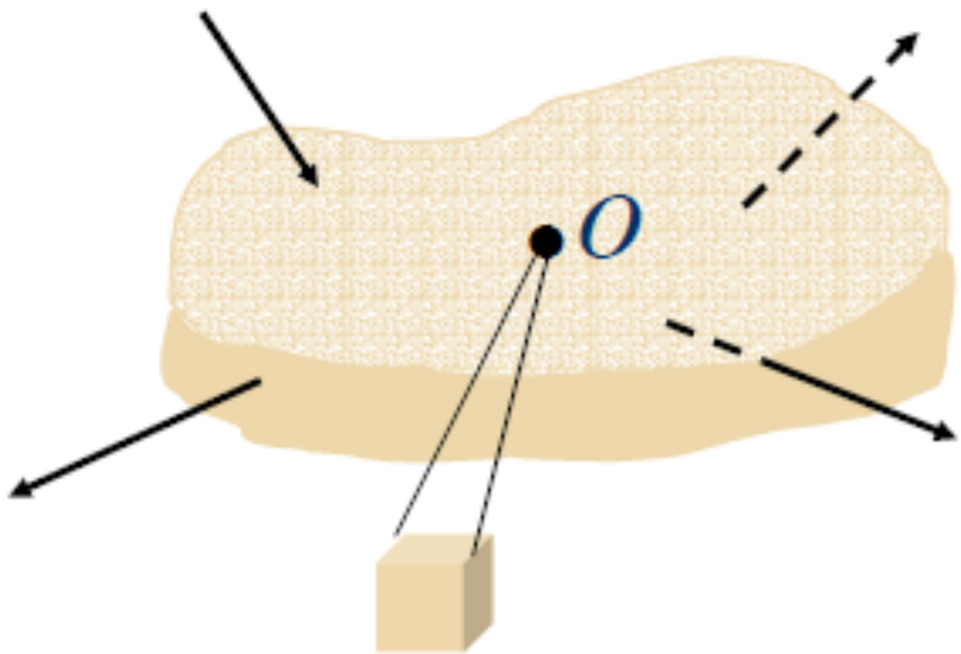
Distribution of Normal Stress in a Beam Cross Section

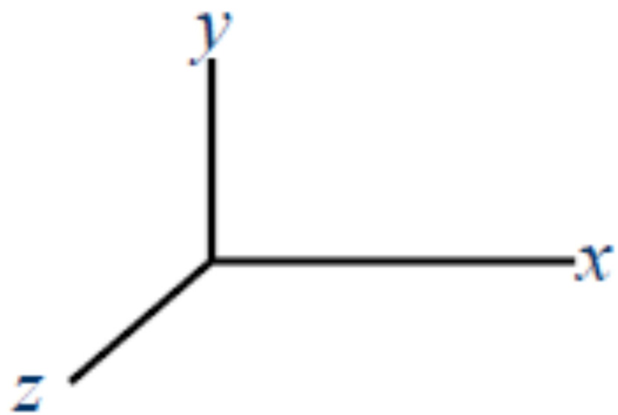




❑ State of Stress

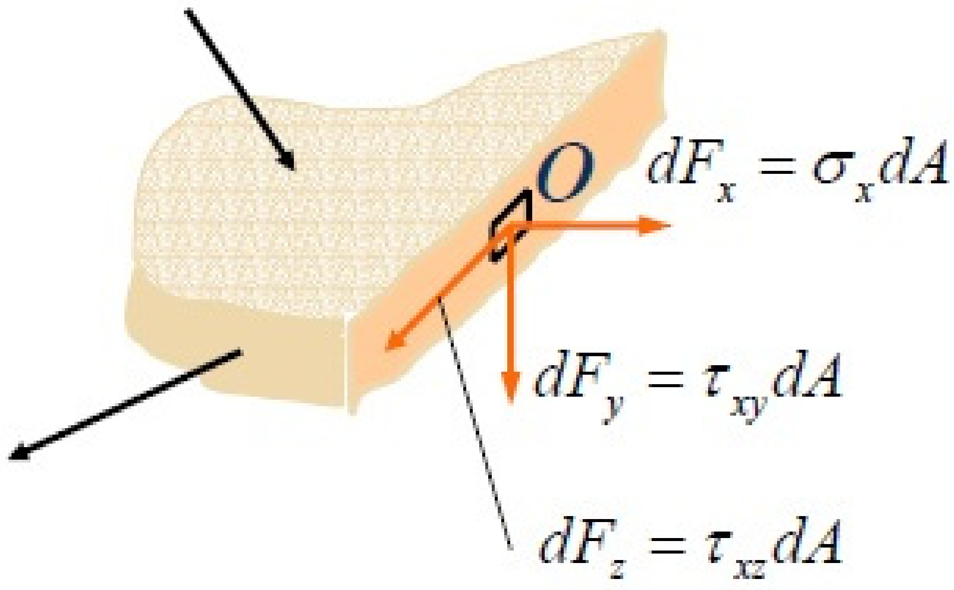
Stress at a point in a material body has been defined as a force per unit area. But this definition is somewhat ambiguous since it depends upon ***what area we consider at the point***.





❑ State of Stress

let us pass a cutting plane through point O perpendicular to the x axis.
If dA is the area, then by definition



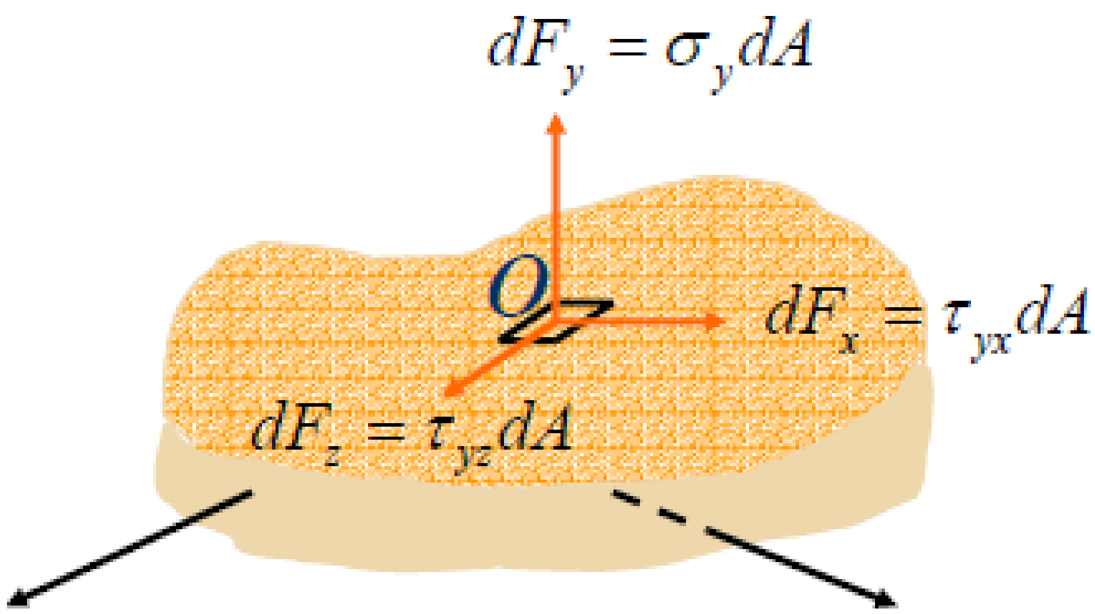
$$\sigma_x = \frac{dF_x}{dA}$$

$$\tau_{xy} = \frac{dF_y}{dA}$$

$$\tau_{xz} = \frac{dF_z}{dA}$$

❑ State of Stress

let us pass a cutting plane through point *O* perpendicular to the *y* axis.
If *dA* is the area, then by definition



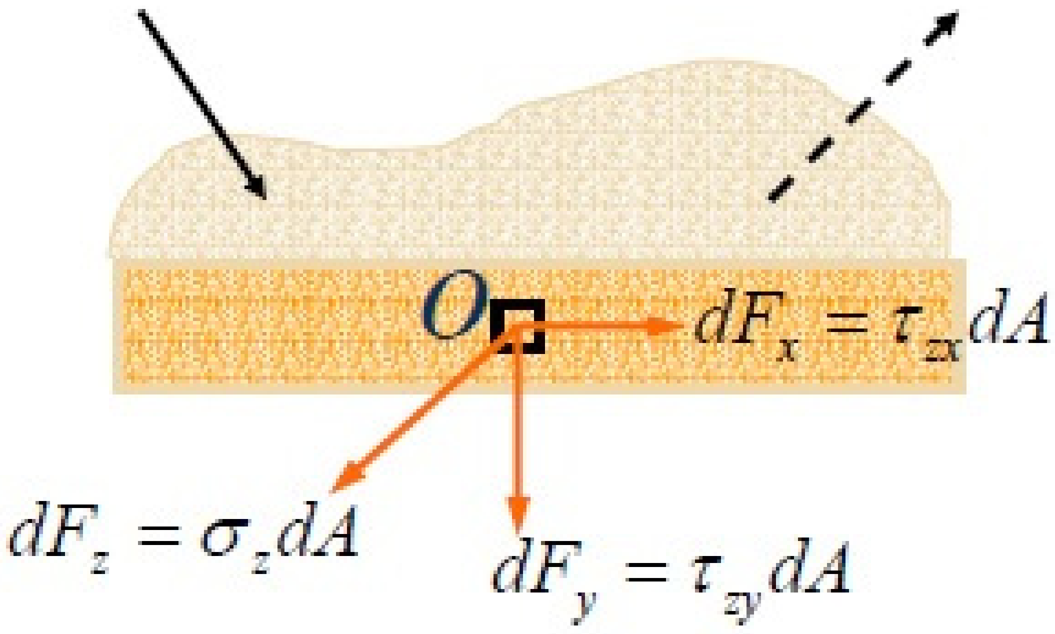
$$\sigma_y = \frac{dF_y}{dA}$$

$$\tau_{yx} = \frac{dF_x}{dA}$$

$$\tau_{yz} = \frac{dF_z}{dA}$$

State of Stress

let us pass a cutting plane through point O perpendicular to the z axis.
If dA is the area, then by definition

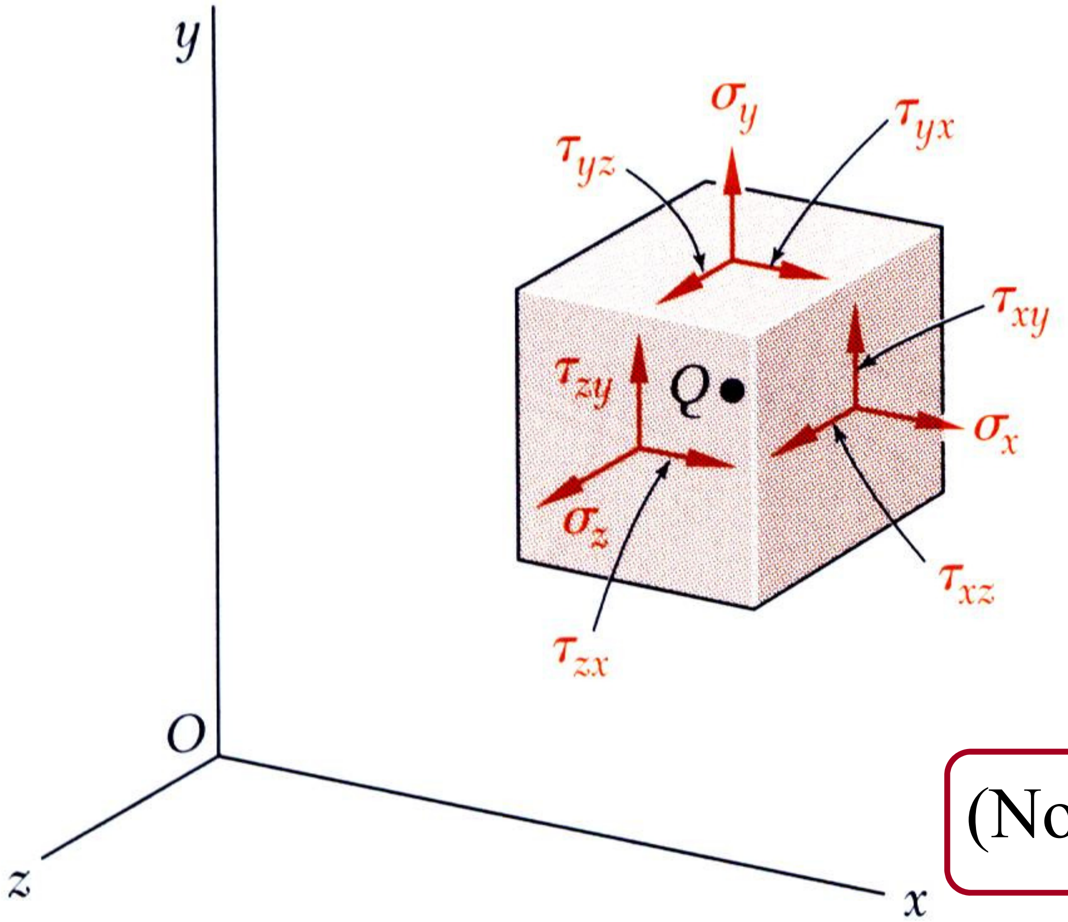


$$\sigma_z = \frac{dF_z}{dA}$$

$$\tau_{zx} = \frac{dF_x}{dA}$$

$$\tau_{zy} = \frac{dF_y}{dA}$$

❑ General or Triaxial State of stress



- **Normal Stresses**

$$\sigma_x, \sigma_y, \sigma_z$$

- **Shear Stress**

$$\tau_{xy}, \ \tau_{yz}, \ \tau_{zx}$$

$$(\text{Note}: \ \tau_{xy} = \tau_{yx}, \ \tau_{yz} = \tau_{zy}, \ \tau_{zx} = \tau_{xz})$$

❑ General or Triaxial State of stress

Sign Conventions

- Normal stresses indicated by the symbol **σ** and a single subscript to indicate the plane (**actually the outward normal to the plane**) on which he stress acts.

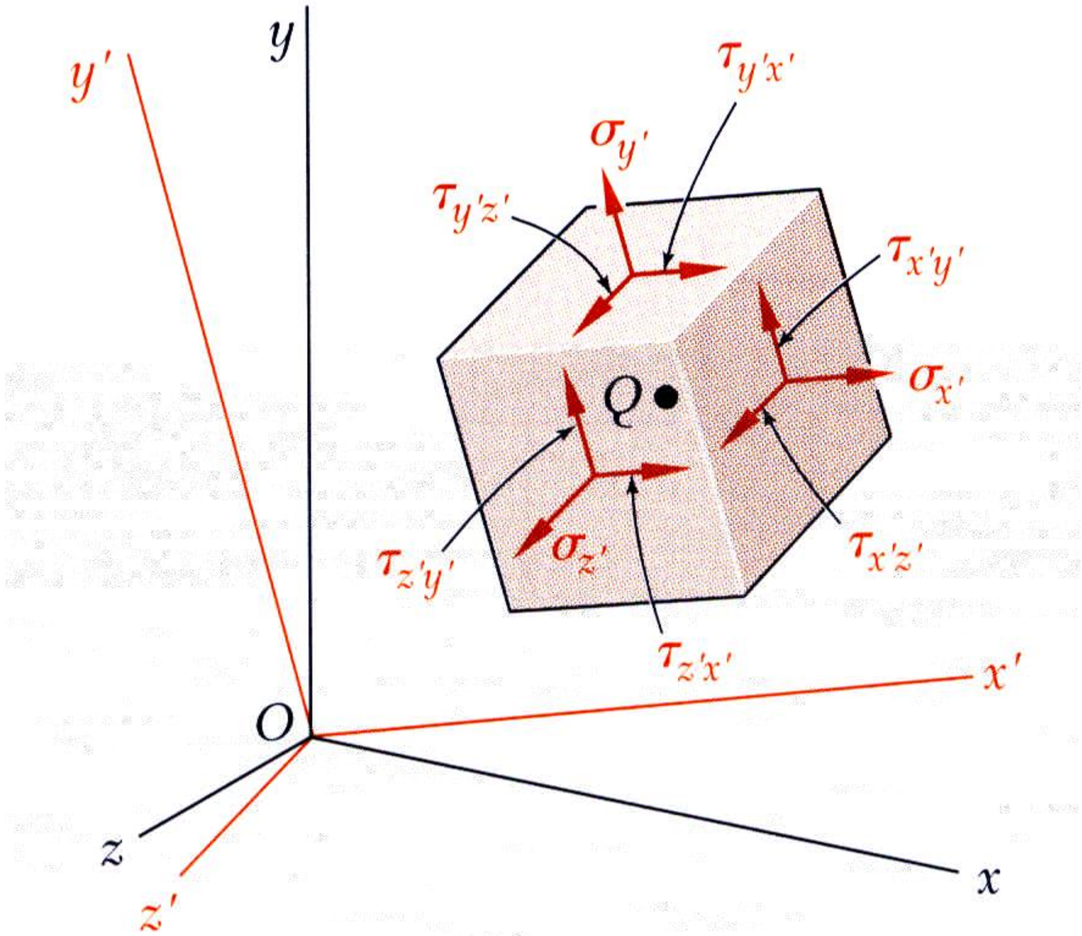
- Normal stresses are positive if they point in the direction of the outward normal. Thus, normal stresses are **positive if tensile and negative if compressive**.

- Shearing stresses are denoted by the symbol **τ** followed by two subscripts, **the first subscript designates the normal to the plane on which the stress acts and the second designate the coordinate axis to which the stress is parallel**.

- A positive shearing stress points in **the positive direction of the coordinate axis of the second subscript if it acts on a surface with an outward normal in the positive direction**.

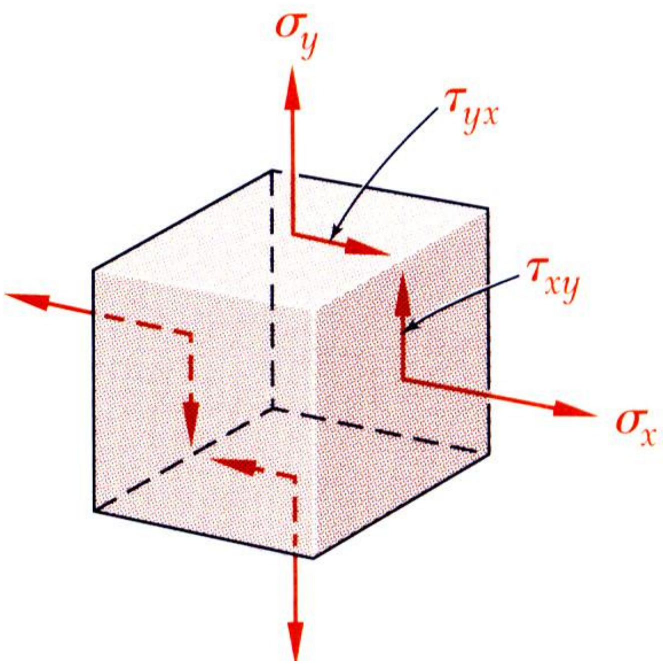
❑ General state of stress

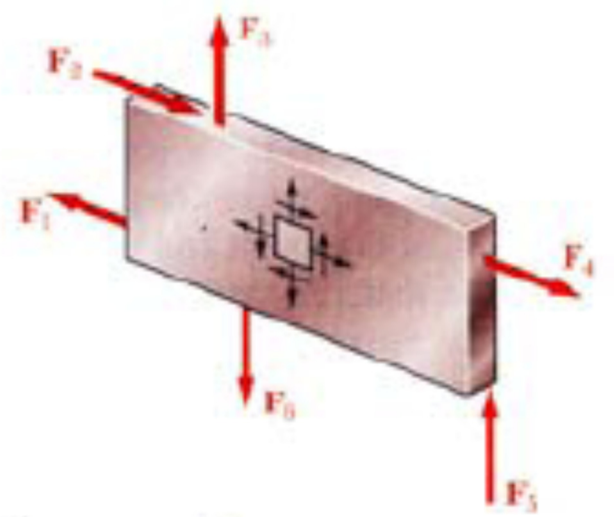
Same state of stress is represented by a **different set of components** if axes are rotated.

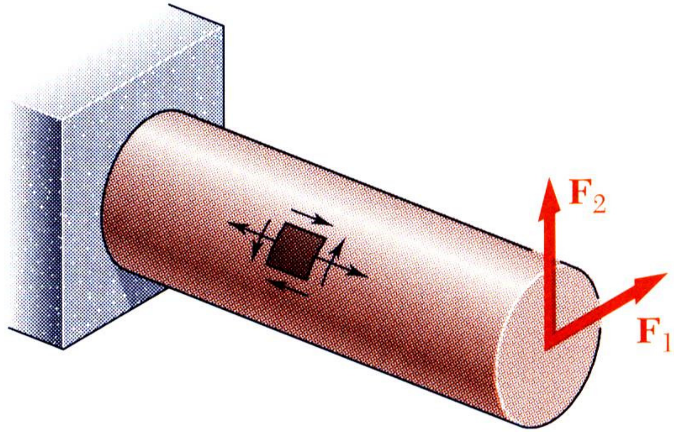


❑Plane Stress

- ***Plane Stress*** - state of stress in which two faces of the cubic element are free of stress.







❑Plane Stress

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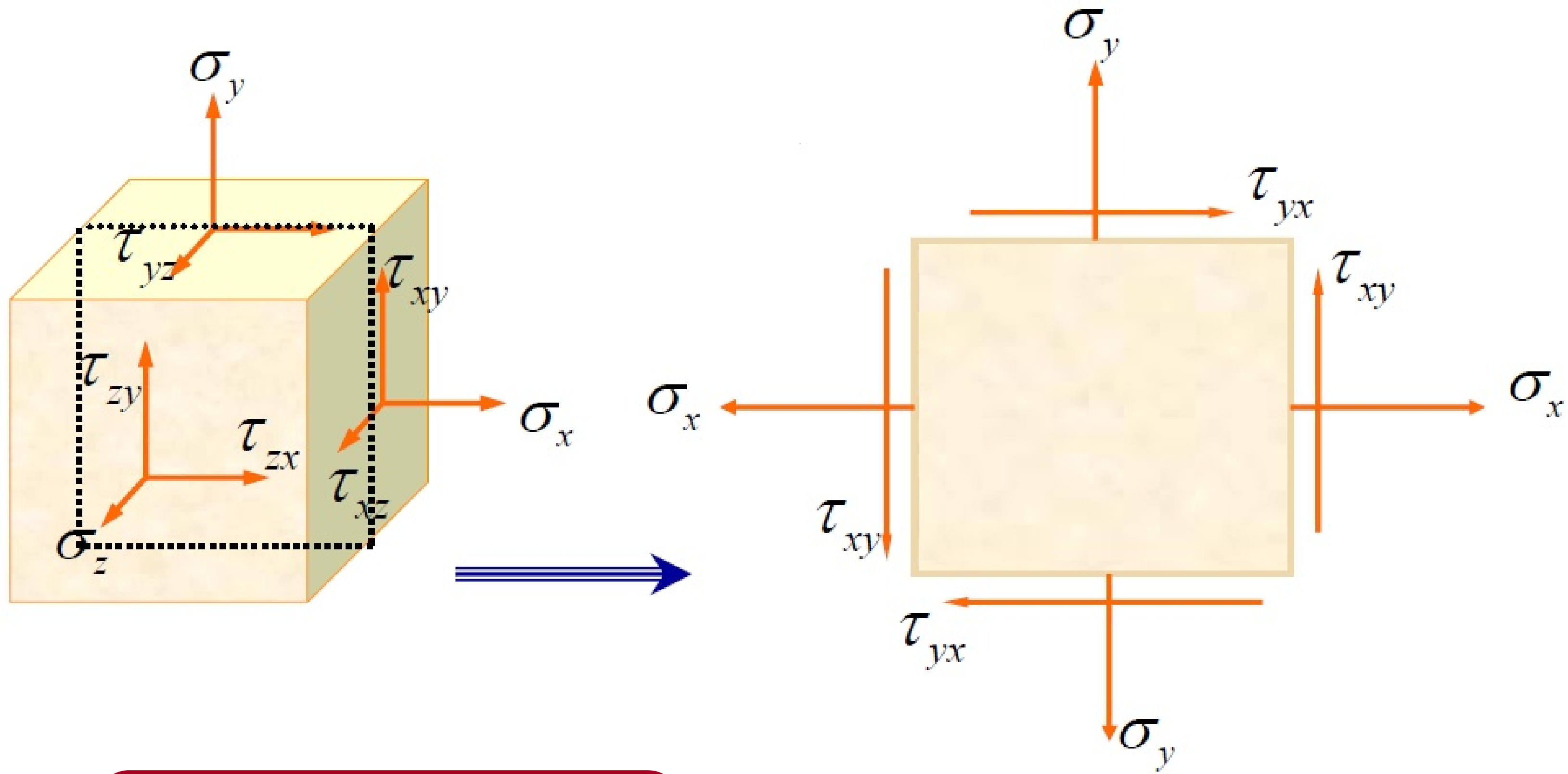
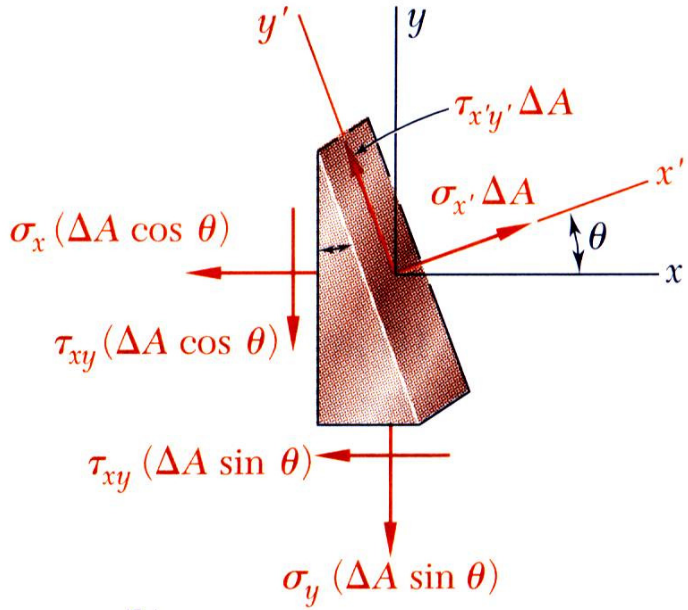


$$\sigma_z = \tau_{zx} = \tau_{zy} = 0.$$

$$\sigma_x,\ \sigma_y,\ \tau_{xy} \neq 0$$

❑Transformation of Plane Stress



Plane Stress Equations
Free-body Diagram

$$\sum F_{x'} = 0 = \sigma_{x'}\Delta A - \sigma_x\left(\Delta A\cos(\theta)\right)\cos(\theta) - \tau_{xy}\left(\Delta A\cos(\theta)\right)\sin(\theta) - \sigma_y\left(\Delta A\sin(\theta)\right)\sin(\theta) - \tau_{xy}\left(\Delta A\sin(\theta)\right)\cos(\theta)$$

$$\sum F_{y'} = 0 = \tau_{x'y'}\Delta A + \sigma_x\left(\Delta A\cos(\theta)\right)\sin(\theta) - \tau_{xy}\left(\Delta A\cos(\theta)\right)\cos(\theta) - \sigma_y\left(\Delta A\sin(\theta)\right)\cos(\theta) + \tau_{xy}\left(\Delta A\sin(\theta)\right)\sin(\theta)$$

❑Transformation of Plane Stress

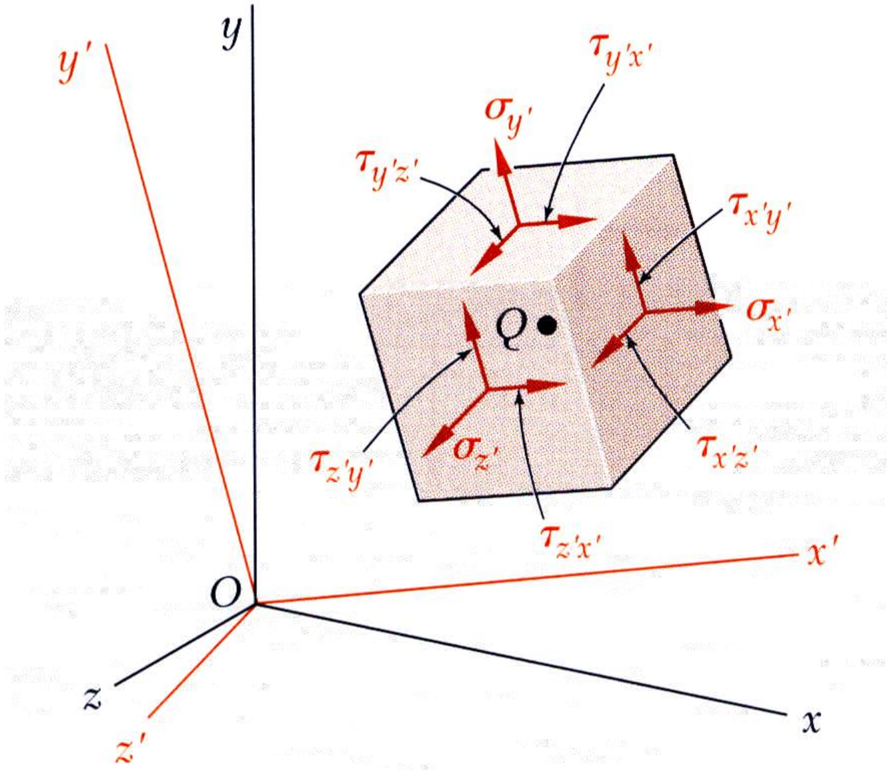
•The equations may be rewritten to yield

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta)$$

$$if \quad \theta \rightarrow \theta + 90 \quad \Rightarrow$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) - \tau_{xy}\sin(2\theta)$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta)$$



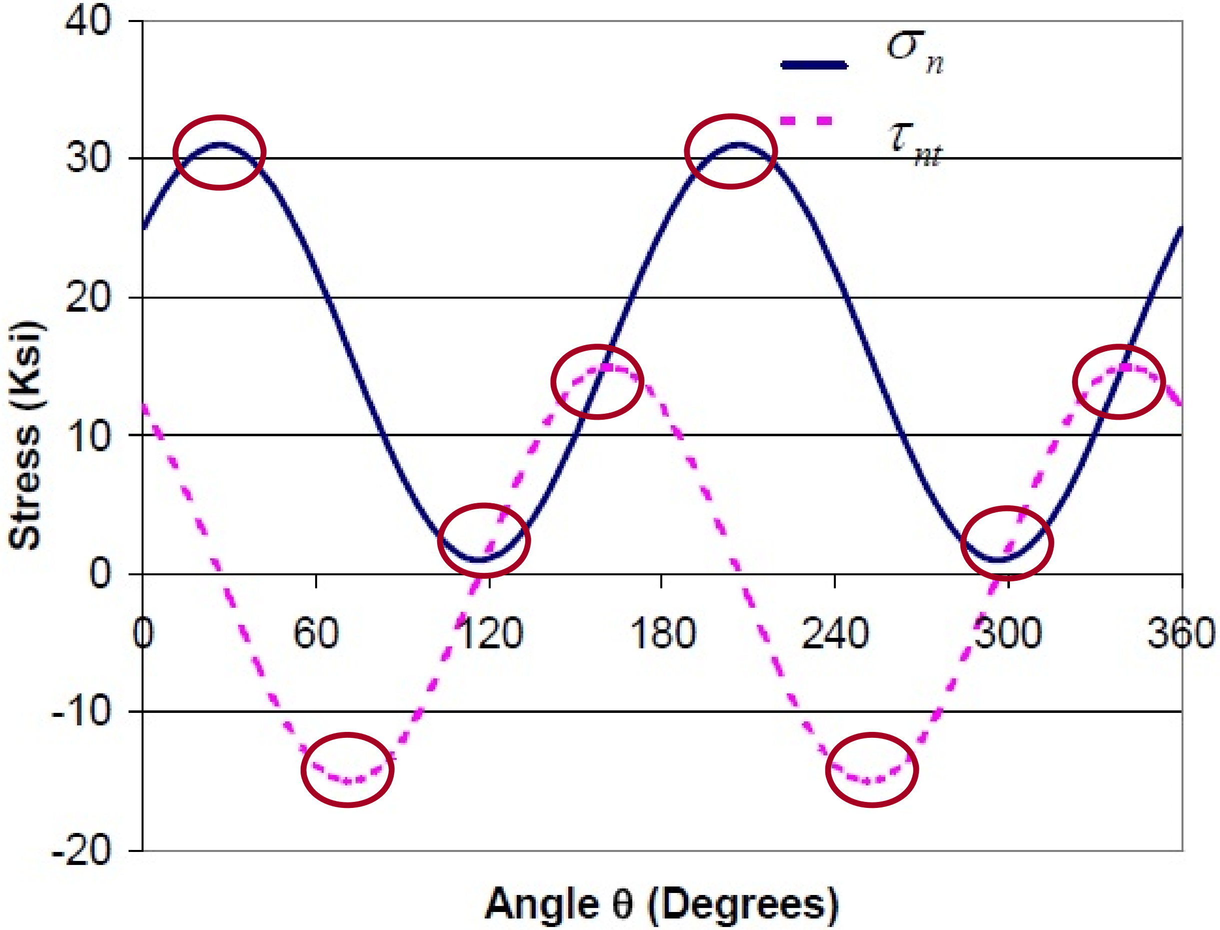
For plane stress, the sum of the normal stresses on any two orthogonal planes through a point in a body is a constant or in invariant.

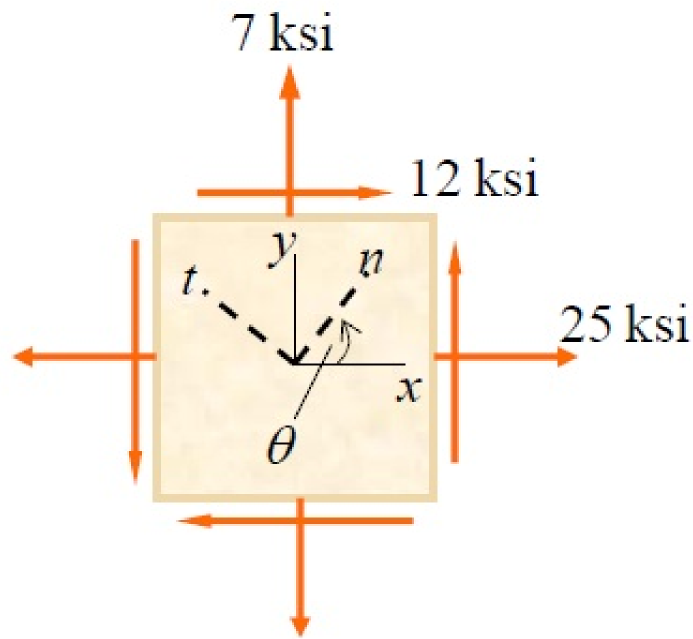
$$\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y$$

Transformations of Stress and Strain

❑ Principal Stresses

- The principal stresses are the maximum and minimum normal stress.
- In general, the principal stresses can be determined by plotting curves.
- This process is time-consuming, and therefore, general methods are needed.





Variation of Stresses as Functions of θ

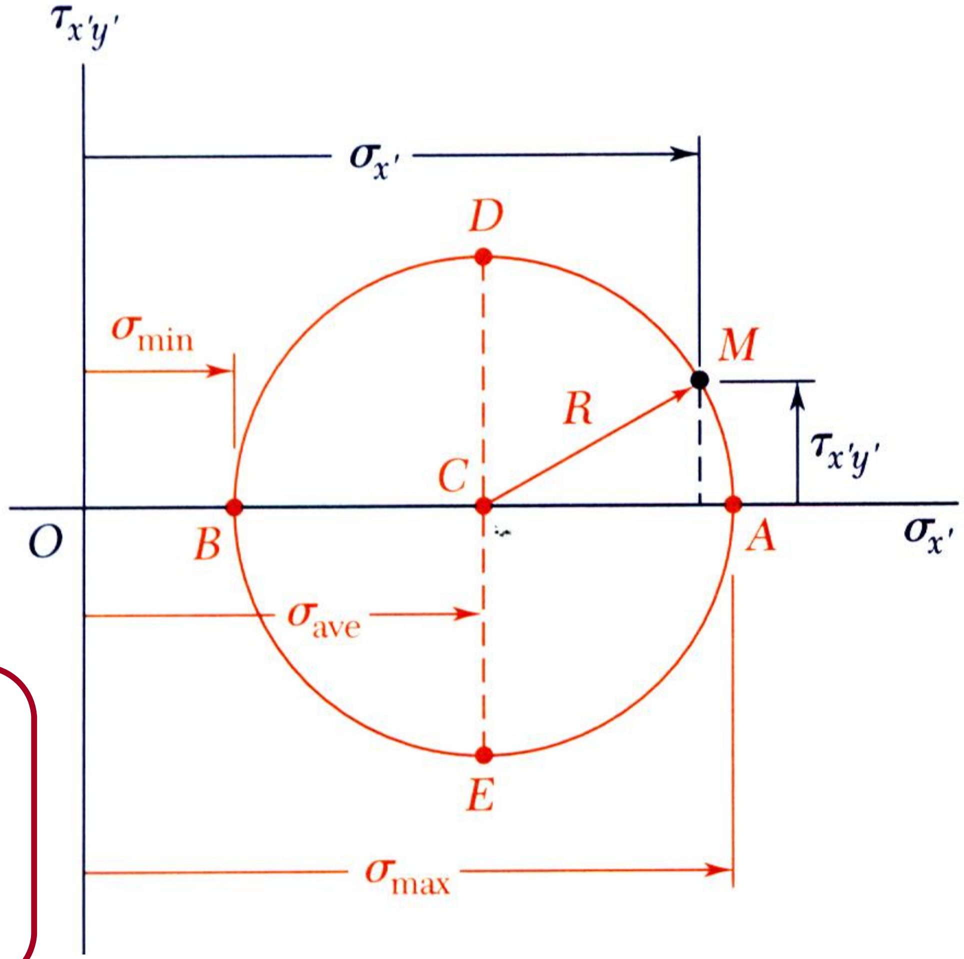
❑Principal Stresses

- The previous equations are combined to yield parametric equations for a circle

$$\left(\sigma_{x'} - \sigma_{ave}\right)^2 + \tau_{x'y'}^2 = R^2$$

where

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} \qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

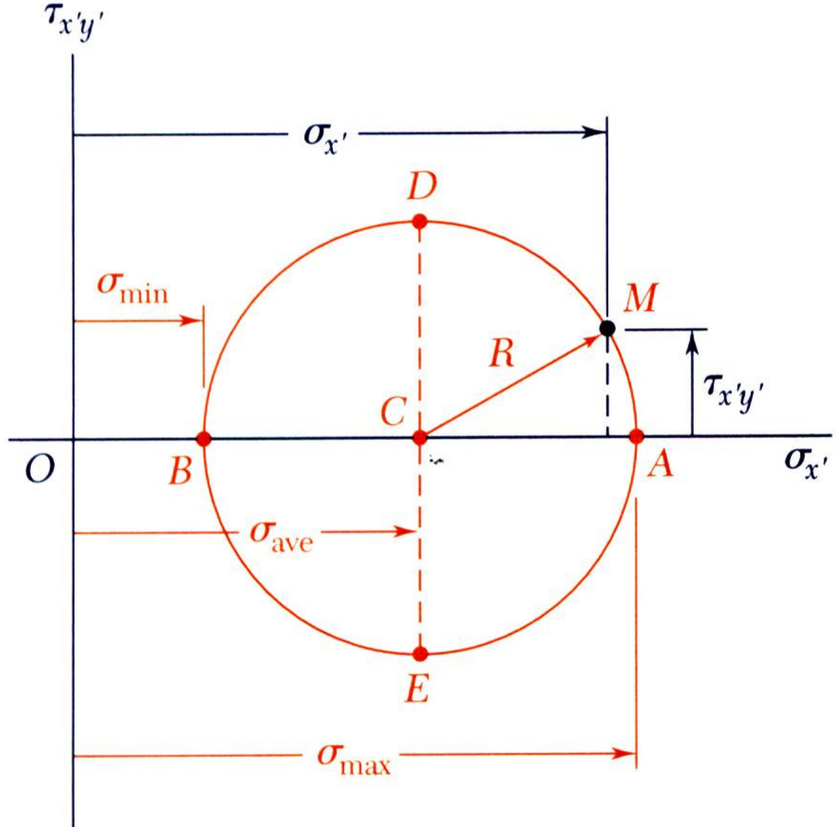


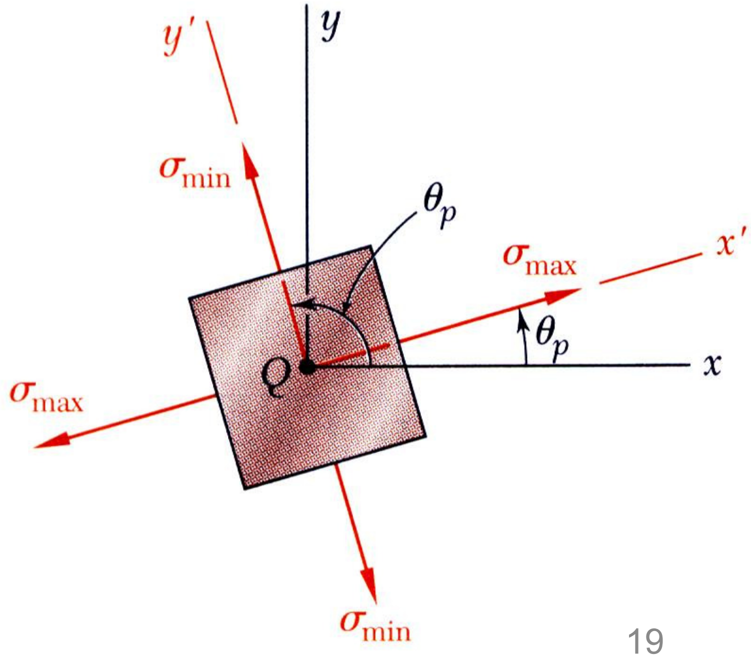
❑Maximum normal stress

- ***Principal stresses*** occur on the principal planes of stress with zero shearing stresses.

$$\sigma_{\max,\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$





❑ Notes on Principal Stresses Equation

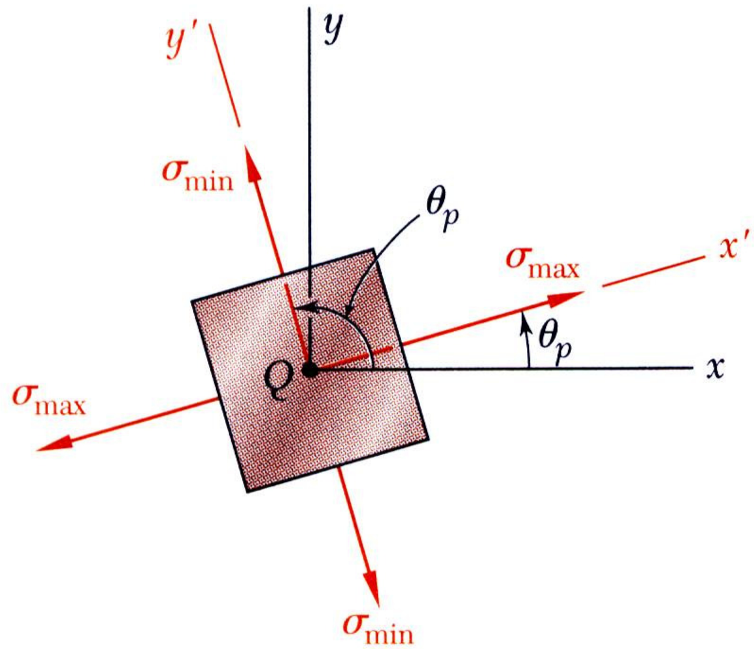
I. The angle θp and θp + 90 between x-plane (or y-plane) and the mutually perpendicular planes on which the principal stresses act.

II. When tan 2θp is positive, θp is **positive**, and the rotation is **counterclockwise.**

III. When tan 2θ*p* is negative, θ*p* is **negative**, and the rotation is **clockwise**.

IV. The shearing stress is zero on planes experiencing maximum and minimum values of normal stresses.

V. If one or both of the principal stresses is negative, the algebraic maximum stress can have a smaller absolute value than the minimum stress.



$$\sigma_{\max,\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

❑Maximum shear stress

Maximum shearing stress occurs for:

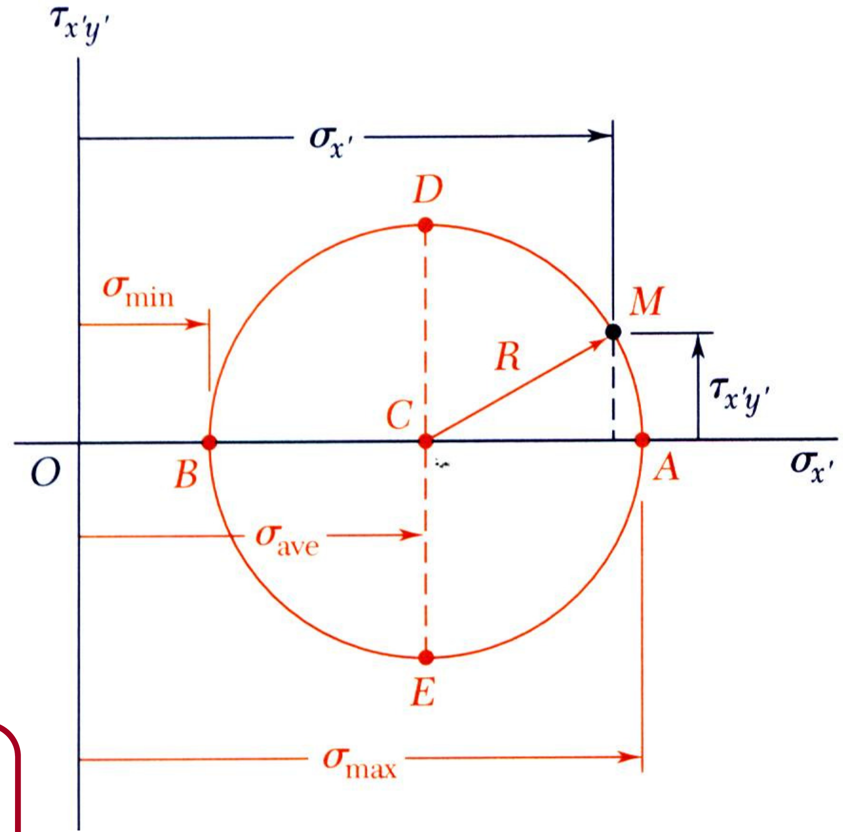
$$\sigma_{x'} = \sigma_{ave} = \frac{\sigma_x + \sigma_y}{2}$$

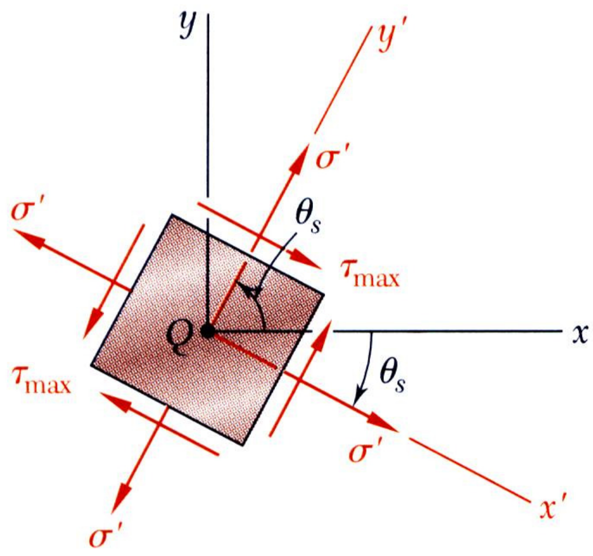
$$\tau_{\max} = R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{\max} = \frac{\sigma_{Max} - \sigma_{Min}}{2}$$

$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

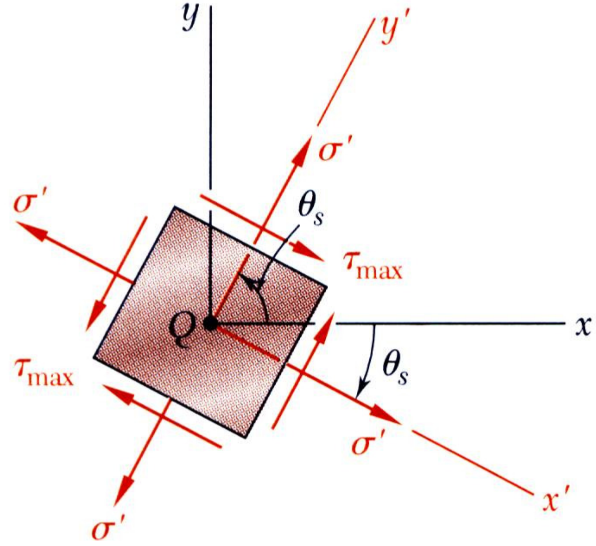
Note : defines two angles separated by 90° and offset from θ_p by 45°





Transformations of Stress and Strain

❑ Notes on Principal Stresses and Maximum In-Plane Shearing Stress Equation



I. The two angles 2θp and 2θs differ by 90, therefore, θp and θs are 45 apart.

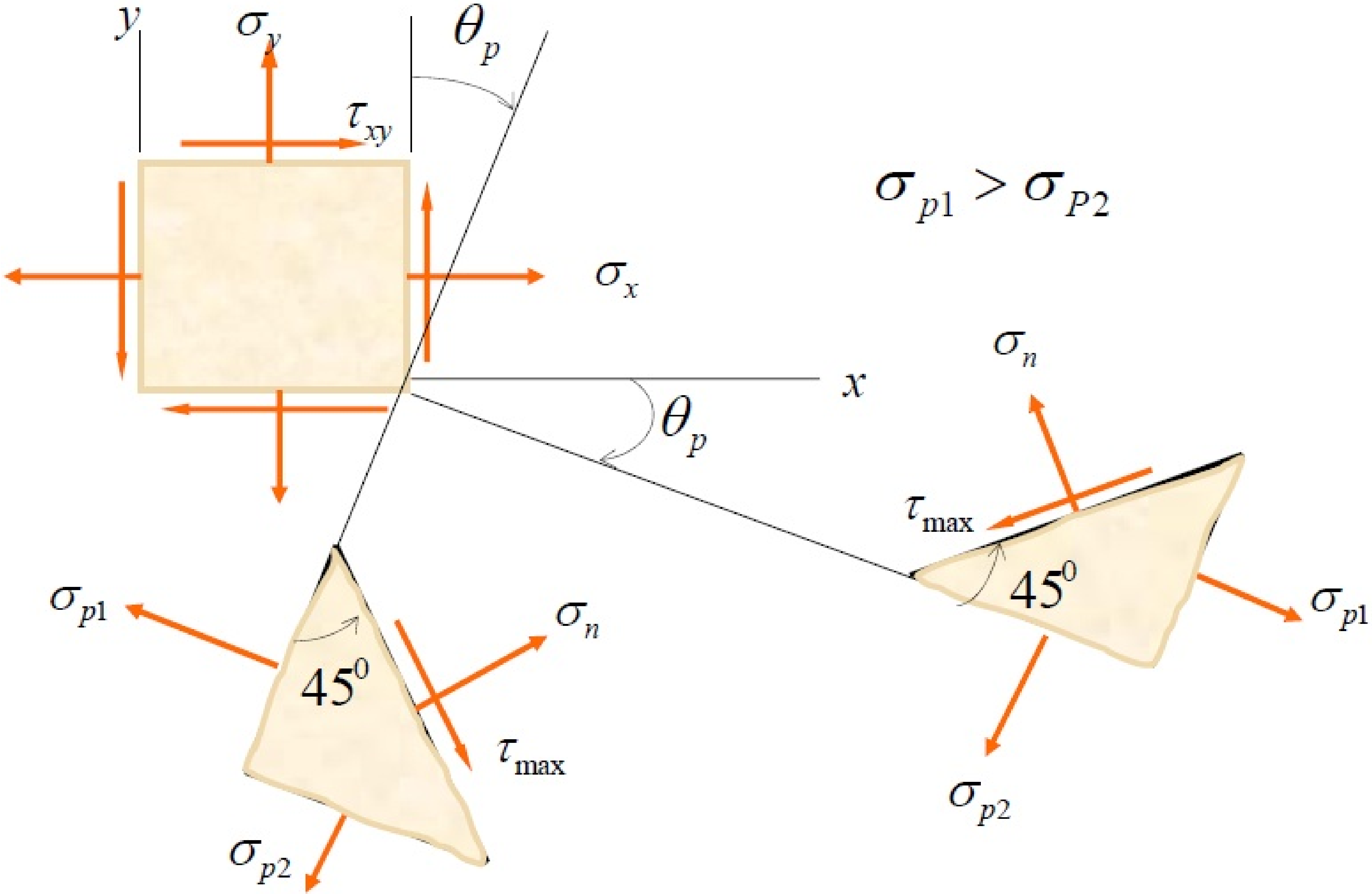
II. This means that the planes in which the maximum in-plane shearing stress occur are 45 from the principal planes.

III. The direction of the maximum shearing stress can be determined by drawing a wedge-shaped block with two sides parallel to the planes having the maximum and minimum principal stresses, and with the third side at an angle of 45. The direction of the maximum shearing stress must oppose the larger of the two principal stresses.

$$\tau_{\max} = R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

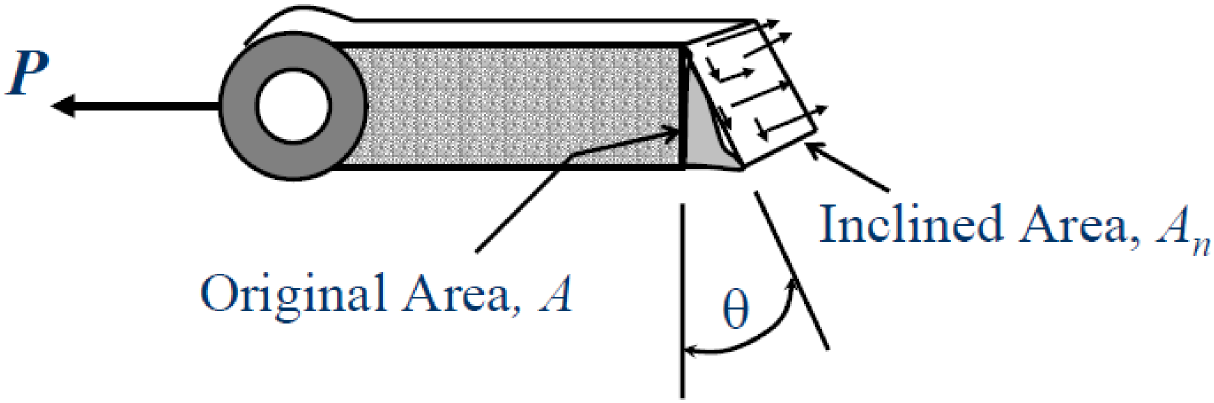
$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

❑Wedge-shaped Block



❑Principal Stresses for Axially Loaded Bar

$$\sigma_x \neq 0$$
$$\sigma_y = 0$$
$$\tau_{xy} = 0$$



$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta) = \frac{\sigma_x + 0}{2} + \frac{\sigma_x - 0}{2}\cos(2\theta) + (0)\sin(2\theta)$$

$$\Rightarrow \quad \sigma_{x'} = \frac{\sigma_x}{2}\left(1 + \cos(2\theta)\right)$$

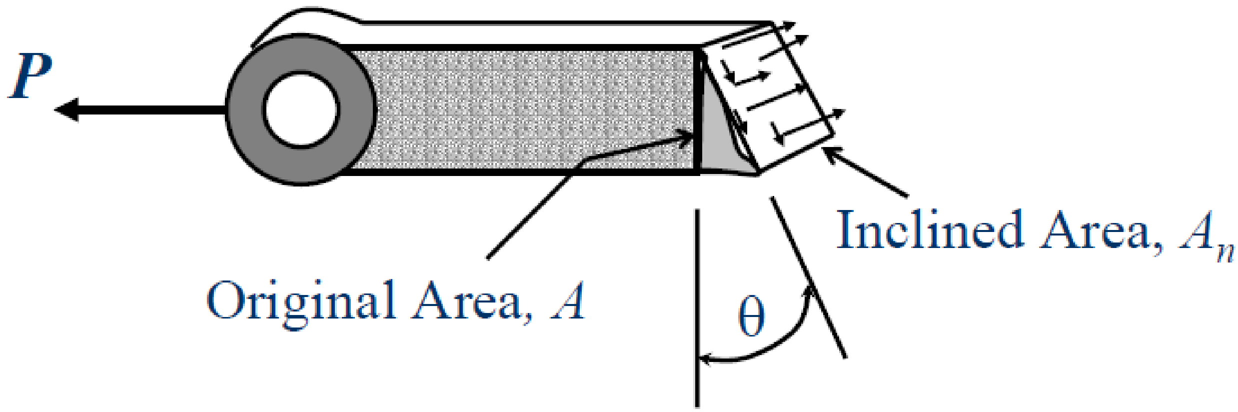
$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta) = -\frac{\sigma_x - 0}{2}\sin(2\theta) + (0)\cos(2\theta)$$

$$\Rightarrow \quad \tau_{x'y'} = -\frac{\sigma_x}{2}\sin(2\theta)$$

❑Principal Stresses for Axially Loaded Bar

$$\sigma_{x'} = \frac{\sigma_x}{2}\left(1 + \cos(2\theta)\right)$$

$$\tau_{x'y'} = -\frac{\sigma_x}{2}\sin(2\theta)$$

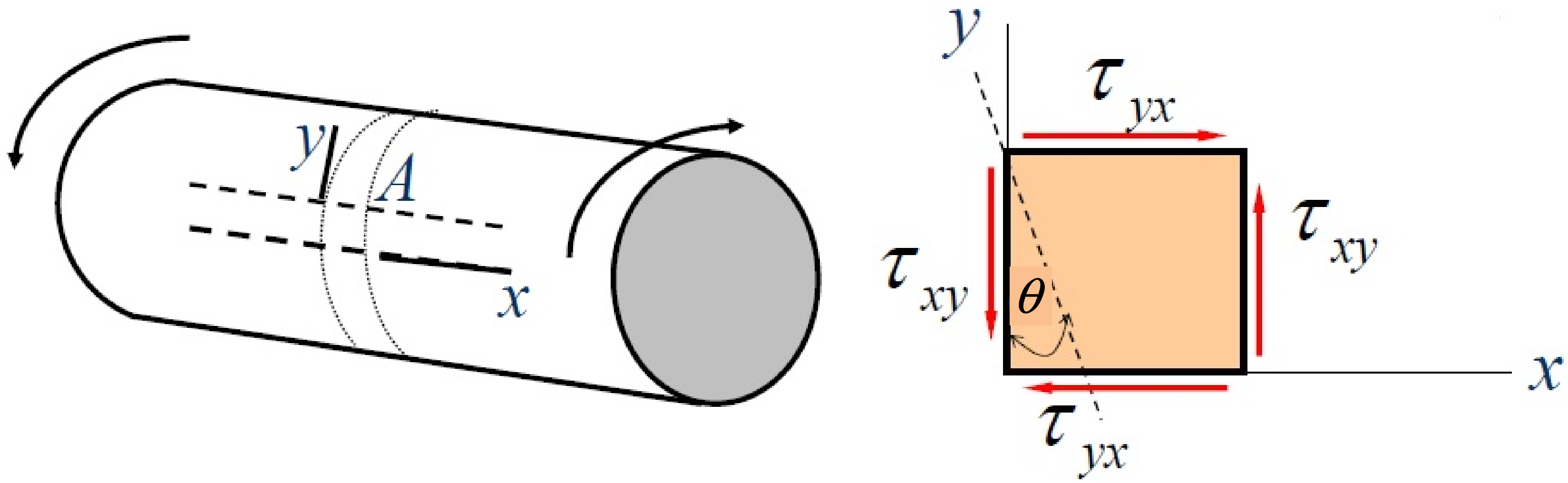


$$\theta = 0^\circ \; or \; 180^\circ \Rightarrow \sigma_{x'} = \sigma_{\max} \quad \Rightarrow \quad \sigma_{Max} = \sigma_x$$

$$\theta = 45^\circ \; or \; 135^\circ \Rightarrow \tau_{x'y'} = \tau_{Max} \quad \Rightarrow \quad \tau_{Max} = \frac{\sigma_x}{2}$$

$$\Rightarrow \quad \tau_{Max} = \frac{\sigma_{Max}}{2} = \frac{P}{2A}$$

❑Principal Stresses for Shaft under Pure Torsion



$$\sigma_x = 0$$
$$\sigma_y = 0$$
$$\tau_{xy} \neq 0$$

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta) = \frac{0+0}{2} + \frac{0-0}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta)$$

$$\Rightarrow \boxed{\sigma_{x'} = \tau_{xy}\sin(2\theta)}$$

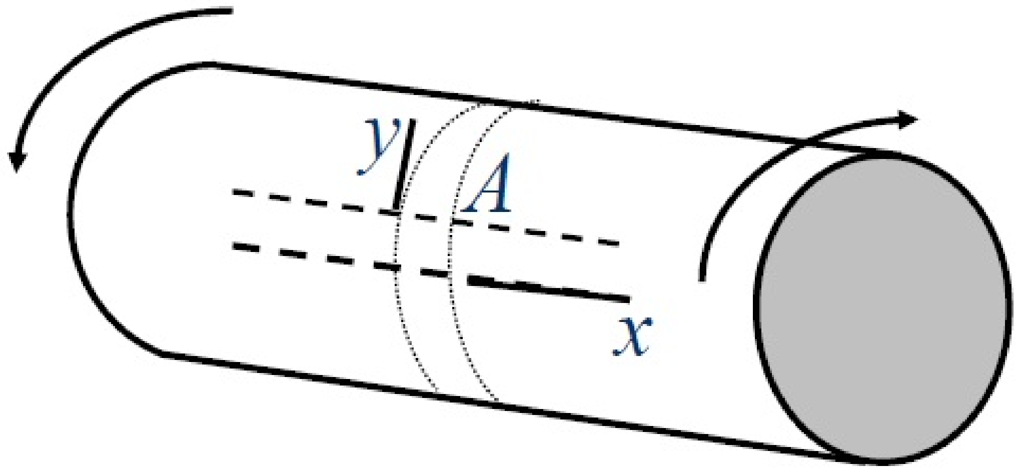
$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta) = -\frac{0-0}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta)$$

$$\Rightarrow \boxed{\tau_{x'y'} = \tau_{xy}\cos(2\theta)}$$

❑Principal Stresses for Shaft under Pure Torsion

$$\sigma_{x'} = \tau_{xy}\sin(2\theta)$$

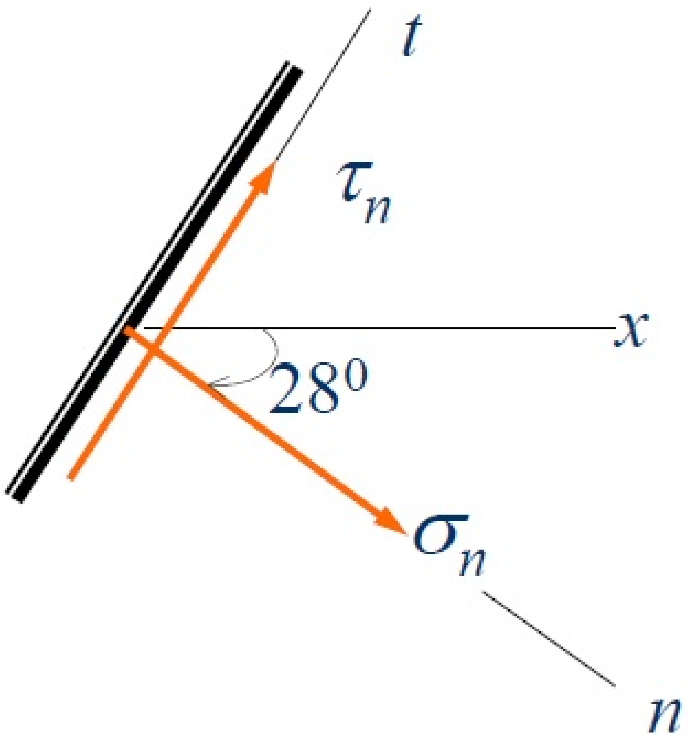
$$\tau_{x'y'} = \tau_{xy}\cos(2\theta)$$

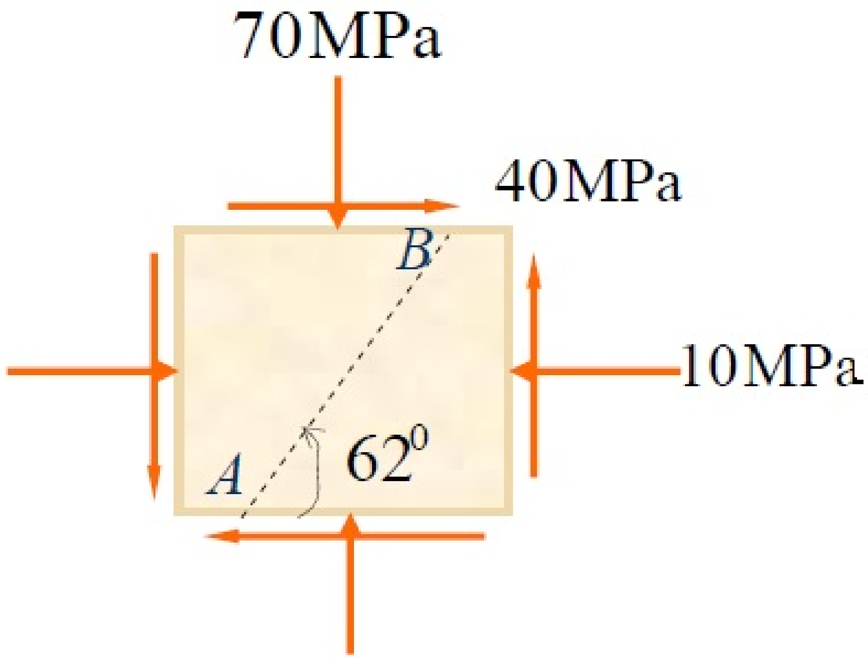


$$\theta = 45^\circ \; or \; 135^\circ \Rightarrow \begin{matrix} \tau_{x'y'} = \tau_{Max} = \tau_{xy} \\ \sigma_{x'} = \sigma_{Max} = \tau_{xy} \end{matrix} \Rightarrow \boxed{\sigma_{Max} = \tau_{Max} = \frac{T_{Max}c}{J}}$$

❑Example 1

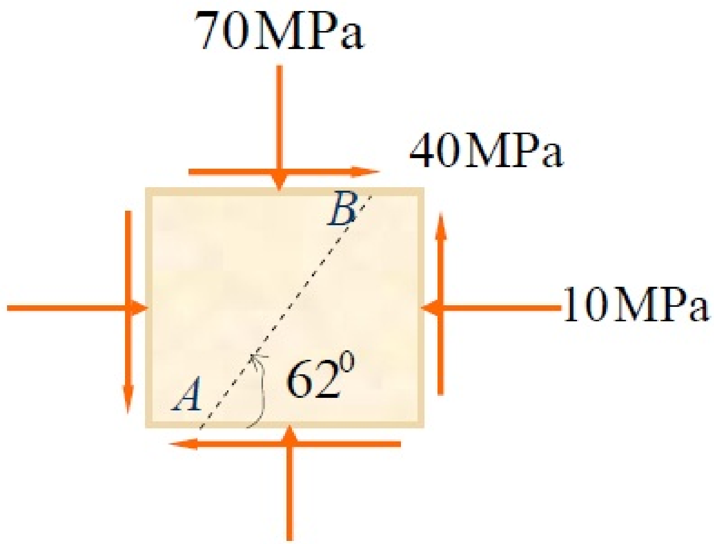
Determine the normal and shearing stresses at this point on the inclined plane AB shown in the figure.



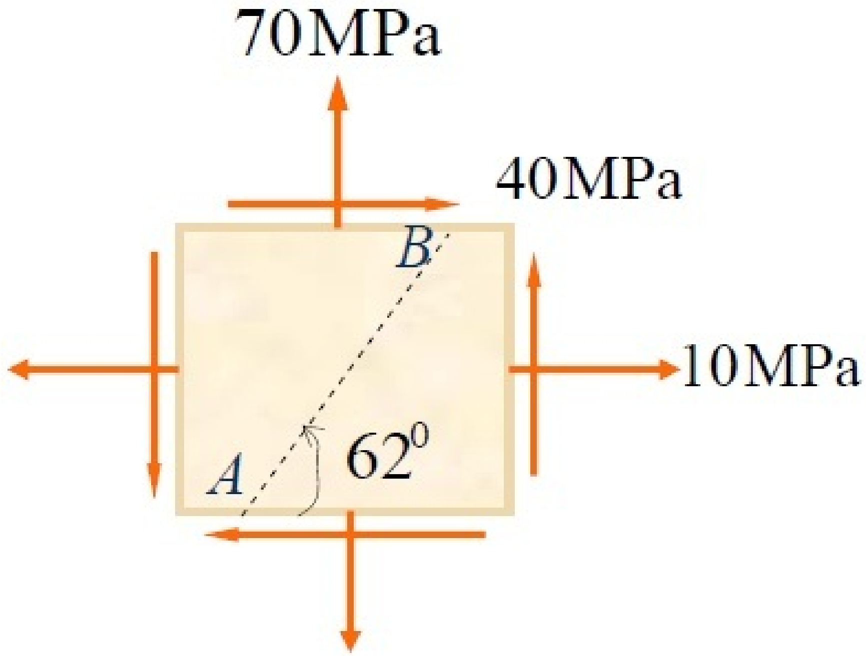


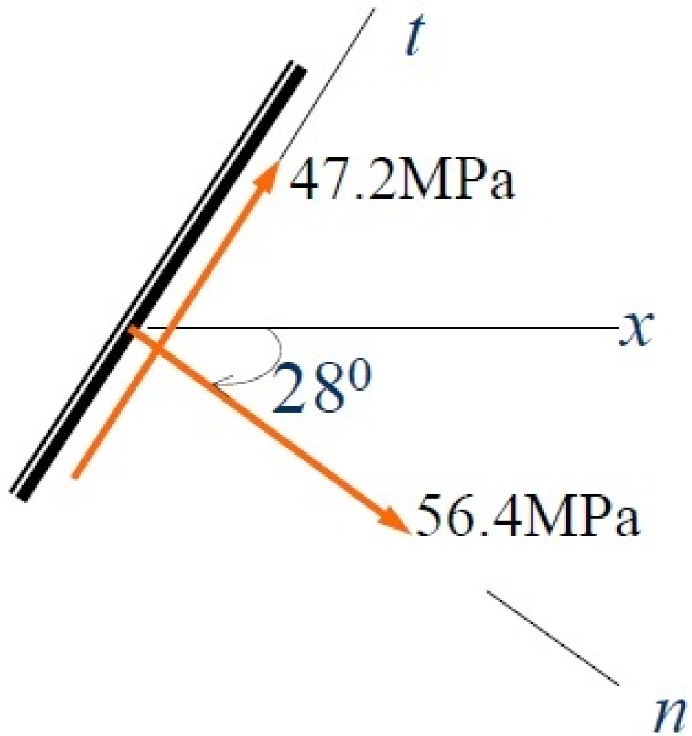
❑Example 1

We have



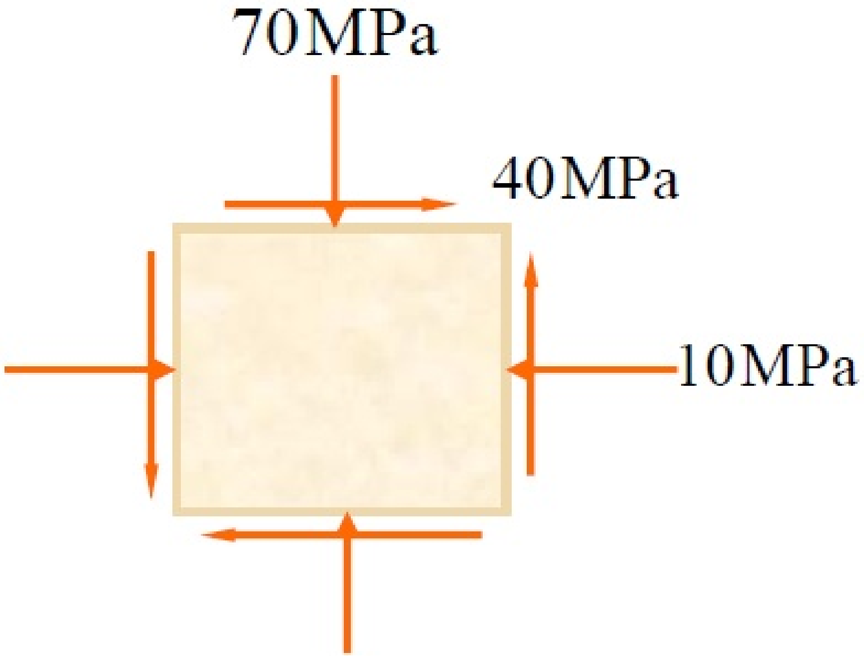
❑Example 1

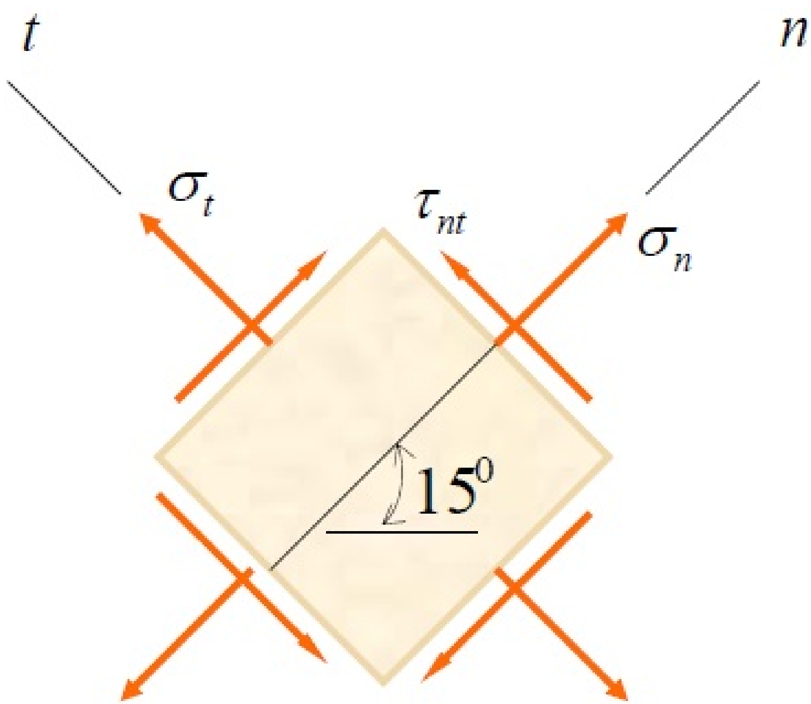




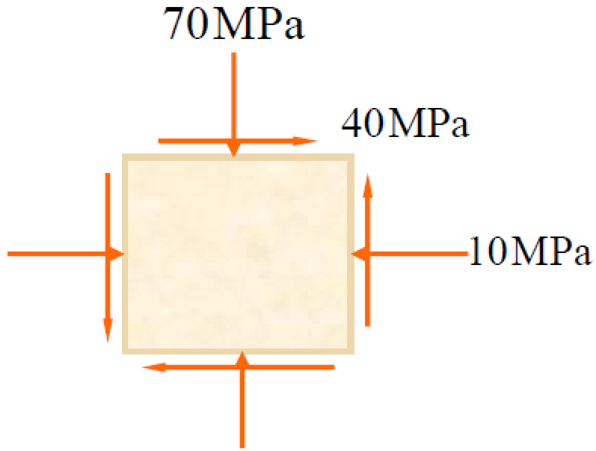
❑Example 2

The stresses shown act at a point on the free surface of a stressed body. Determine ***the normal stresses and the shearing stress*** at this point if they act on the rotated stress element.



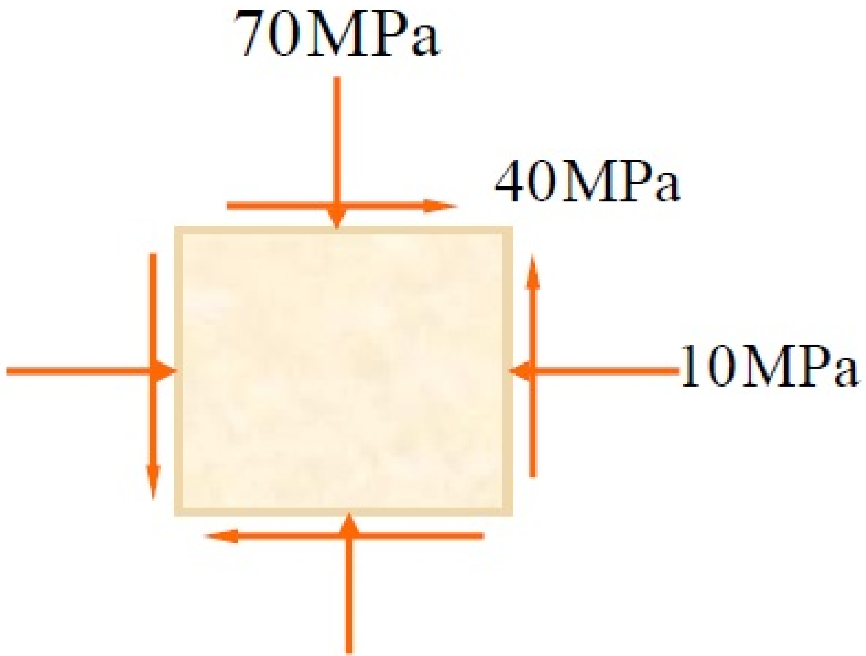


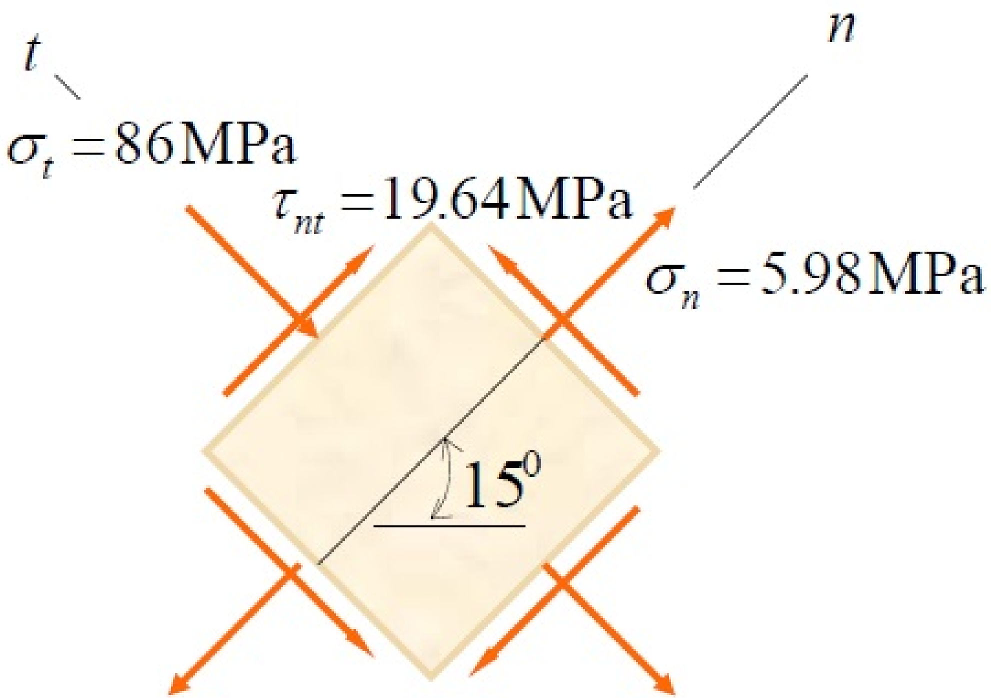
❑Example 2



We have

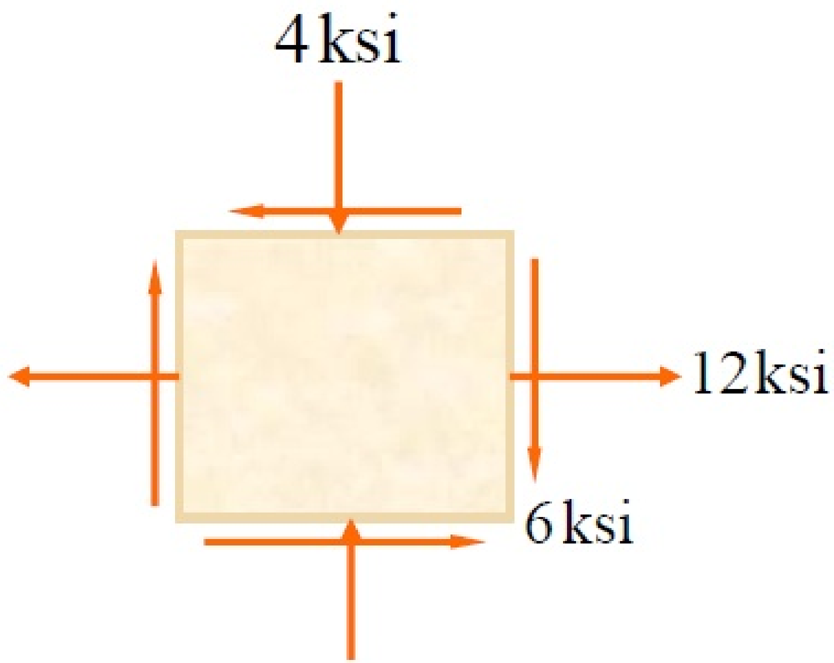
Example 2


70MPa
40MPa
10MPa

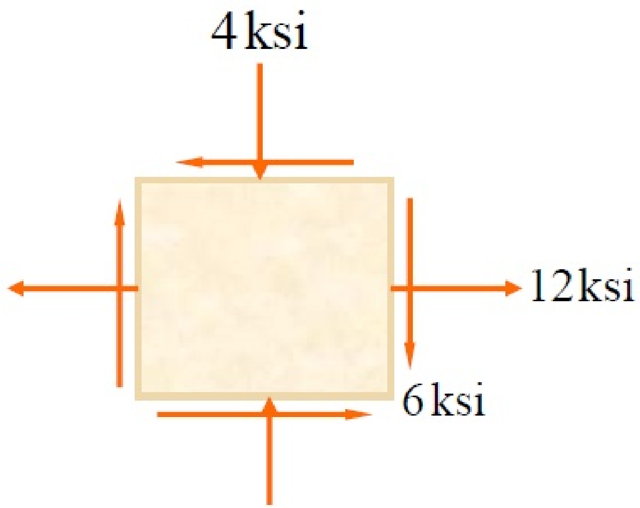

t
n
$\sigma_t = 86\text{MPa}$
$\tau_{nt} = 19.64\text{MPa}$
$\sigma_n = 5.98\text{MPa}$
15^0

❑Example 3

Determine and show on a sketch the principal and maximum shearing stresses.

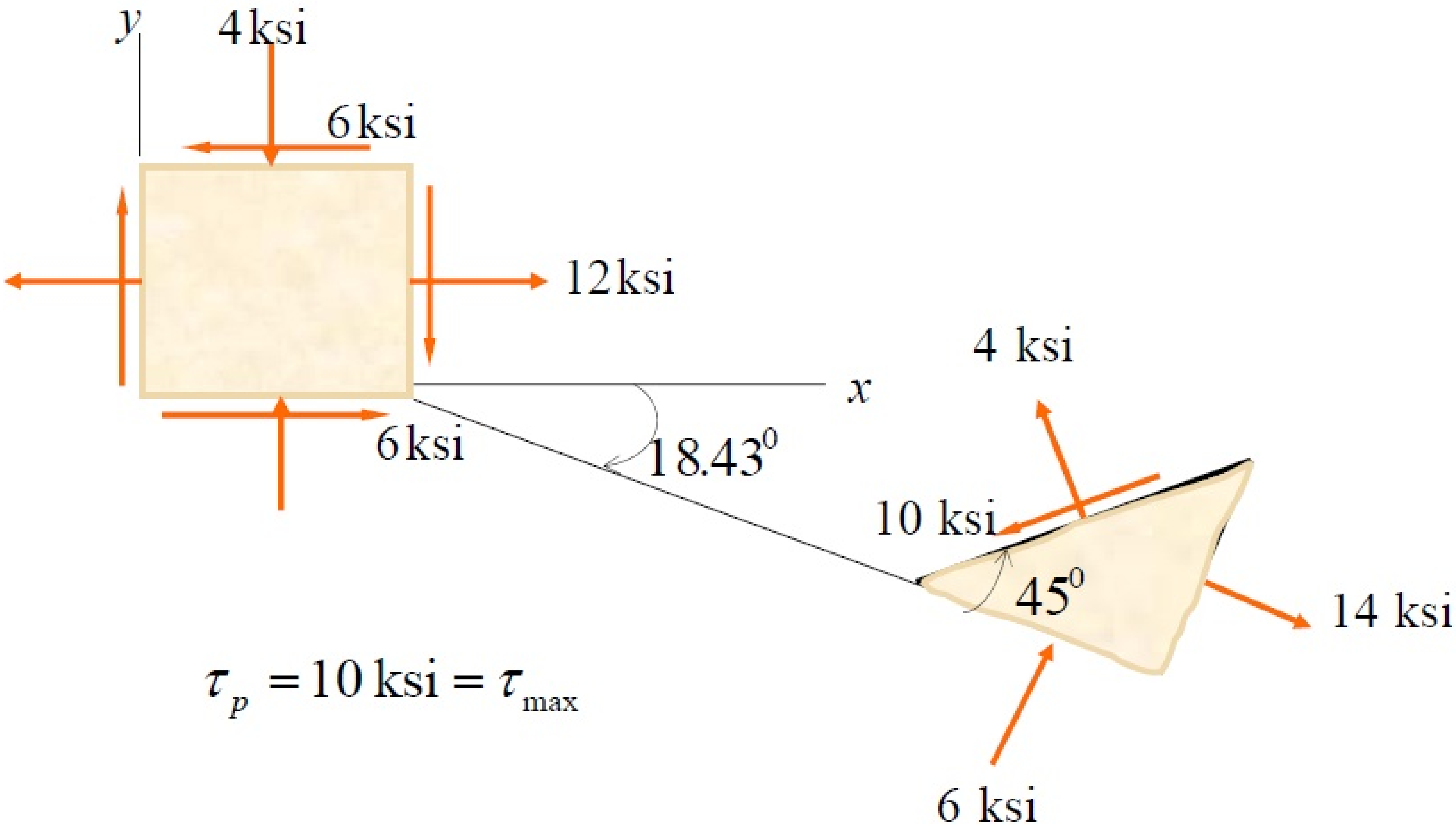


❑Example 3



We have

❑Example 3



$$\tau_p = 10\,\text{ksi} = \tau_{max}$$

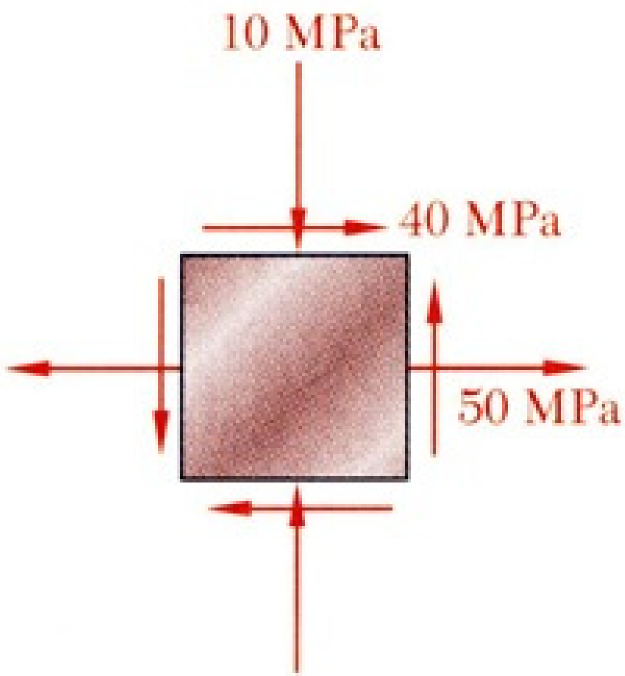
❑Example 4

For the state of plane stress shown, determine:

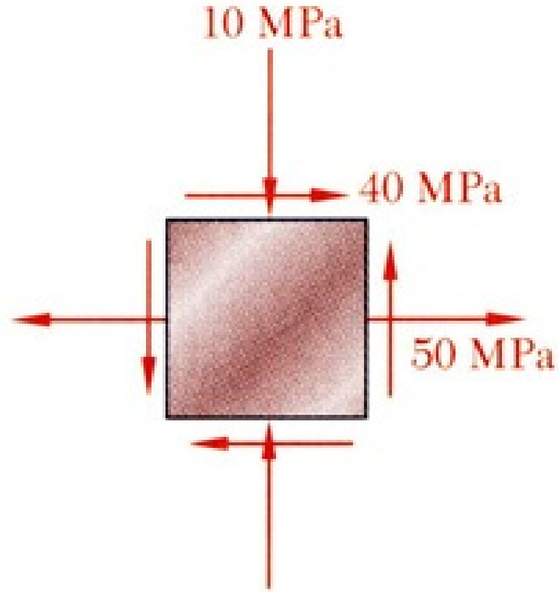
(a) The principal planes

(b) The principal stresses

(c) The maximum shearing stress and the corresponding normal stress.



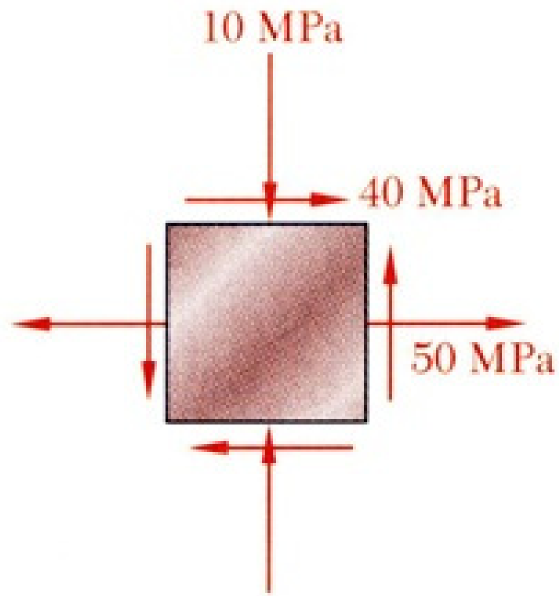
❑Example 4



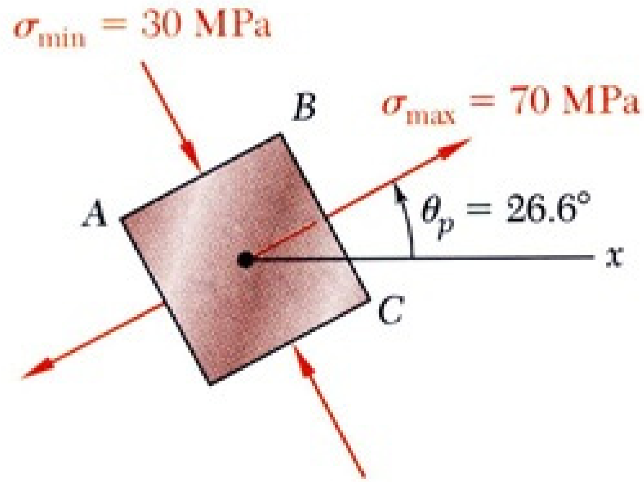
SOLUTION:

- Find the element orientation for the principal stresses from

❑Example 4



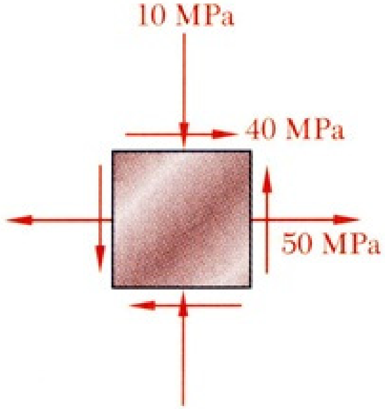
- Determine the principal stresses from



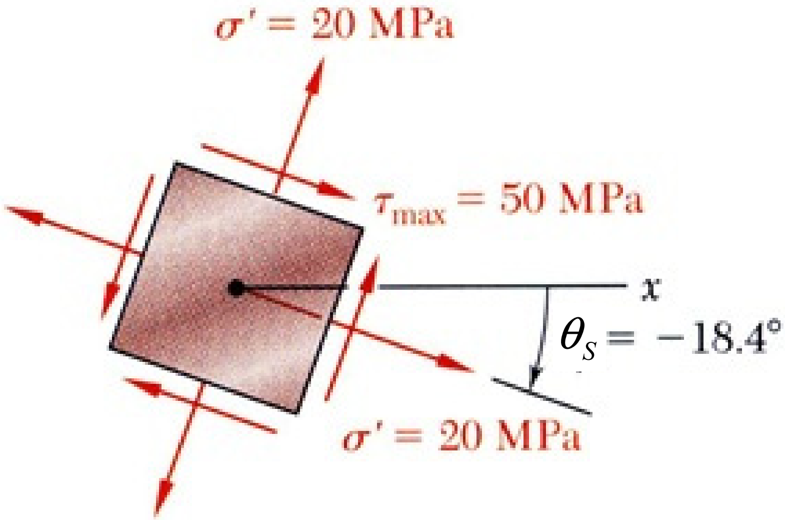
❑Example 4

$\sigma_x = +50\,\text{MPa}$ $\tau_{xy} = +40\,\text{MPa}$

$\sigma_y = -10\,\text{MPa}$



- Calculate the maximum shearing stress with



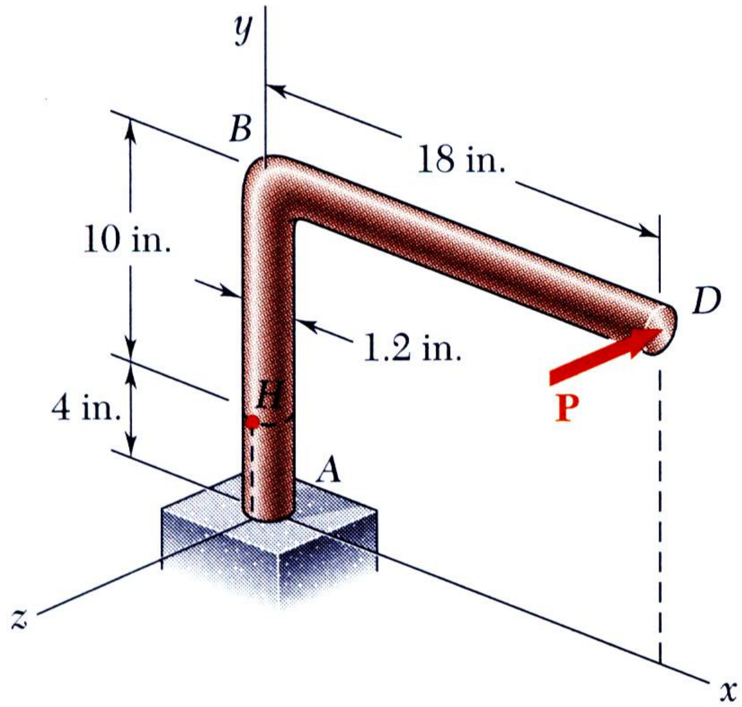
- The corresponding normal stress is

❑Example 5

A single horizontal force *P* of 150 lb magnitude is applied to end D of lever *ABD*. Determine:

(a) The normal and shearing stresses on an element at point *H* having sides parallel to the *x* and *y* axes

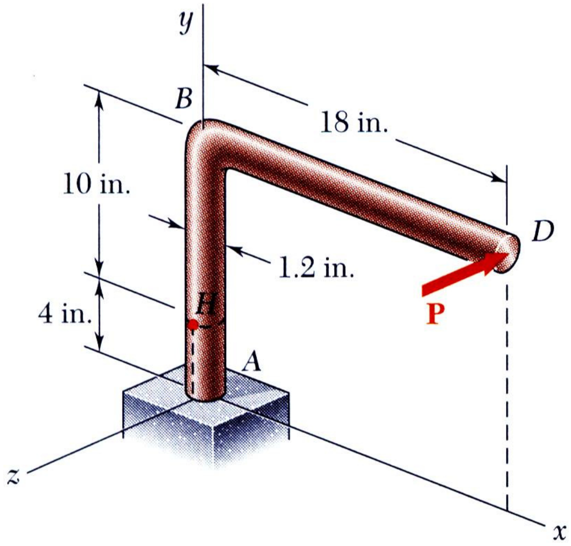
(b) The principal planes and principal stresses at the point *H*.

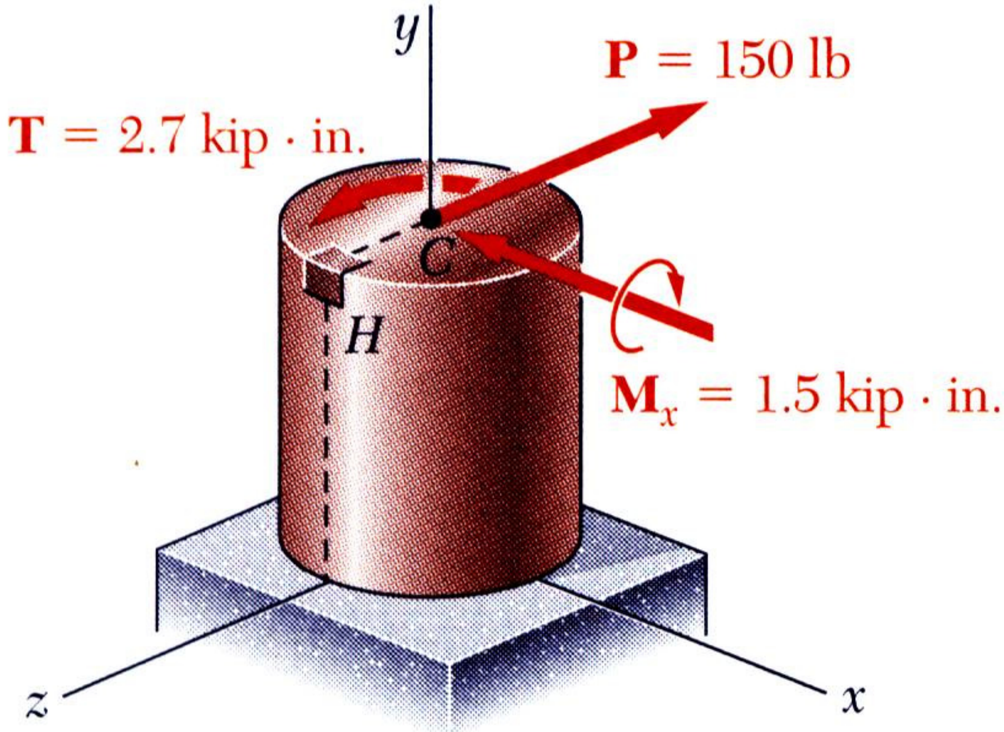


❑Example 5

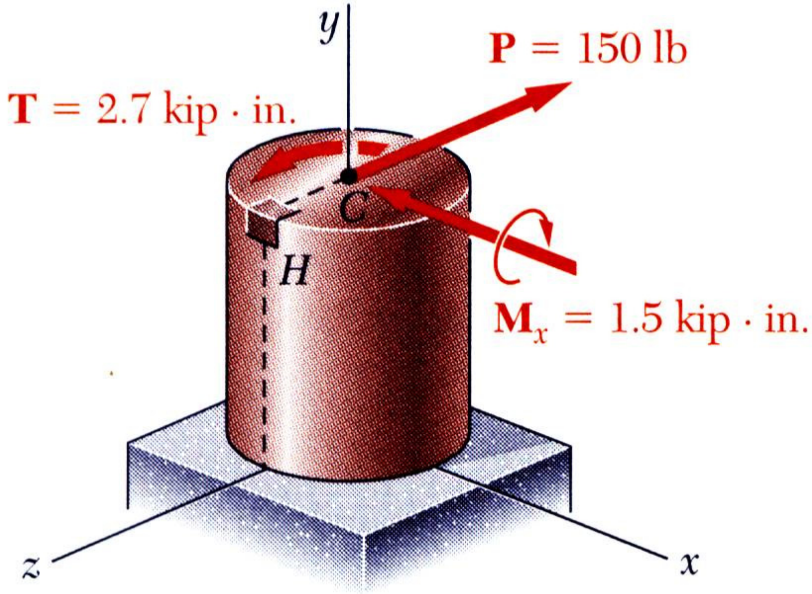
SOLUTION:

- Determine an equivalent force-couple system at the center of the transverse section passing through *H*.

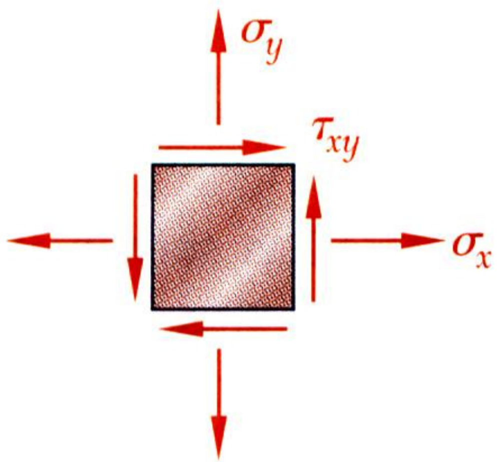




❑Example 5

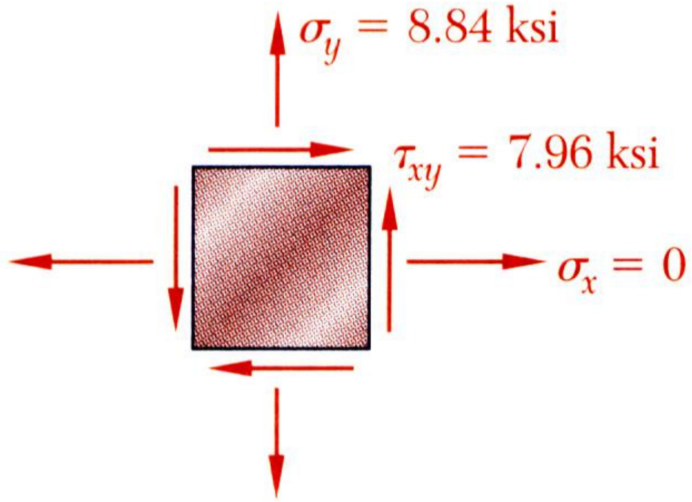


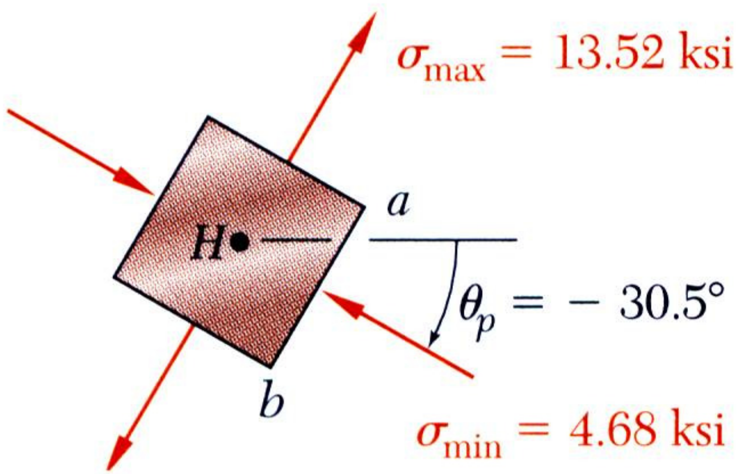
- Evaluate the normal and shearing stresses at *H*.



❑Example 5

- Determine the principal planes and calculate the principal stresses.



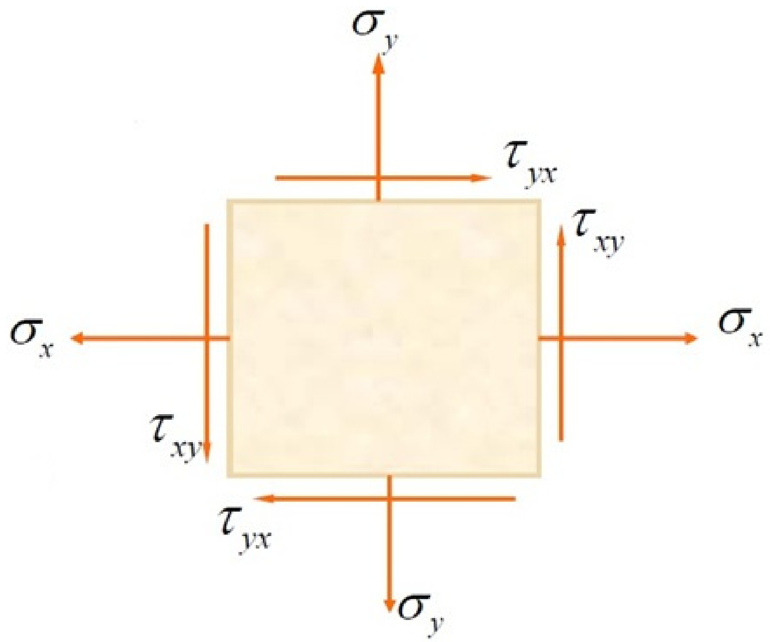


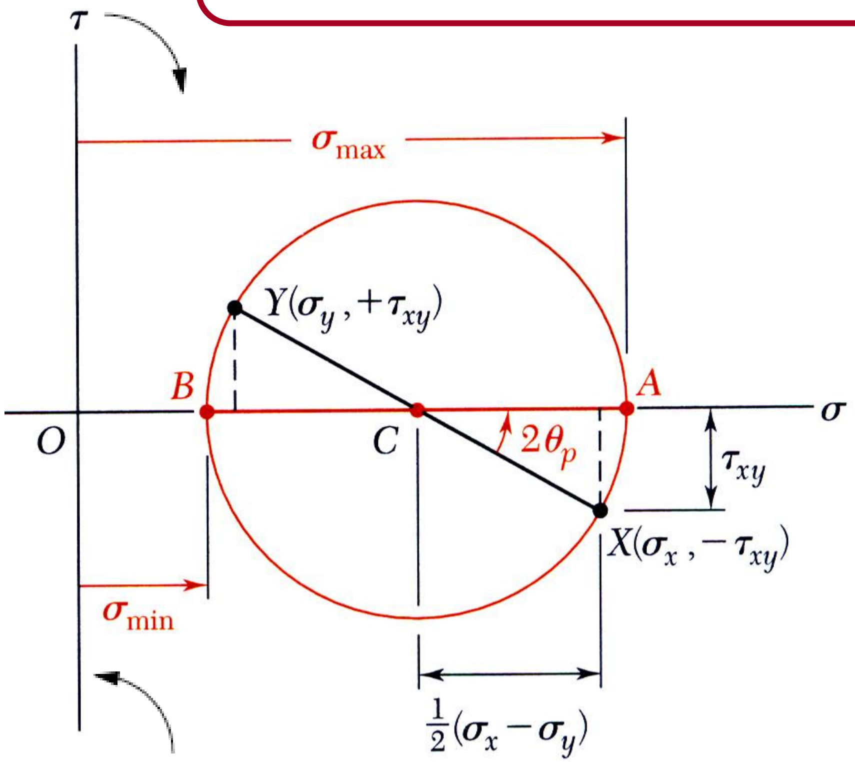
Transformations of Stress and Strain

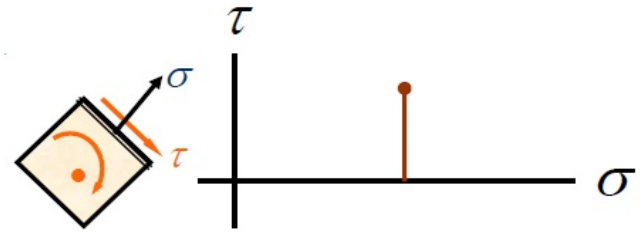
❑ Mohr's Circle for Plane Stress

- For a known state of plane stress $\sigma_x, \sigma_y, \tau_{xy}$ plot the points X and Y and construct the circle centered at C.

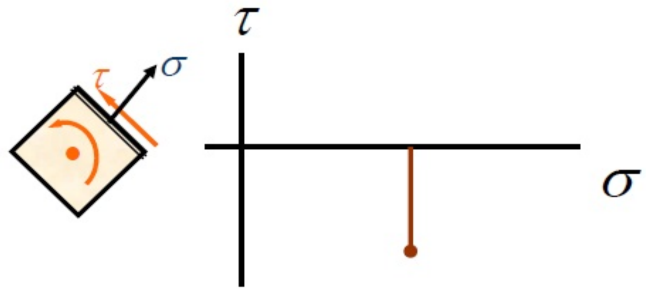
$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} \qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$







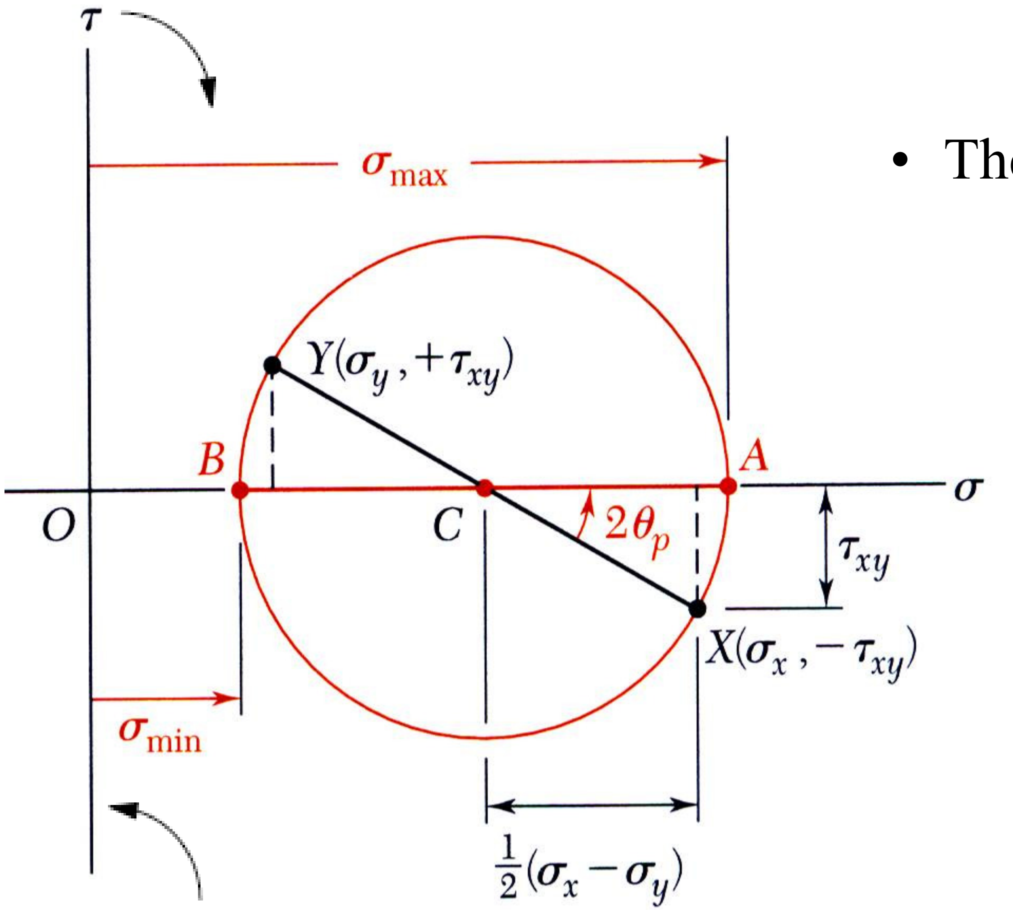
(a) Clockwise → **Above**



(b) Counterclockwise → **Below**

Transformations of Stress and Strain

❑ Mohr's Circle for Plane Stress

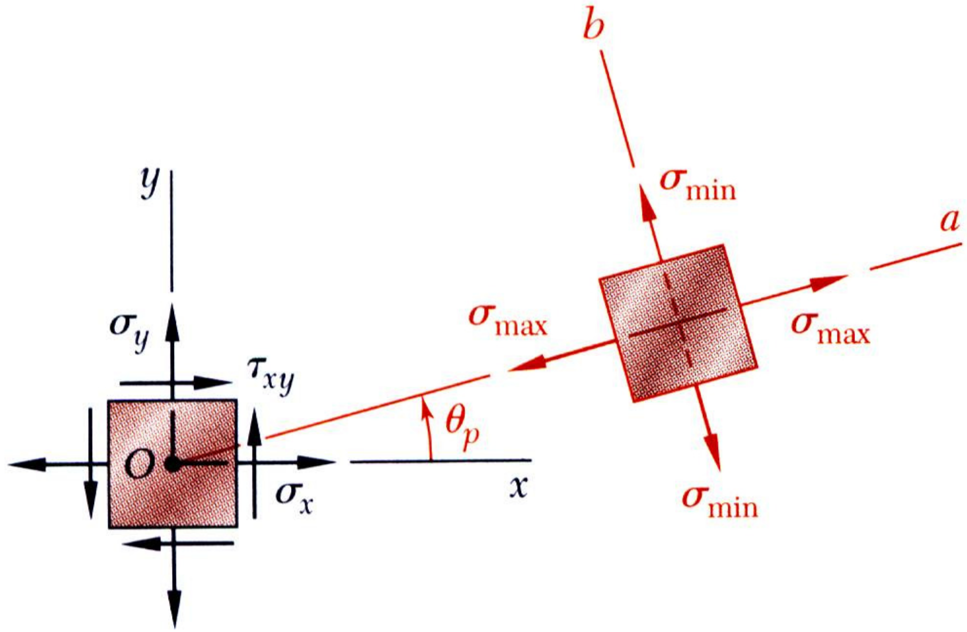


- The principal stresses are obtained at *A* and *B*.

$$\sigma_{\max,\min} = \sigma_{ave} \pm R$$

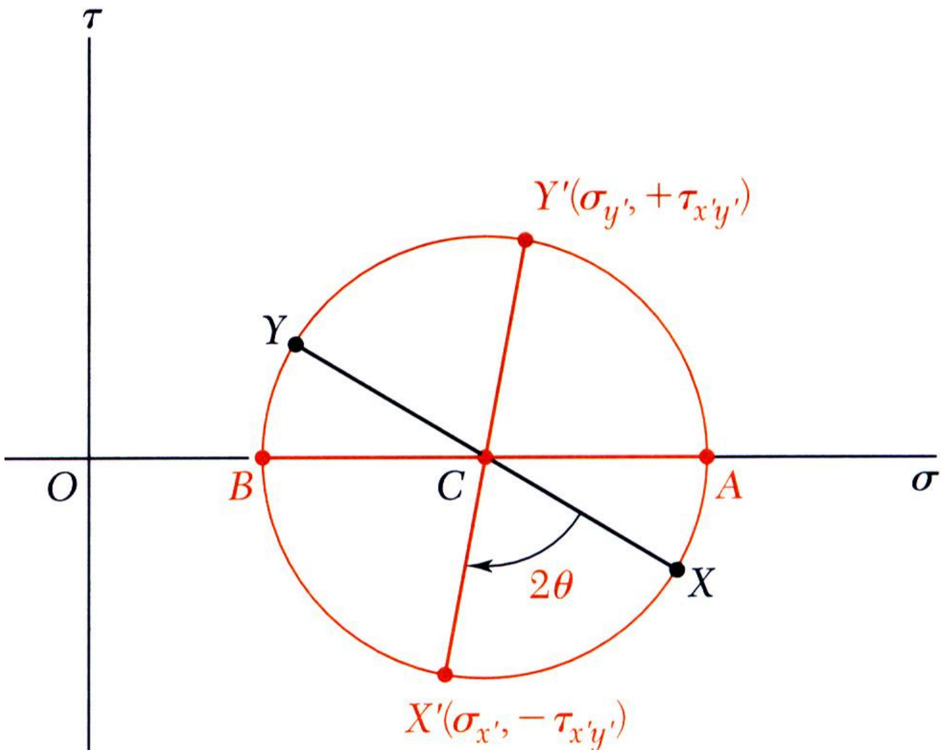
$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

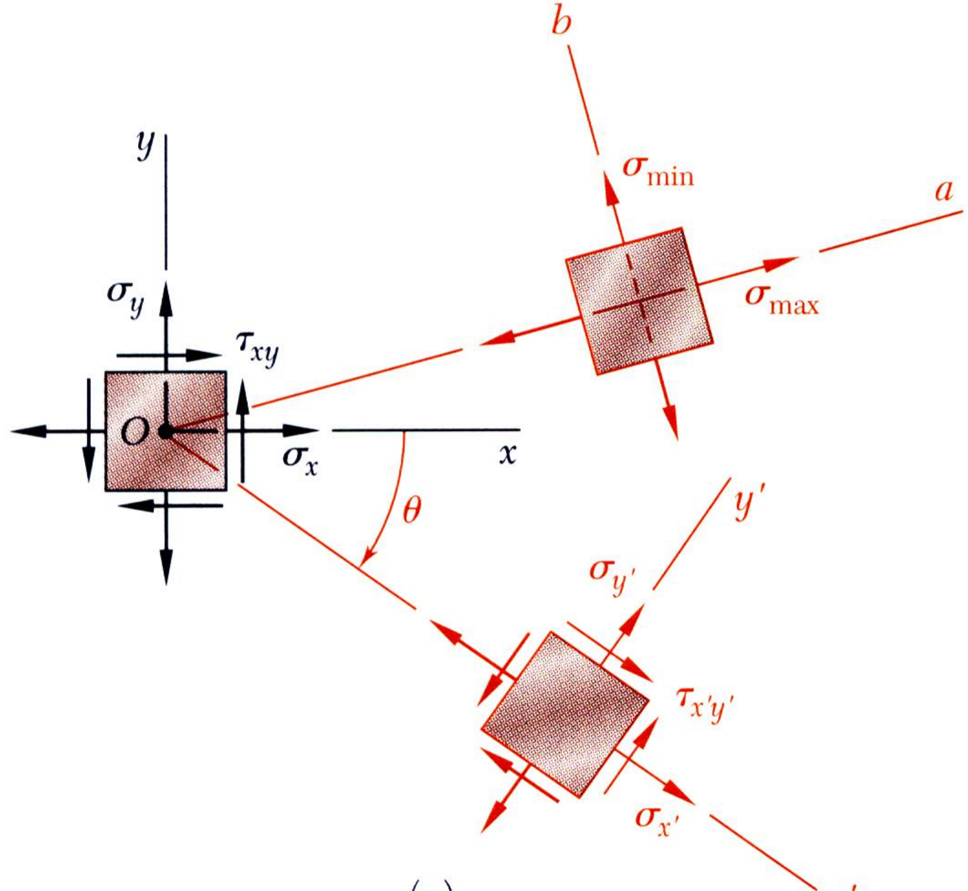
The direction of rotation of *Ox* to *Oa* is the same as *CX* to *CA*.



Transformations of Stress and Strain

❑ Mohr's Circle for Plane Stress





(a)

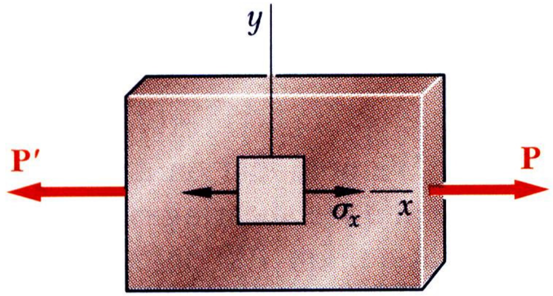
- Mohr's circle ***uniquely*** defines, the state of stress at ***other axes orientations***.
- For the state of stress ***at an angle θ*** with respect to ***the xy axes***, construct a new diameter *X'Y'* at an ***angle 2θ*** with ***respect to XY***.

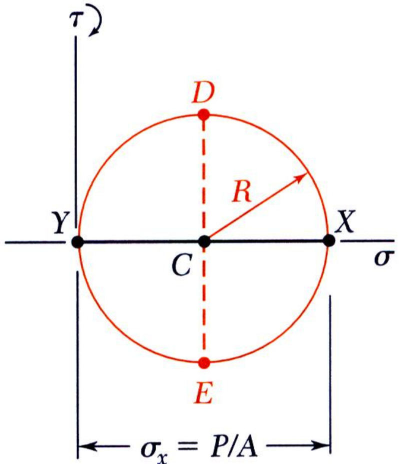
Transformations of Stress and Strain

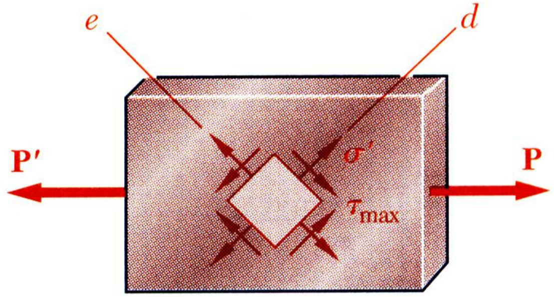
❑ Mohr's Circle for Plane Stress

- Mohr's circle for centric axial loading:

$$\begin{pmatrix}\sigma_x = \dfrac{P}{A} \\ \tau_{xy} = 0\end{pmatrix}, \begin{pmatrix}\sigma_y = 0 \\ \tau_{yx} = 0\end{pmatrix}$$



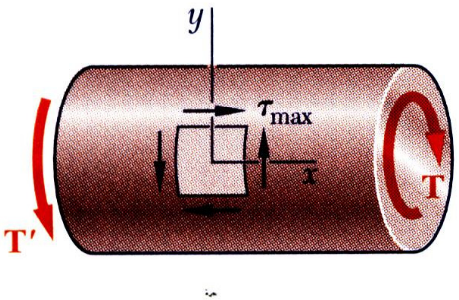


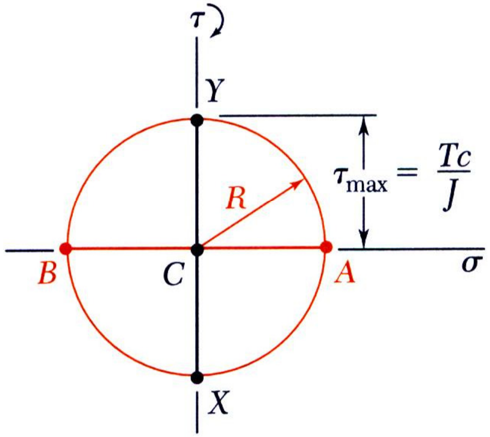


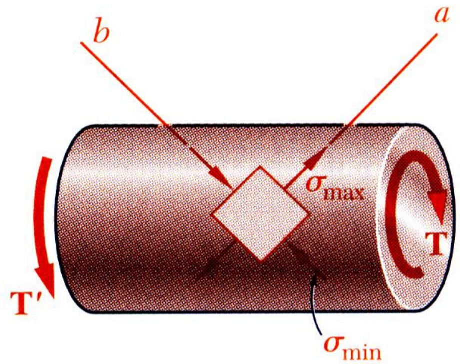
$$\tau_{Max} = \frac{\sigma_{Max}}{2} = \frac{P}{2A}$$

- Mohr's circle for torsional loading:

$$\begin{pmatrix}\sigma_x = 0 \\ \tau_{xy} = -\dfrac{T_{\max}c}{J}\end{pmatrix}, \begin{pmatrix}\sigma_y = 0 \\ \tau_{yx} = \dfrac{T_{\max}c}{J}\end{pmatrix}$$

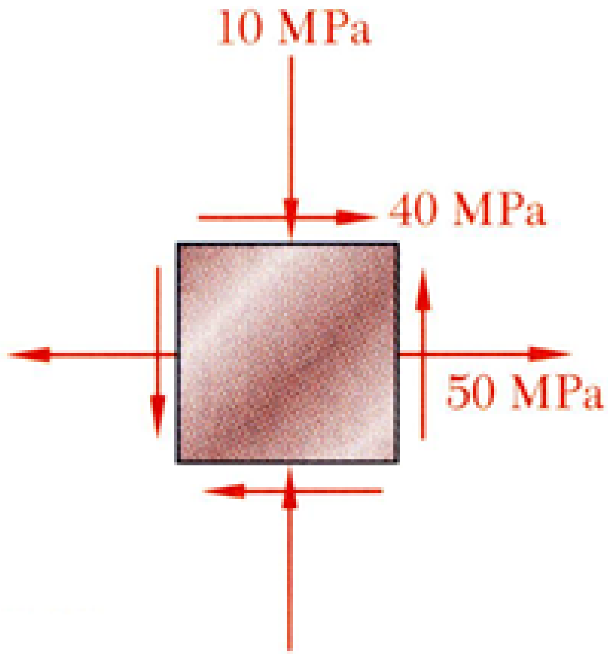






$$\sigma_{Max} = \tau_{Max} = \frac{T_{Max}c}{J}$$

Example 6



For the state of plane stress shown,

(a) Construct Mohr's circle, determine

(b) The principal planes,

(c) The principal stresses,

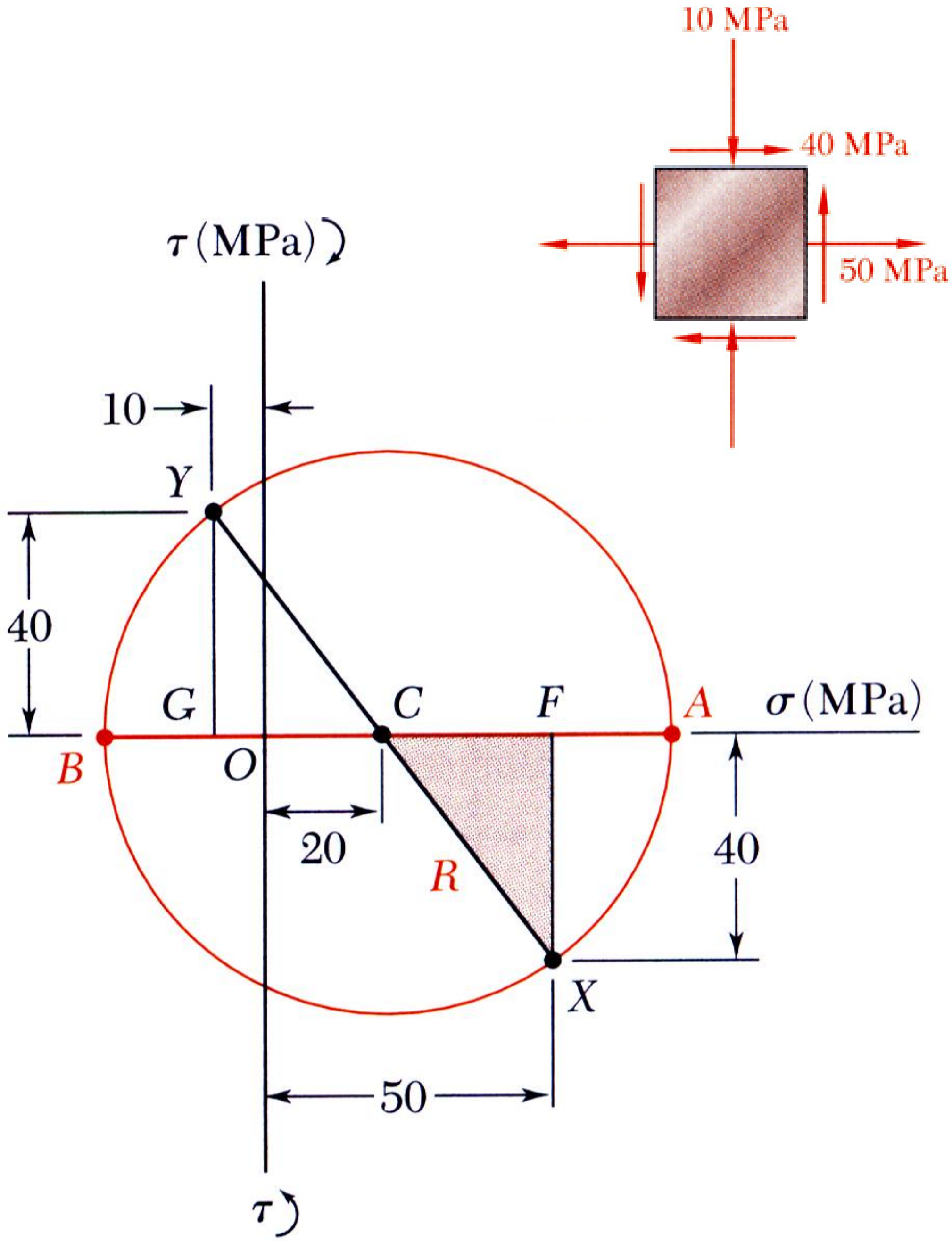
(d) The maximum shearing stress and the corresponding normal stress.

❑ Example 6

SOLUTION:

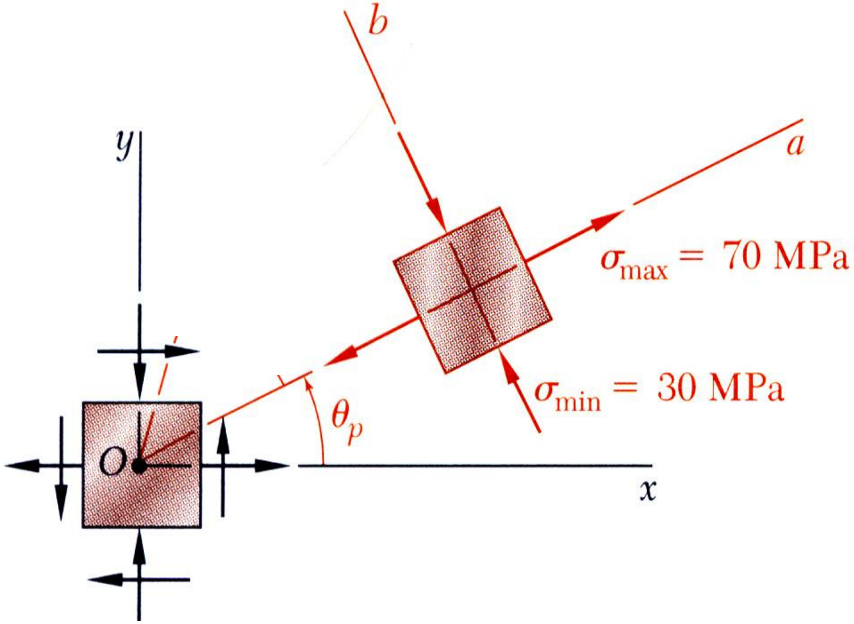
- Construction of Mohr's circle

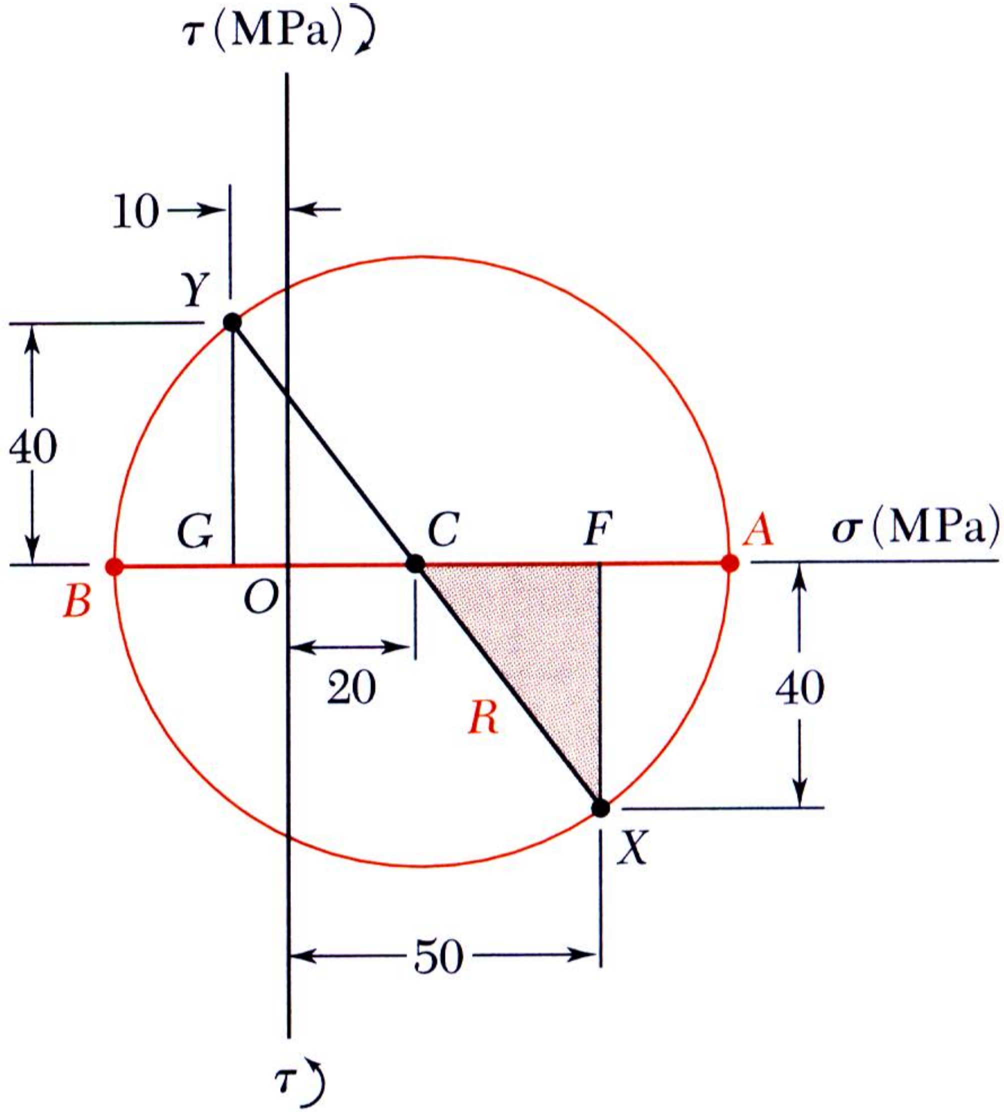
We have



Example 6

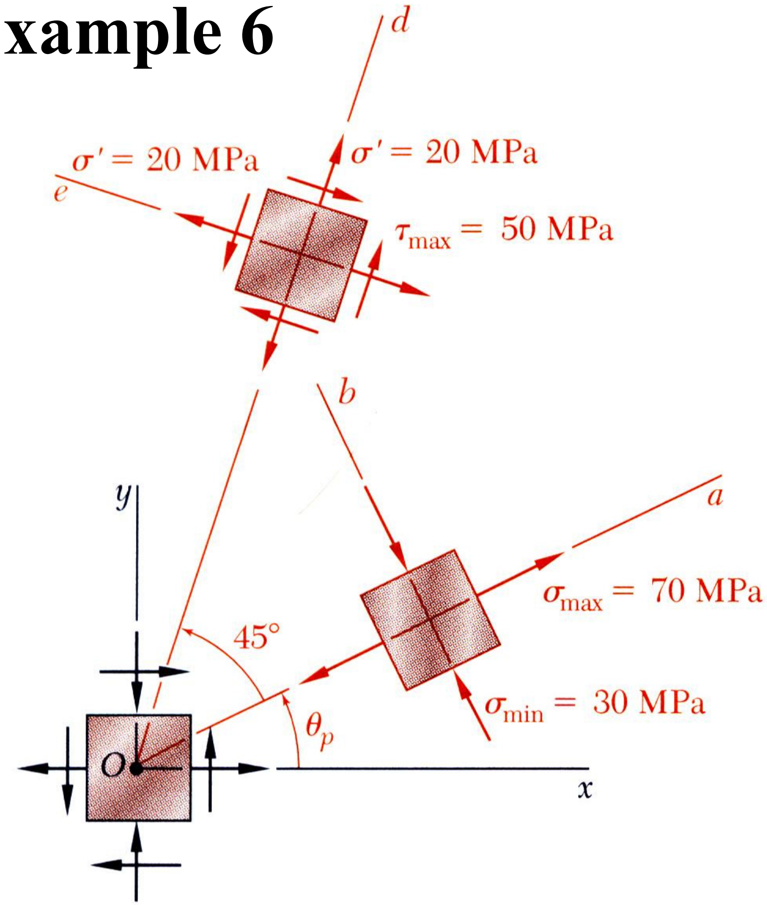
- Principal planes and stresses

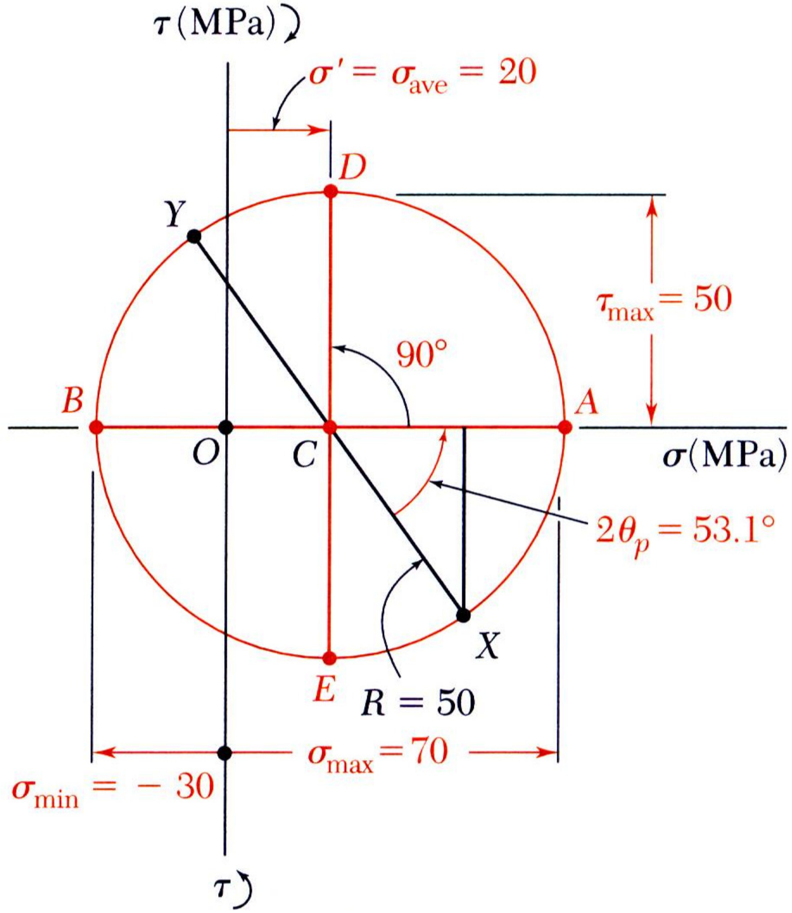




Transformations of Stress and Strain

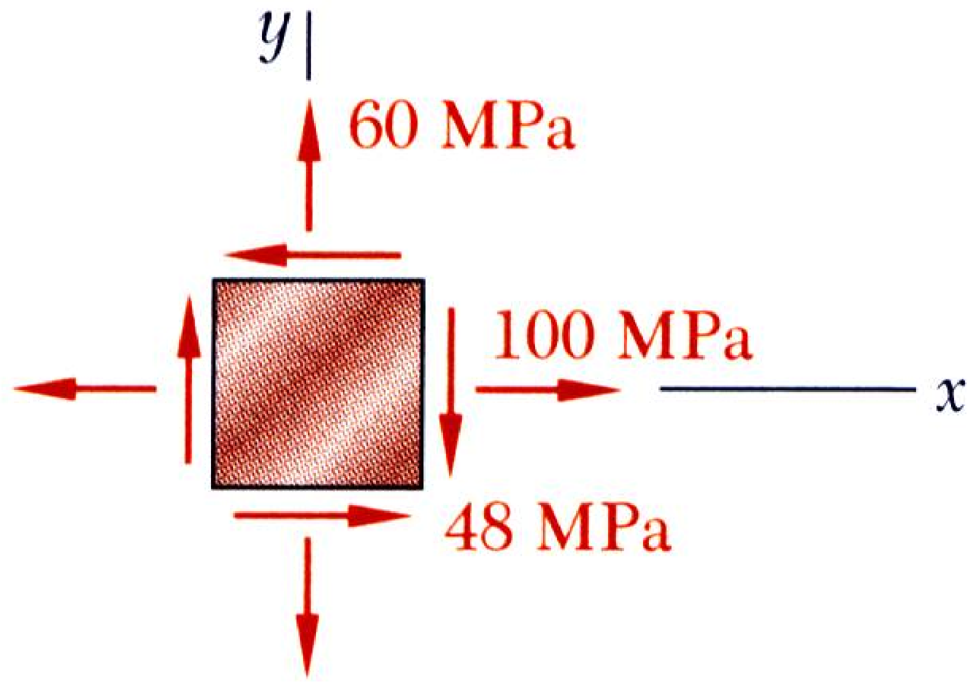
❑ Example 6





•Maximum shear stress

❑ Example 7



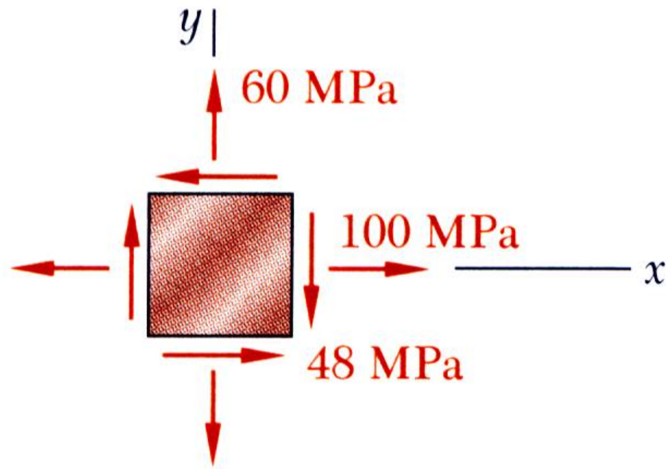
For the state of stress shown, determine

(a) The principal planes and the principal stresses.

(b) The stress components exerted on the element obtained by rotating the given element counterclockwise through 30 degrees.

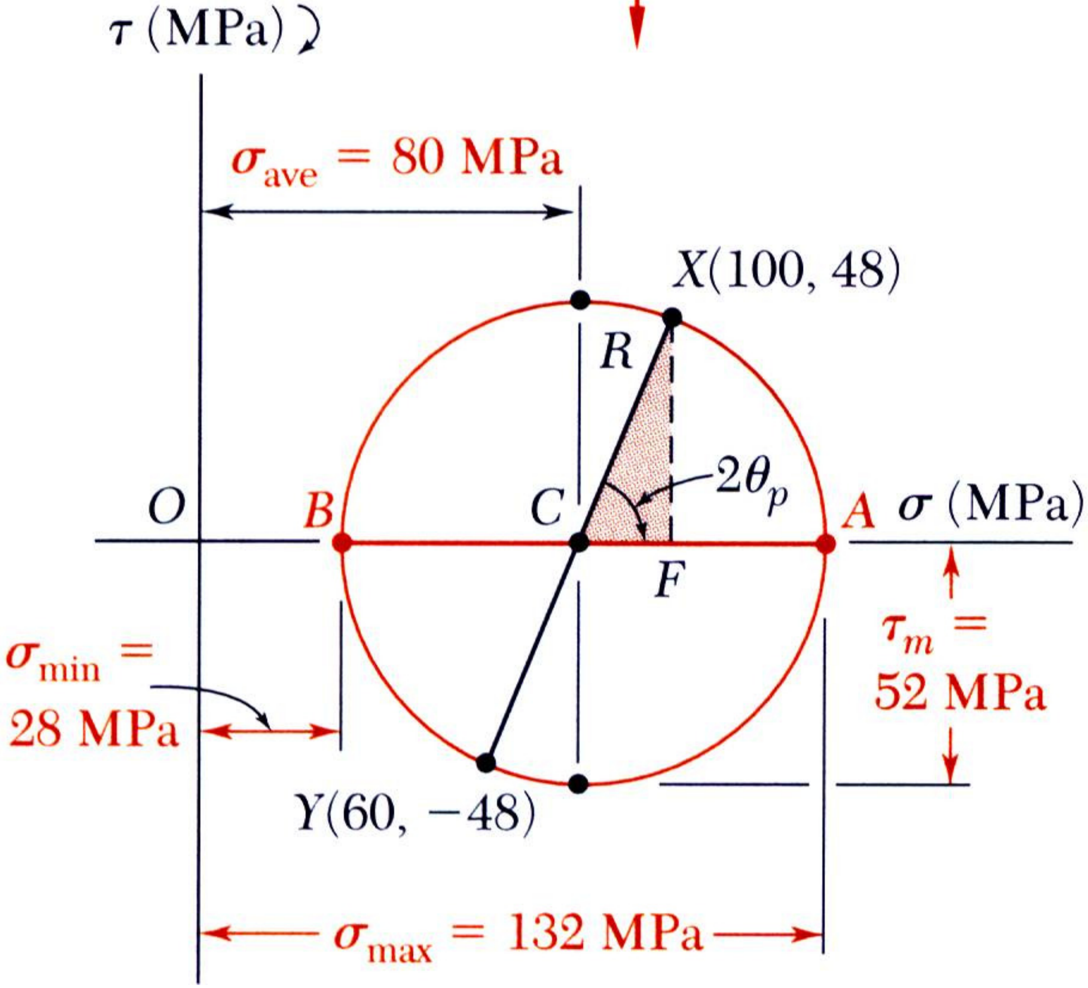
Transformations of Stress and Strain

❑ Example 7



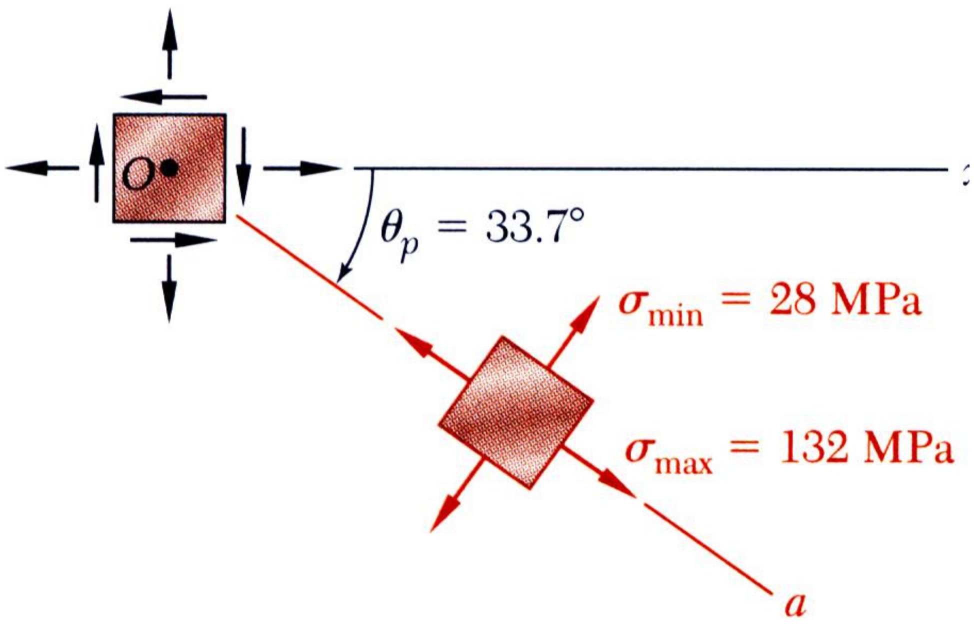
SOLUTION:

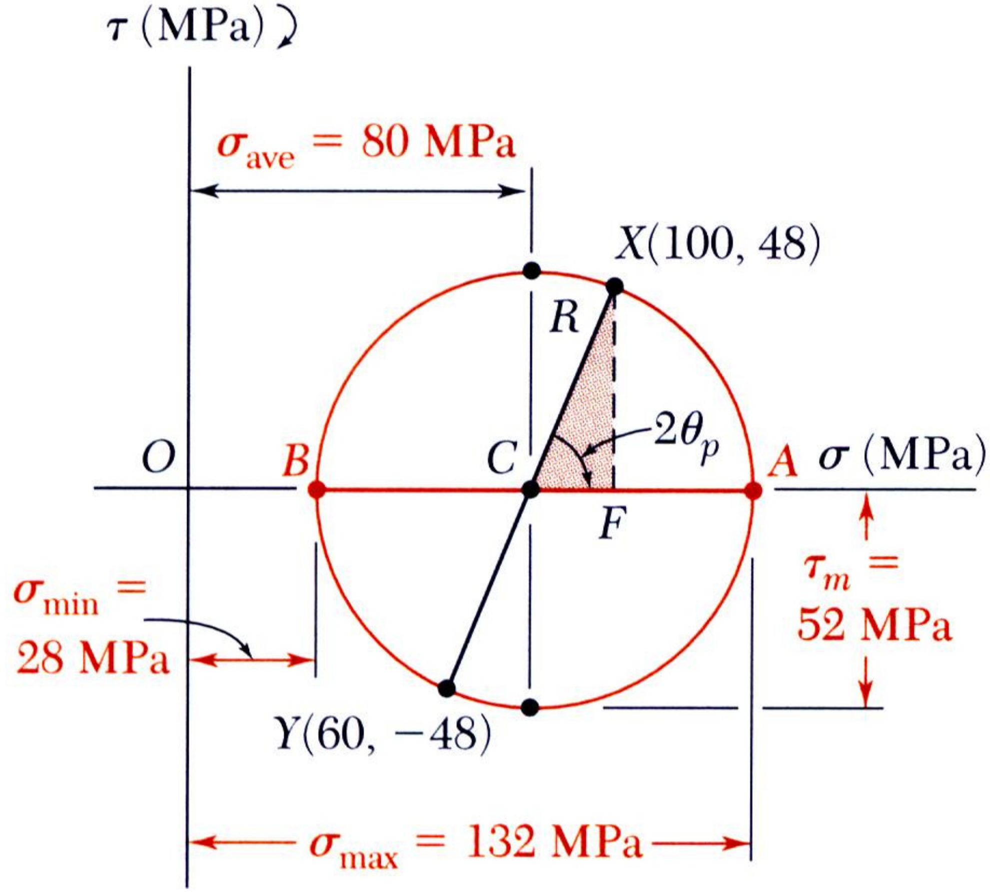
- Construction of Mohr's circle



We have

❑ Example 7

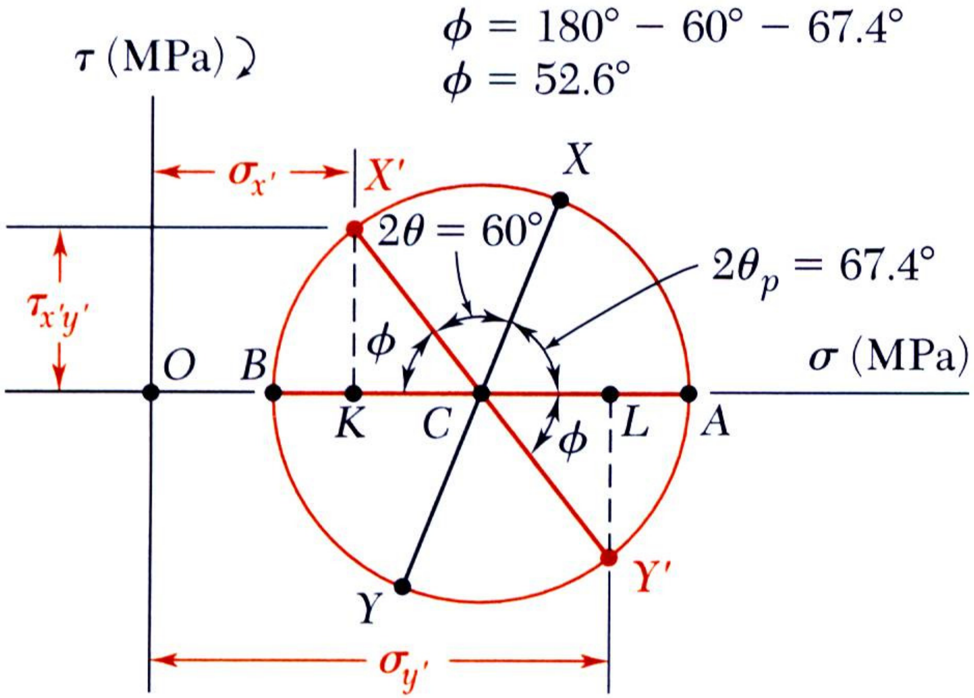


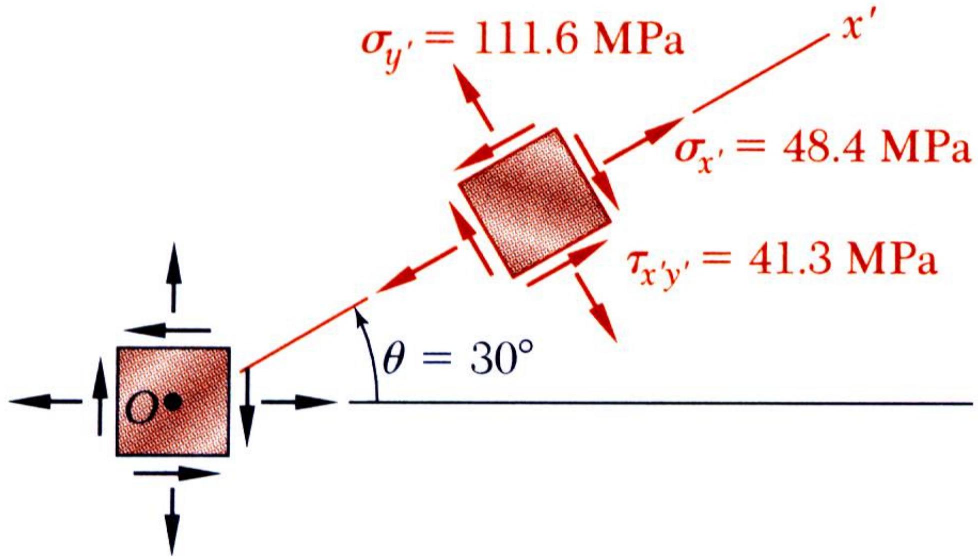


- Principal planes and stresses

Transformations of Stress and Strain

❑ Example 7

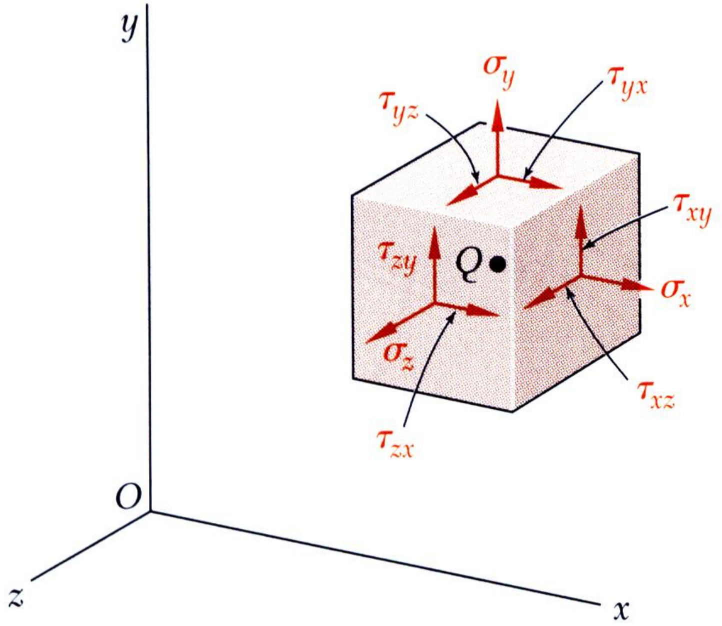




•Stress components after rotation by 30^o Points *X'* and *Y'* on Mohr's circle that correspond to stress components on the rotated element are obtained by rotating *XY* counterclockwise through $2\theta = 60°$

Transformations of Stress and Strain

General State of Stress

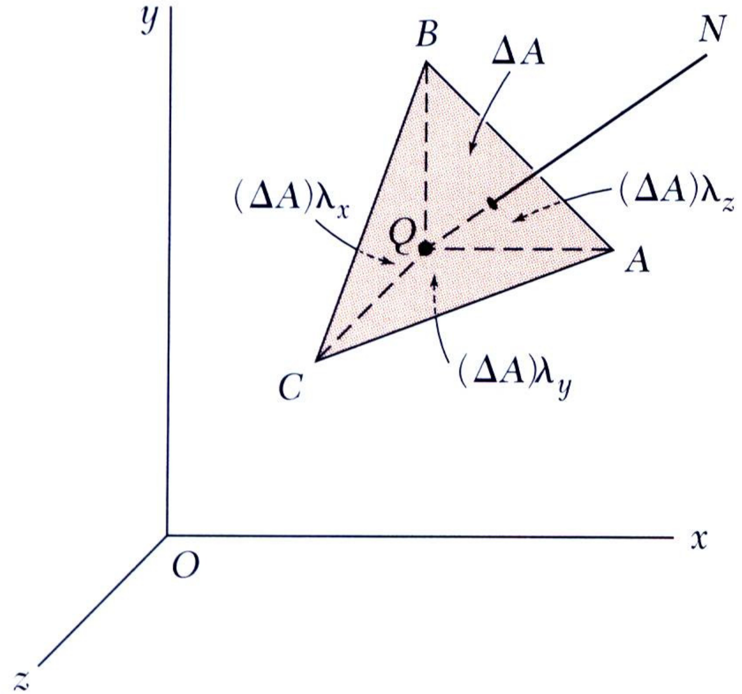


•Consider the general 3D state of stress at a point and the transformation of stress from element rotation

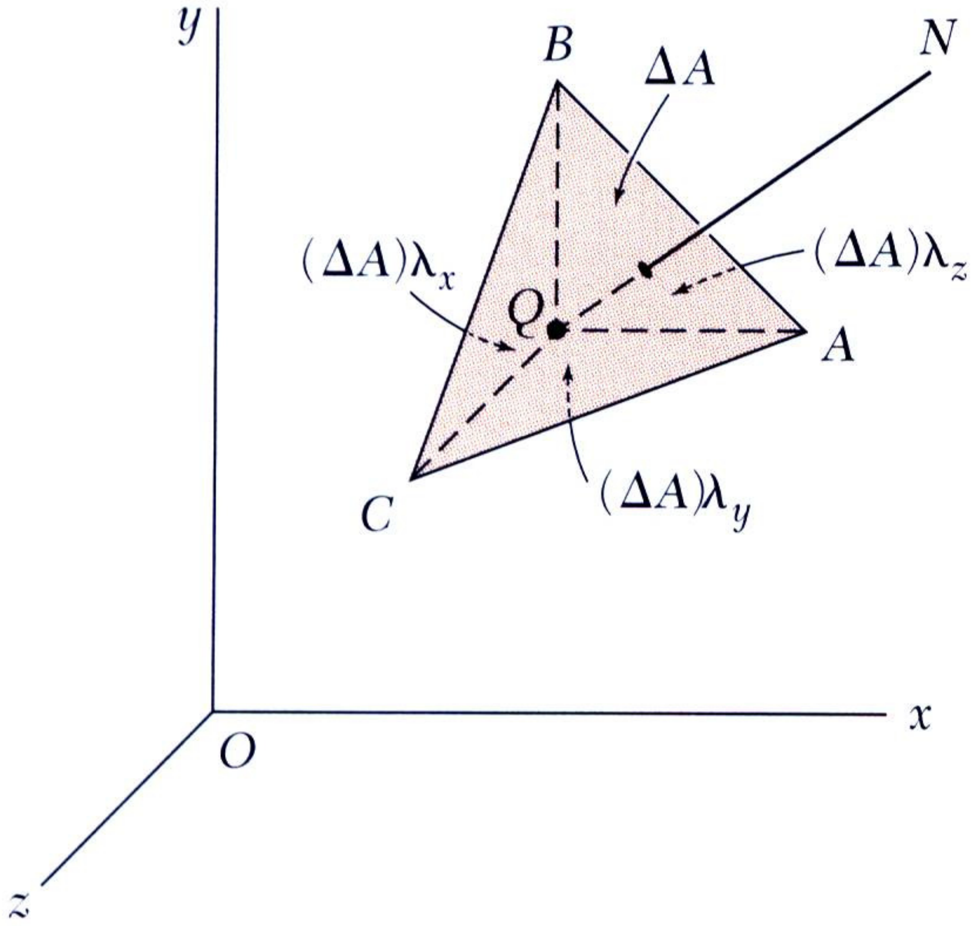
•State of stress at Q defined by: $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$

•Consider tetrahedron with face perpendicular to the line QN with ***direction cosines***:

$$\lambda_x, \lambda_y, \lambda_z$$



General State of Stress

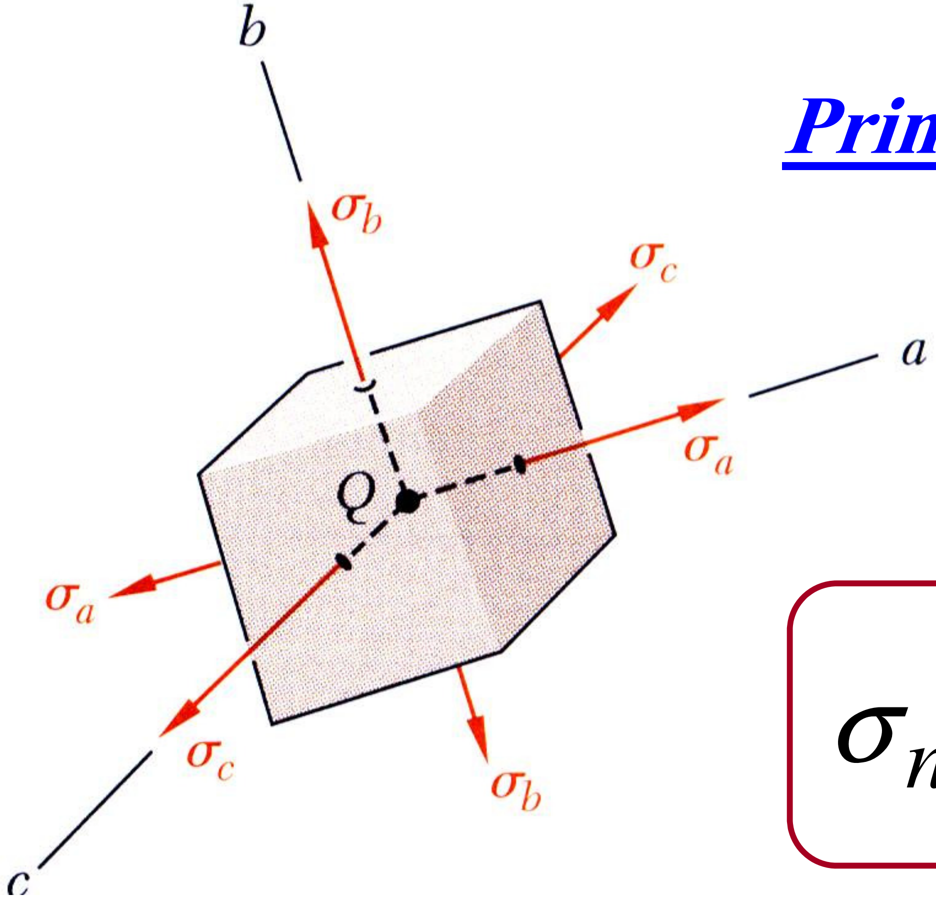


- The requirement leads to

$$\sum F_n = 0$$

$$\sigma_n = \sigma_x \lambda_x^2 + \sigma_y \lambda_y^2 + \sigma_z \lambda_z^2 + 2\tau_{xy}\lambda_x\lambda_y + 2\tau_{yz}\lambda_y\lambda_z + 2\tau_{zx}\lambda_z\lambda_x$$

General State of Stress



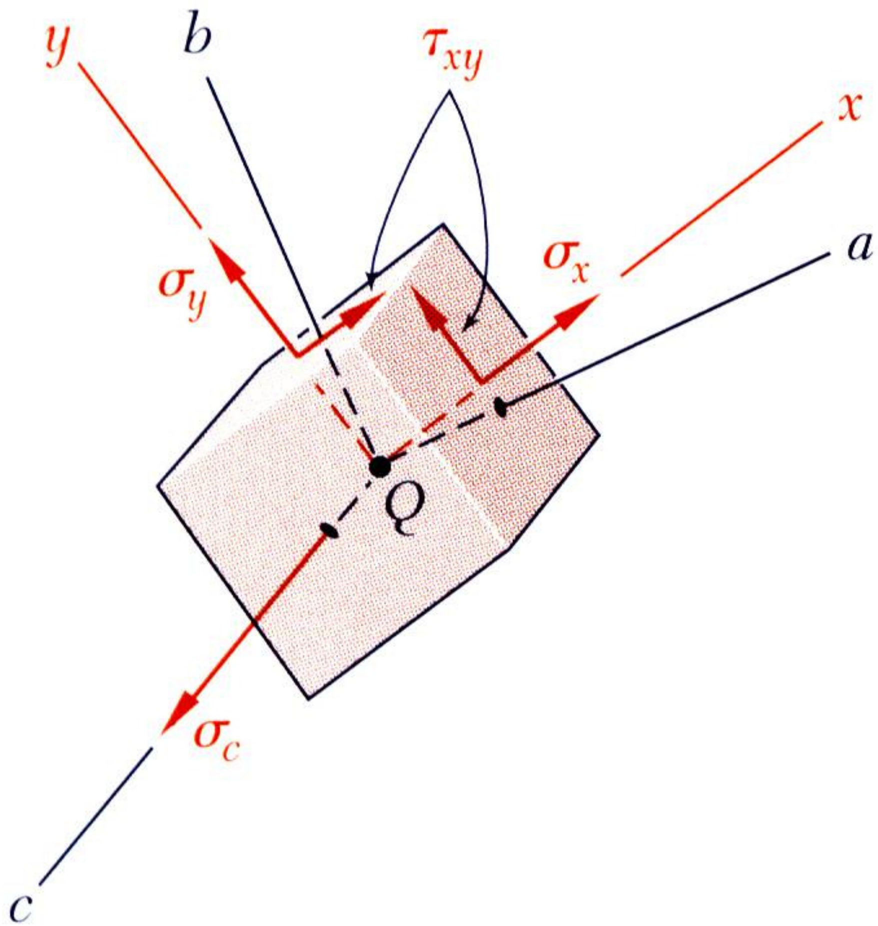
Principal axes and principal planes

•Form of equation guarantees that an element orientation can be found such that

$$\sigma_n = \sigma_a \lambda_a^2 + \sigma_b \lambda_b^2 + \sigma_c \lambda_c^2$$

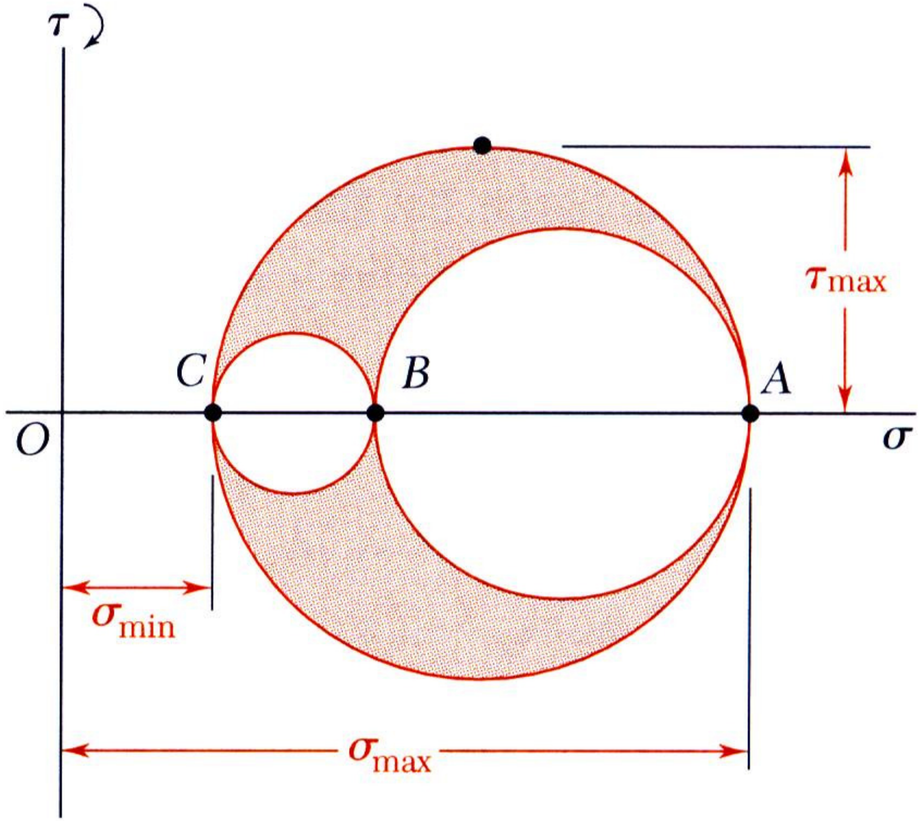
These are the principal axes and principal planes and the normal stresses are the principal stresses.

❑ Application of Mohr's Circle to the Three- Dimensional Analysis of Stress



•Transformation of stress for an element rotated around a principal axis may be represented by Mohr's circle.

Application of Mohr's Circle to the Three- Dimensional Analysis of Stress



- Points *A*, *B*, and *C* represent the ***principal stresses on the principal planes*** (shearing stress is zero)

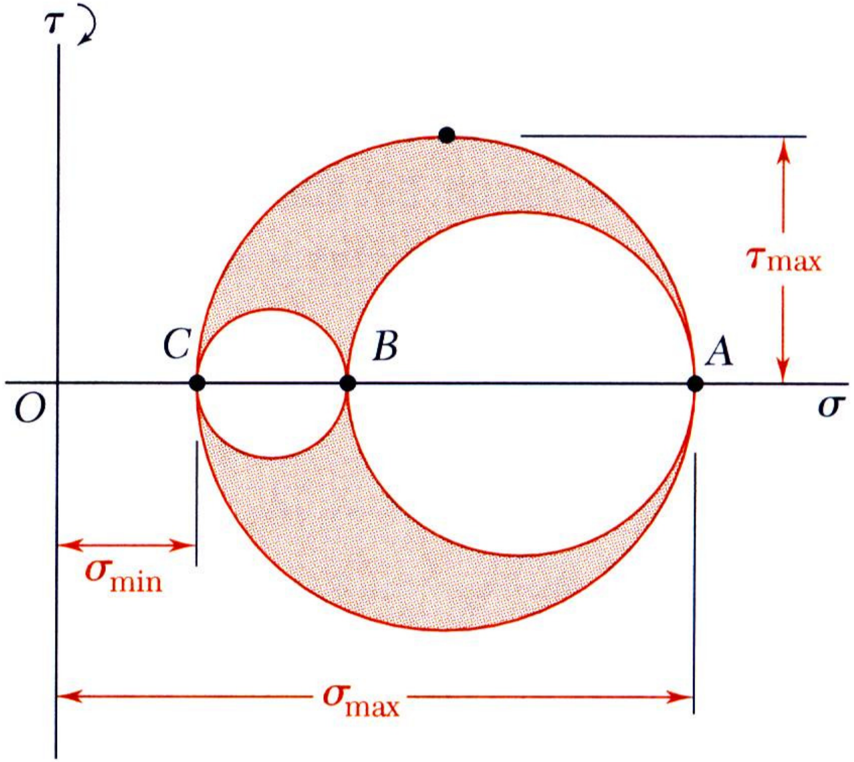
- The three circles represent the normal and shearing stresses for rotation around each principal axis.

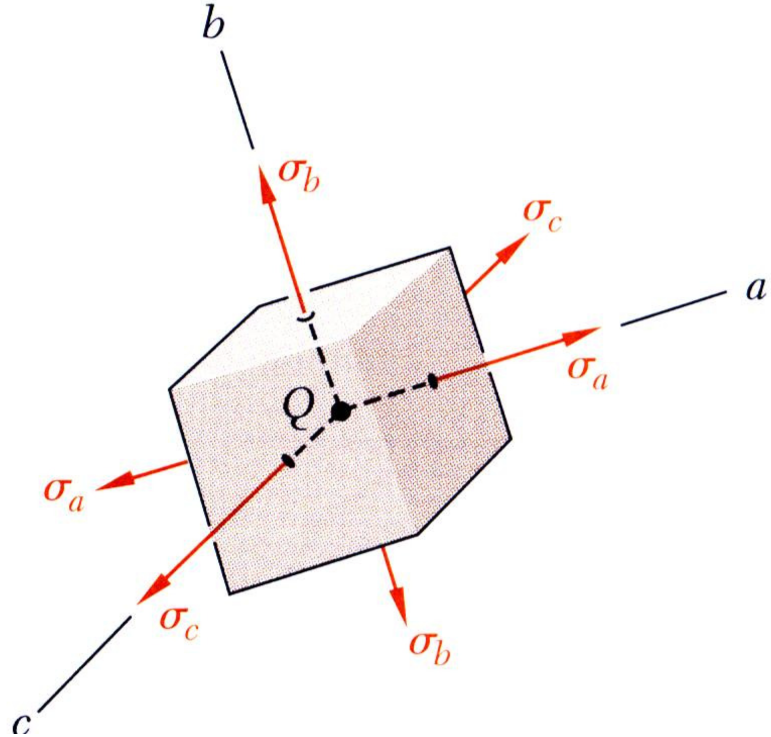
- Radius of the largest circle yields the maximum shearing stress.

$$\tau_{max} = \frac{1}{2}\left|\sigma_{max} - \sigma_{min}\right|$$

Transformations of Stress and Strain

❑ Application of Mohr's Circle to the Three- Dimensional Analysis of Stress

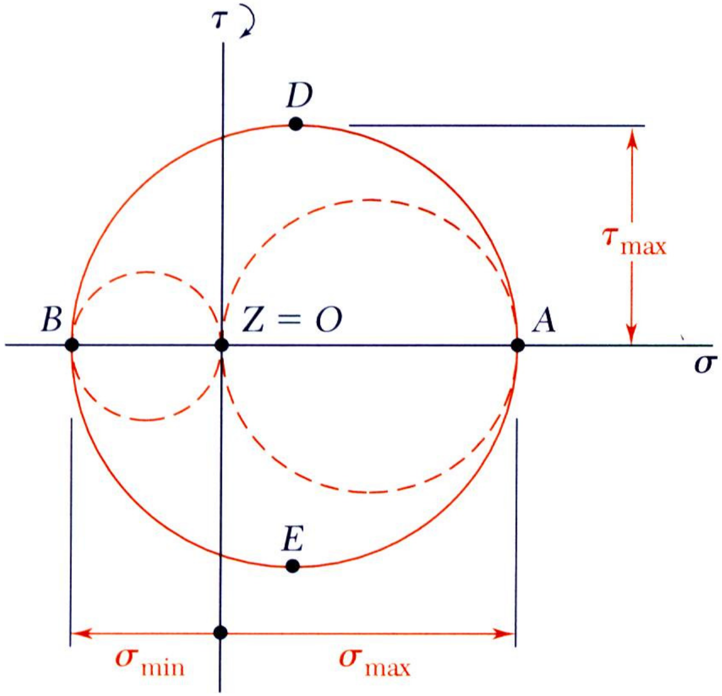


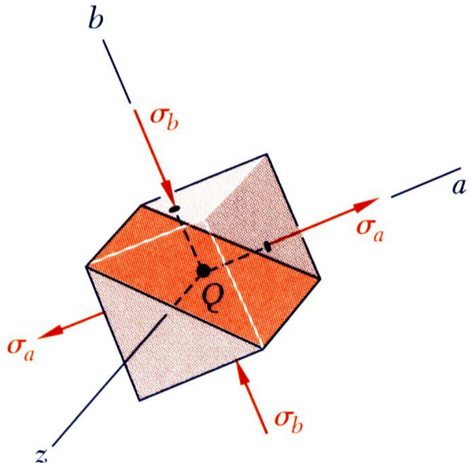


- ***Circle AB*** corresponding to surfaces which ***rotates around c*** axis.
- ***Circle BC*** corresponding to surfaces which ***rotates around a*** axis.
- ***Circle CA*** corresponding to surfaces which ***rotates around b*** axis.

Transformations of Stress and Strain

❑ Application of Mohr's Circle to the Three- Dimensional Analysis of Stress





•**In the case of plane stress**, the axis perpendicular to the plane of stress is a principal axis (shearing stress equal zero).

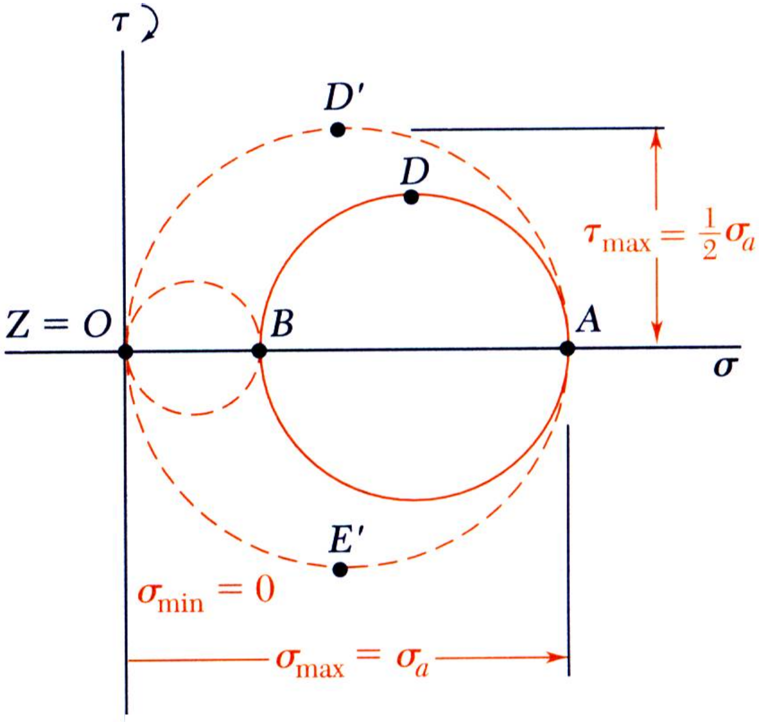
•If the points *A* and *B* (representing the principal planes) are on opposite sides of the origin, then

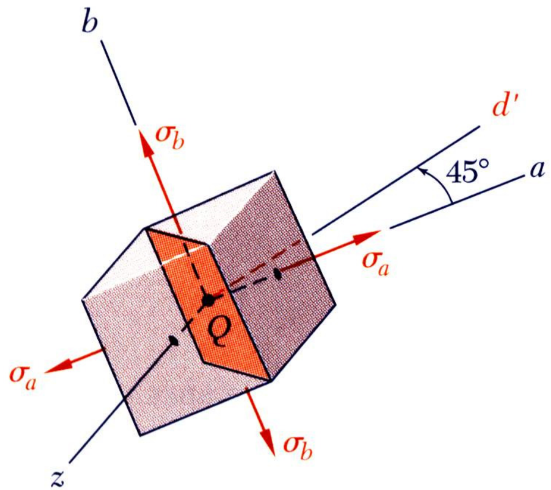
a) The corresponding principal stresses are the maximum and minimum normal stresses for the element

b) The maximum shearing stress for the element is equal to the maximum "in-plane" shearing stress

c) Planes of maximum shearing stress are at 45° to the principal planes.

Application of Mohr's Circle to the Three- Dimensional Analysis of Stress





- If A and B are on the same side of the origin (i.e., ***have the same sign***), then

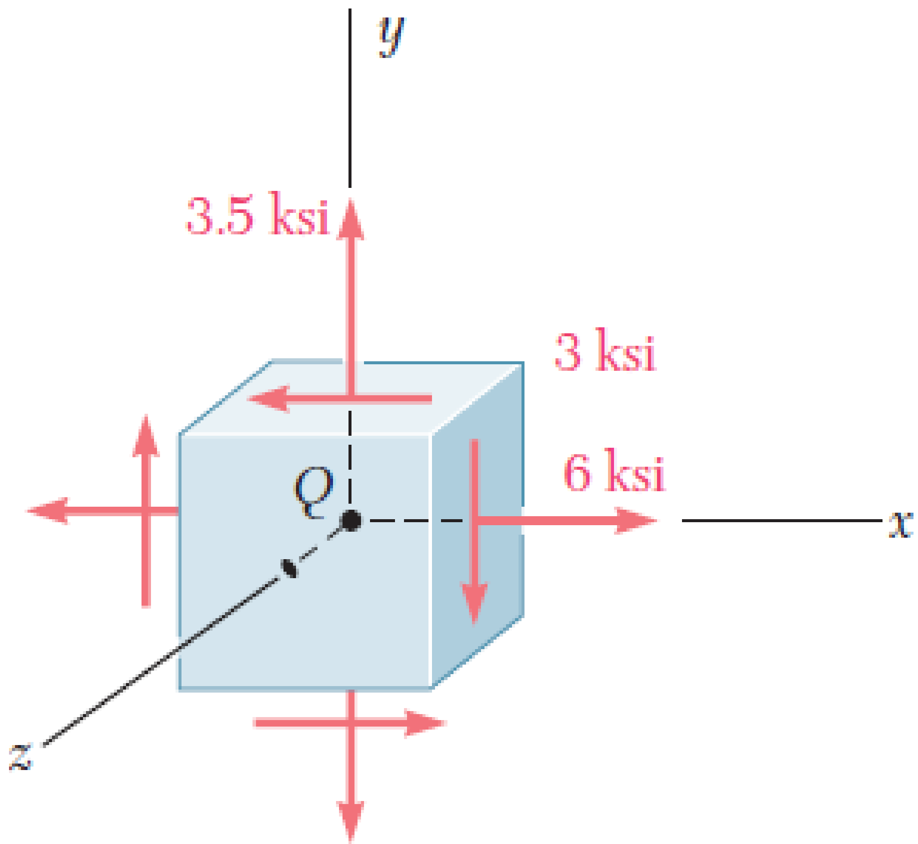
  a) The circle defining σ_{max}, σ_{min}, and τ_{max} for the element is not the circle corresponding to transformations within the plane of stress

  b) Maximum shearing stress for the element is equal to ***half of the maximum stress***

  c) Planes of maximum shearing stress are at 45 degrees to the plane of stress

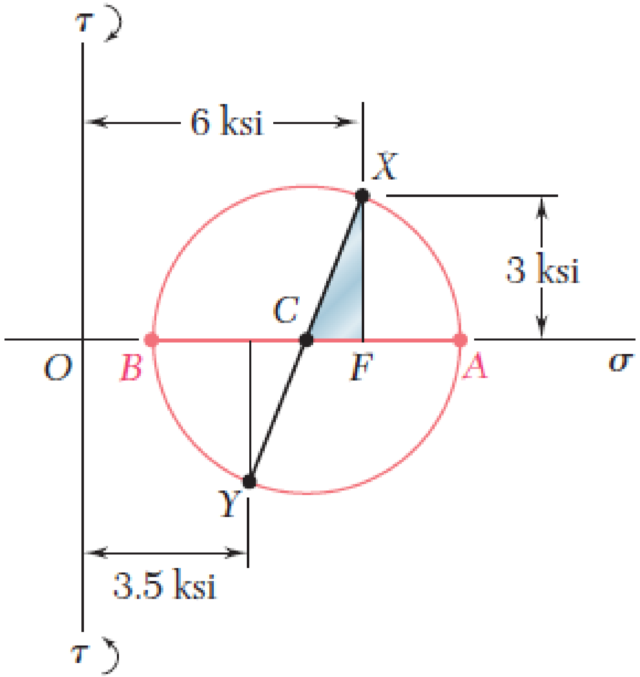
❑ Example 8

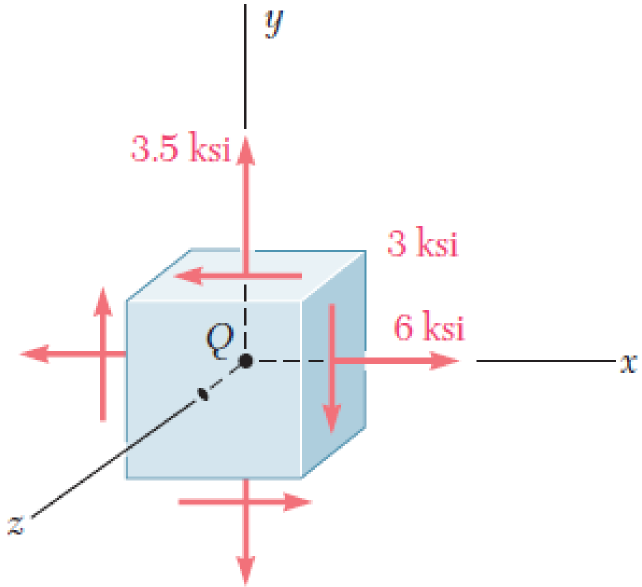
For the state of plane stress shown, determine (*a*) the three principal planes and principal stresses, (*b*) the maximum shearing stress.



Transformations of Stress and Strain

❑ Example 8

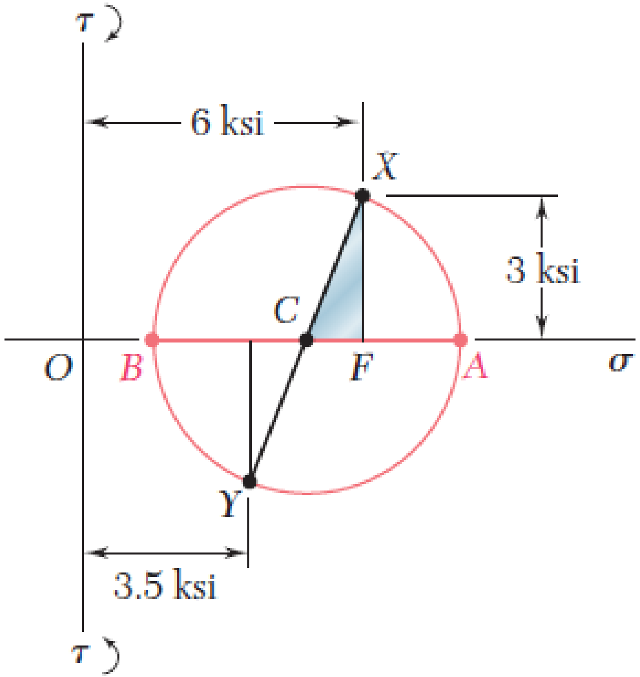


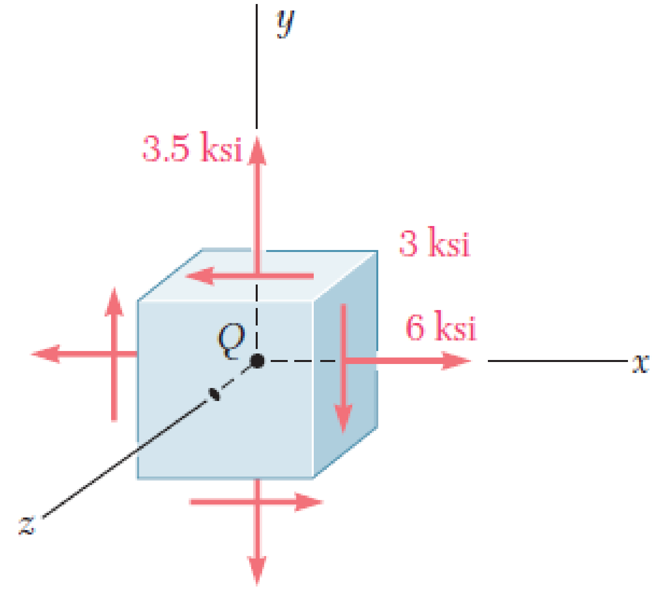


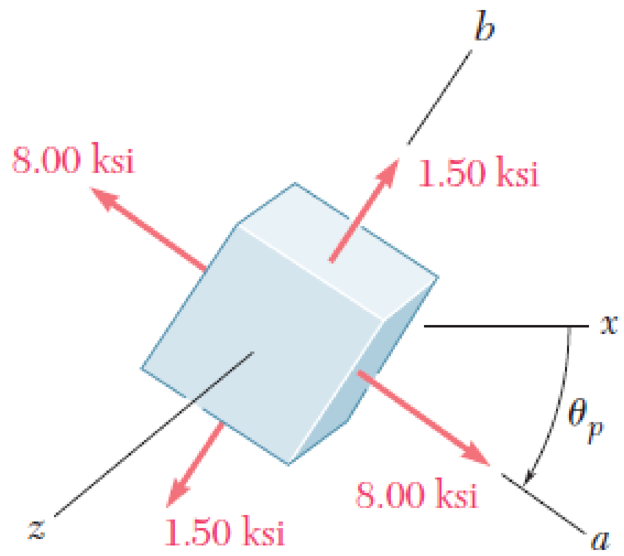
Since the sides of the right triangle CFX are CF=6-4.75=1.25 ksi and FX= 3 ksi, the radius of the circle is

Transformations of Stress and Strain

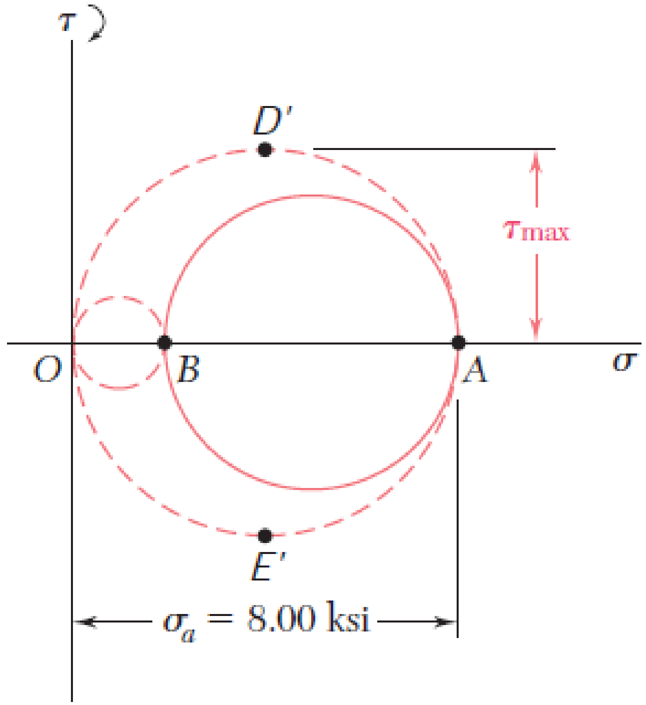
❑ Example 8

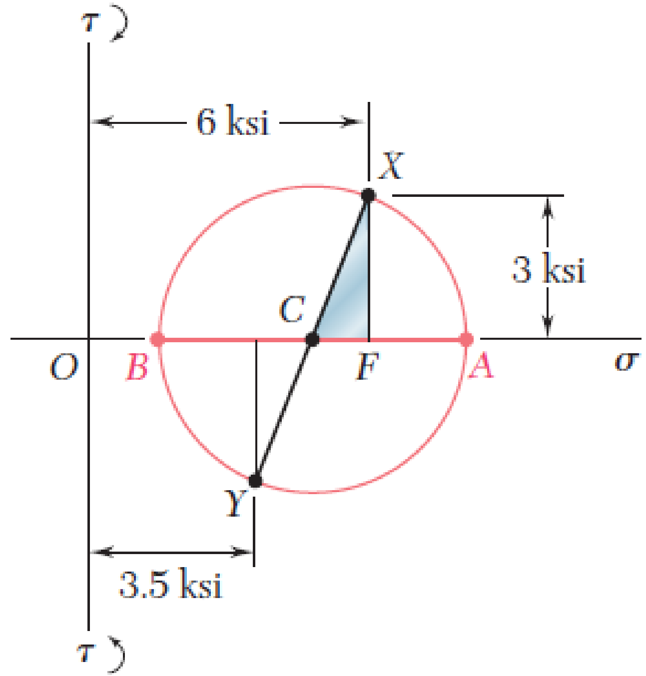






❑ Example 8

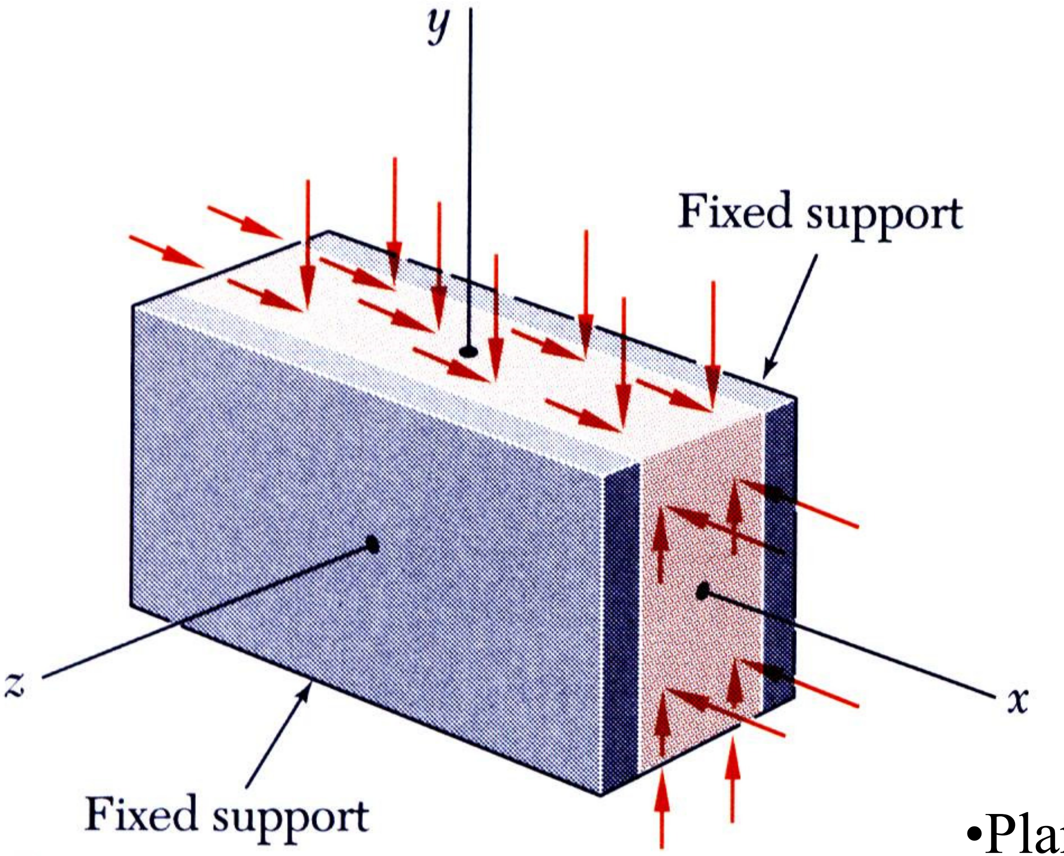




Since points D' and E', which define the planes of maximum shearing stress, are located at the ends of the vertical diameter of the circle corresponding to a rotation about the b axis, the faces of the element can be brought to coincide with the planes of maximum shearing stress through a rotation of 458 about the b axis.

Transformations of Stress and Strain

❑ Transformation of Plane Strain



•*Plane strain* :

I. Deformations of the material take place in **parallel planes**.

II. Deformations of the material are **the same** in each of those planes.

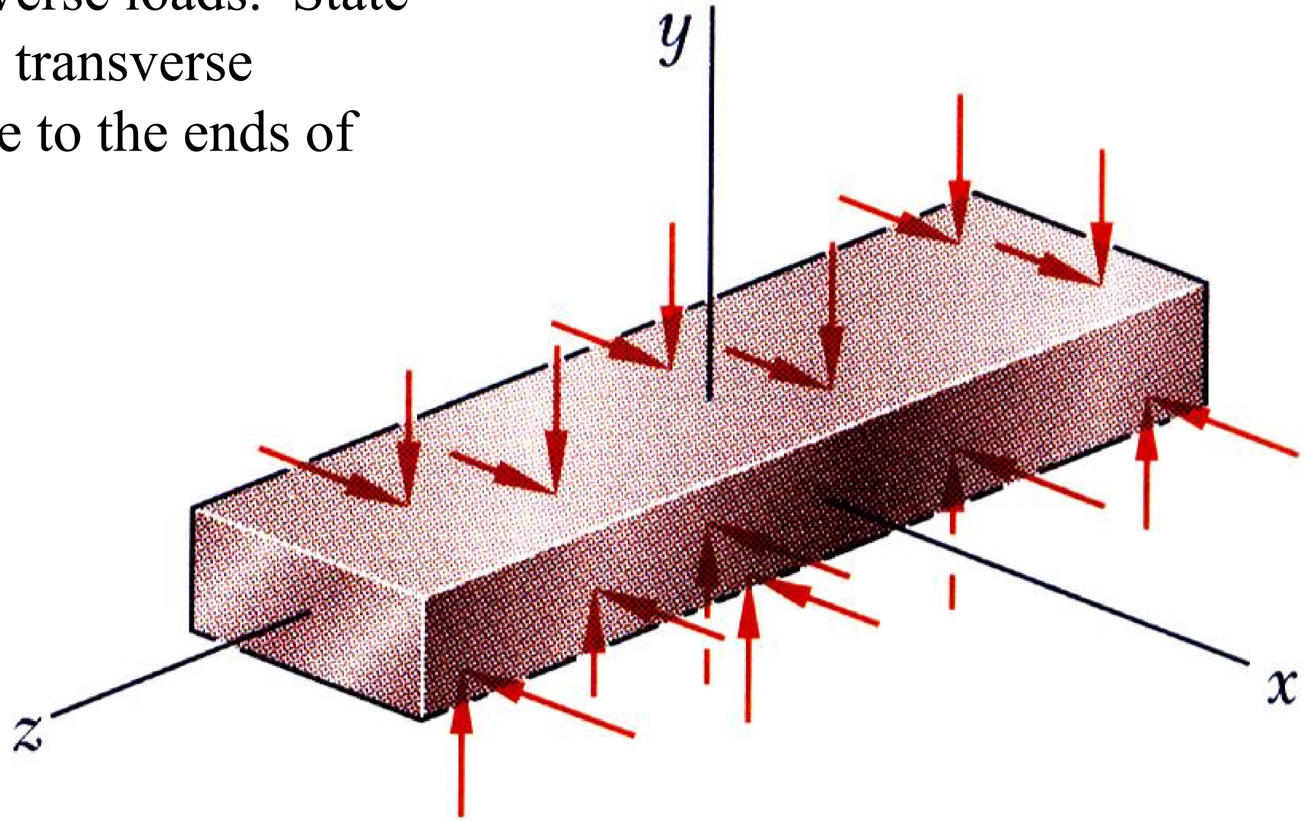
•Plane strain occurs in a plate subjected along its edges to a uniformly distributed load and restrained from expanding or contracting laterally by smooth, rigid and fixed supports

Components of Strain

$$\varepsilon_x = \varepsilon_y = \gamma_{xy} \neq 0$$
$$\varepsilon_z = \gamma_{zx} = \gamma_{zy} = 0$$

Transformation of Plane Strain

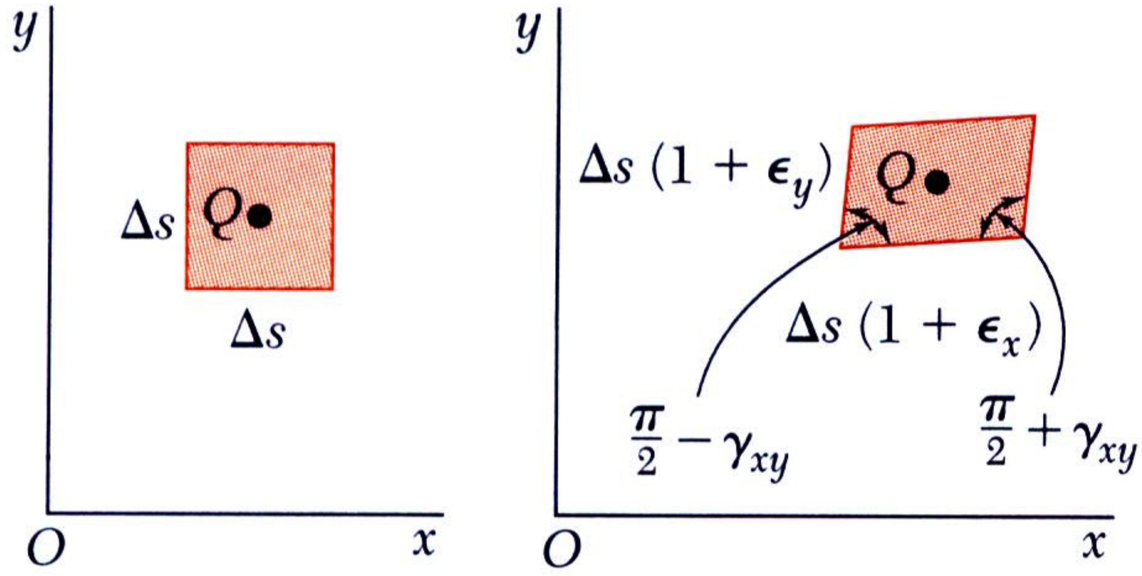
•Example: Consider a long bar subjected to uniformly distributed transverse loads. State of plane stress exists in any transverse section not located too close to the ends of the bar.



Transformations of Stress and Strain

❑ Transformation of Plane Strain

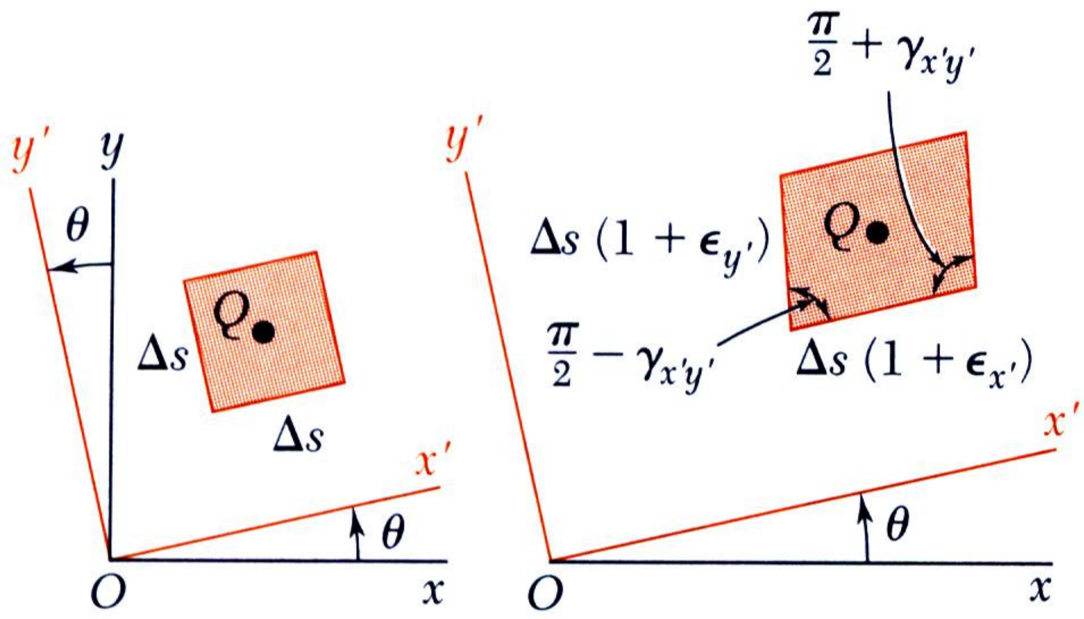
•State of strain at the point Q results in different strain components with respect to the xy reference frames.



$$\varepsilon(\theta) = \varepsilon_x \cos^2(\theta) + \varepsilon_y \sin^2(\theta) + \gamma_{xy} \sin(\theta)\cos(\theta)$$

$$\varepsilon_{OB} = \varepsilon_{(45)} = \frac{1}{2}(\varepsilon_x + \varepsilon_y + \gamma_{xy}) \quad \Rightarrow \quad \gamma_{xy} = 2\varepsilon_{OB} - (\varepsilon_x + \varepsilon_y)$$

Transformations of Stress and Strain



❑ Transformation of Plane Strain

•Applying the trigonometric relations used for the transformation of stress,

$$\varepsilon_{x'} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos(2\theta) + \frac{\gamma_{xy}}{2}\sin(2\theta)$$

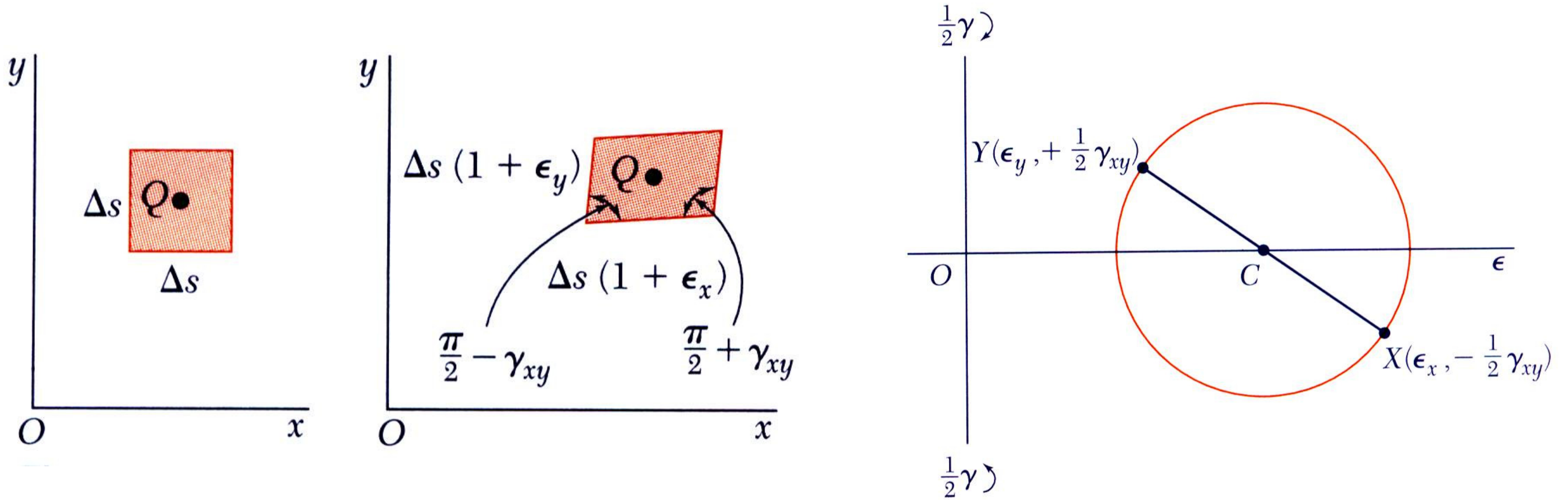
$$\theta \rightarrow \theta + 90 \quad \Rightarrow \quad \varepsilon_{y'} = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2}\cos(2\theta) - \frac{\gamma_{xy}}{2}\sin(2\theta)$$

$$\theta \rightarrow \theta + 45 \quad \Rightarrow \quad \varepsilon_{OB'} = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2}\sin(2\theta) + \frac{\gamma_{xy}}{2}\cos(2\theta)$$

$$\frac{\gamma_{x'y'}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin(2\theta) + \frac{\gamma_{xy}}{2}\cos(2\theta)$$

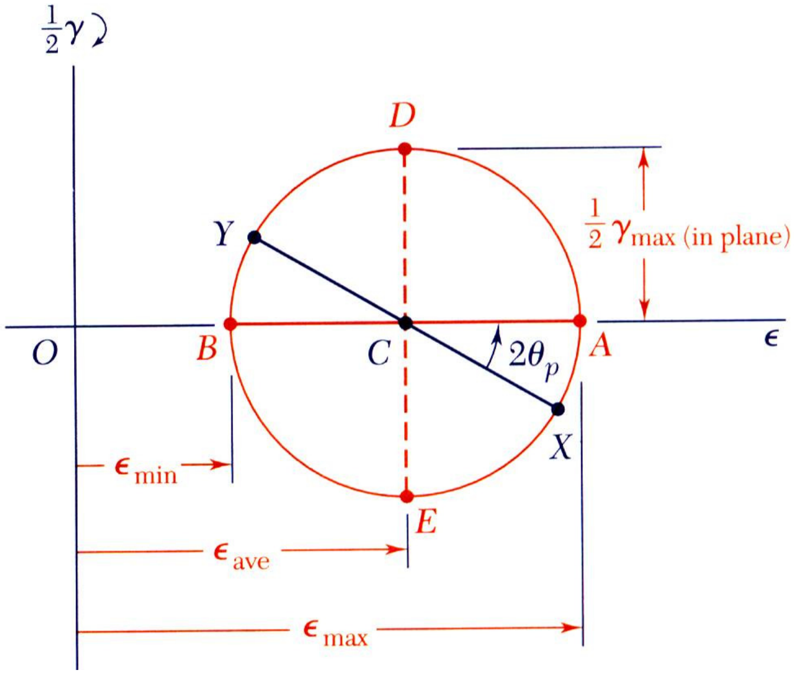
❑ Mohr’s Circle for Plane Strain

•The equations for the transformation of plane strain are of the same form as the equations for the transformation of plane stress - *Mohr’s circle techniques apply*.



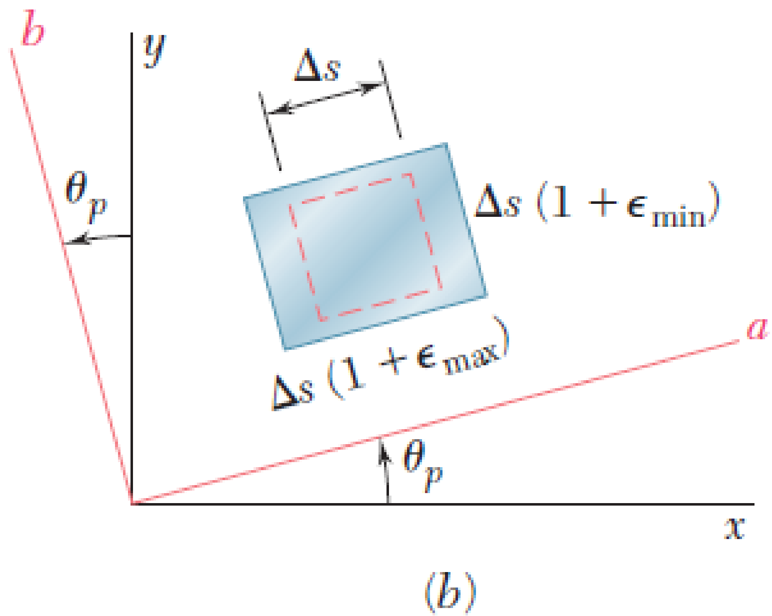
If the shear deformation causes a ***given side to rotate clockwise***, the corresponding point on Mohr's circle for plane strain is plotted ***above the horizontal axis***, and if the deformation causes ***the side to rotate counterclockwise***, the corresponding point is plotted ***below the horizontal axis***. We note that this convention matches the convention used to draw Mohr's circle for plane stress.

Mohr's Circle for Plane Strain



•The center C and radius R,

$$\varepsilon_{ave} = \frac{\varepsilon_x + \varepsilon_y}{2} \qquad R = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

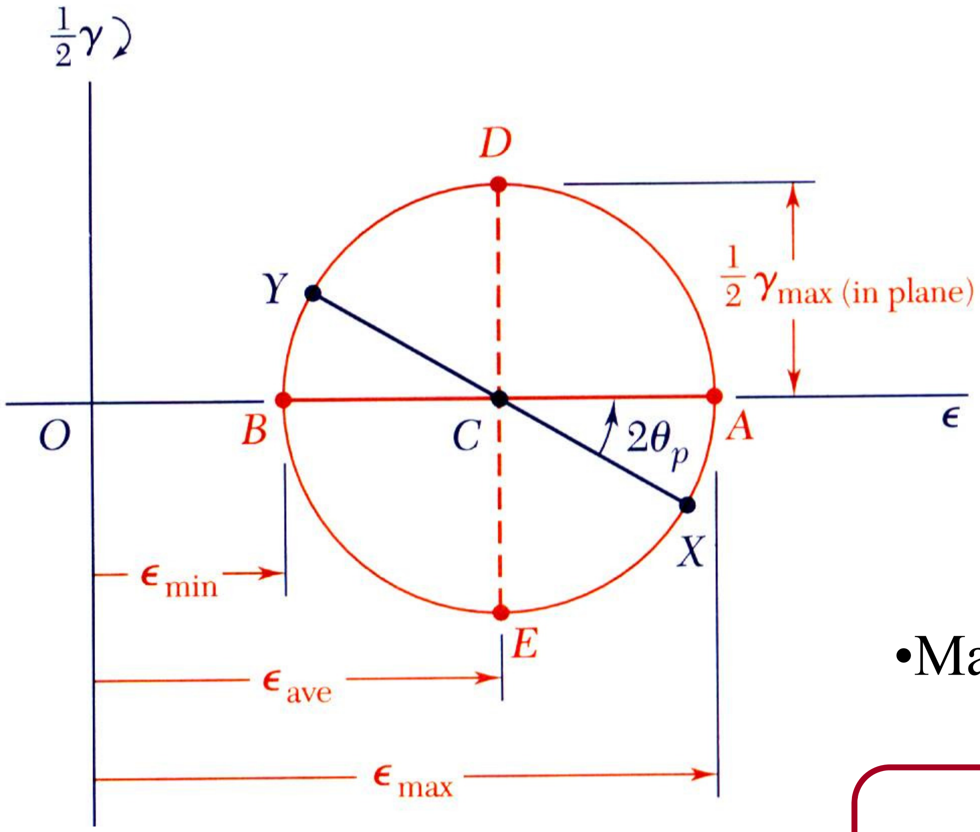


•Principal axes of strain and principal strains,

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}$$

$$\varepsilon_{max} = \varepsilon_{ave} + R \qquad \varepsilon_{min} = \varepsilon_{ave} - R$$

❑ Mohr's Circle for Plane Strain



•The center C and radius R,

$$\varepsilon_{ave} = \frac{\varepsilon_x + \varepsilon_y}{2} \qquad R = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

•Maximum in-plane shearing strain,

$$\gamma_{max} = 2R = \sqrt{(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2}$$

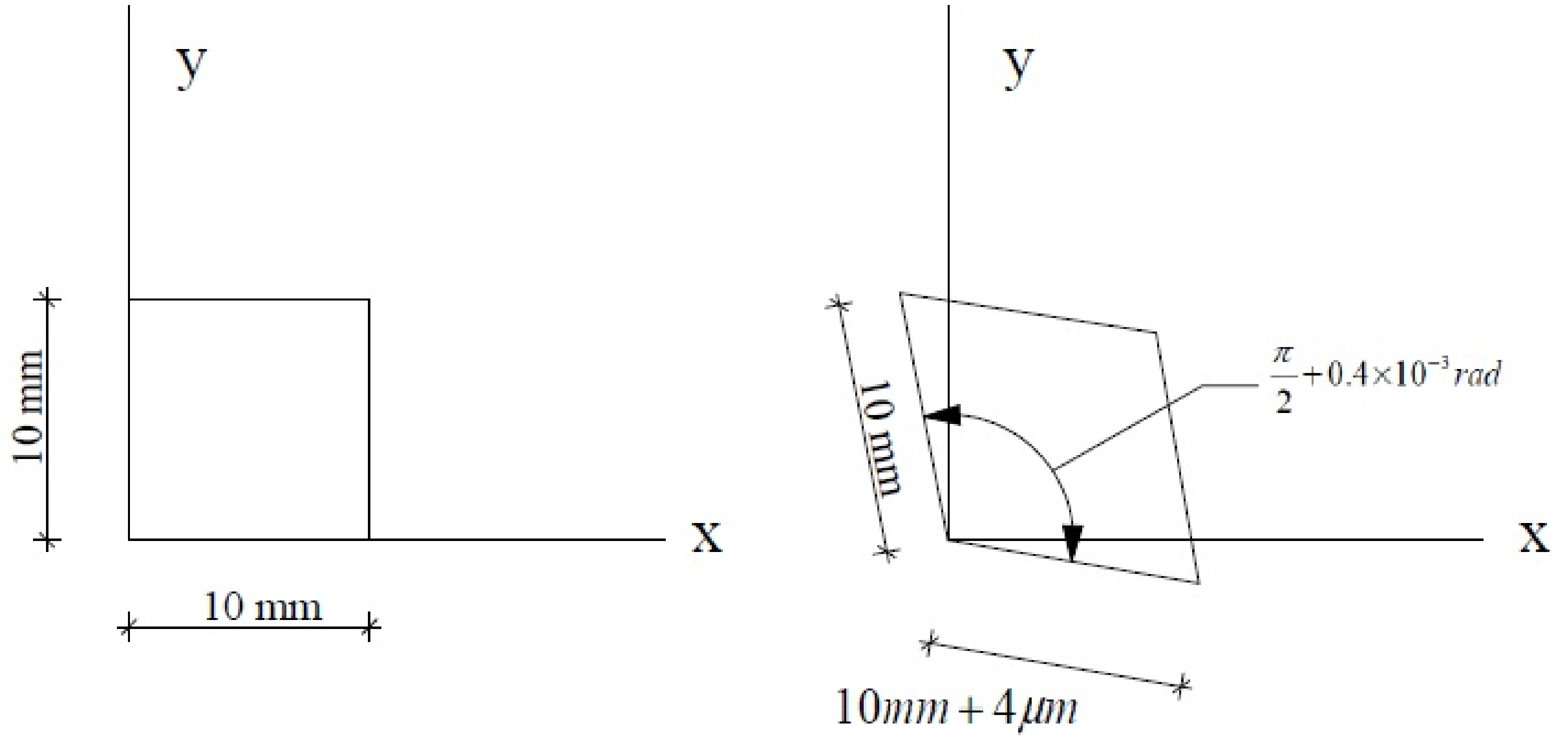
❑ Mohr's Circle for Plane Strain

Example 9

Determine

(a) The principal axes and principal strains,

(b) The maximum shearing strain and the corresponding normal strain.

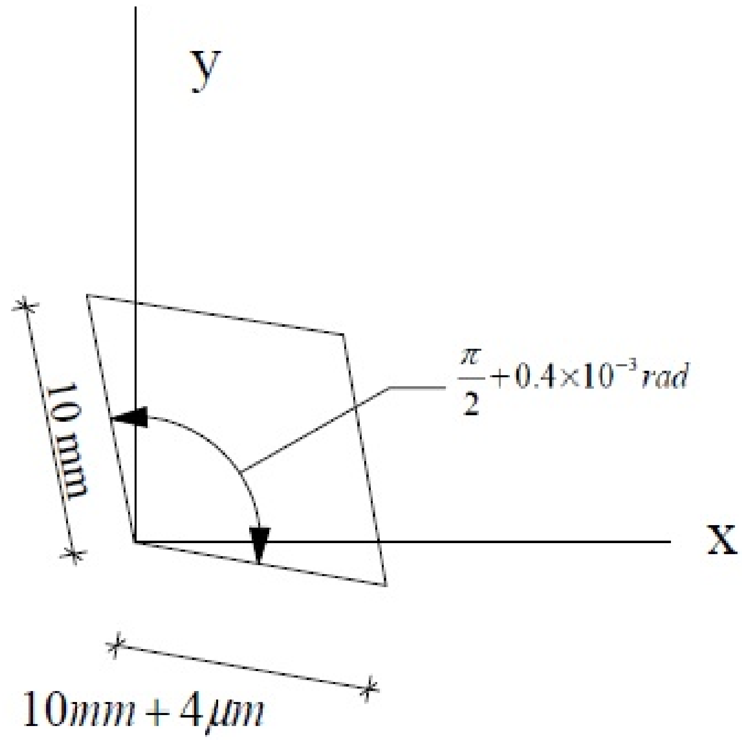


Mohr's Circle for Plane Strain

Example 9

Principal Axes and Principal Strains:

We first determine the coordinates of points X and Y



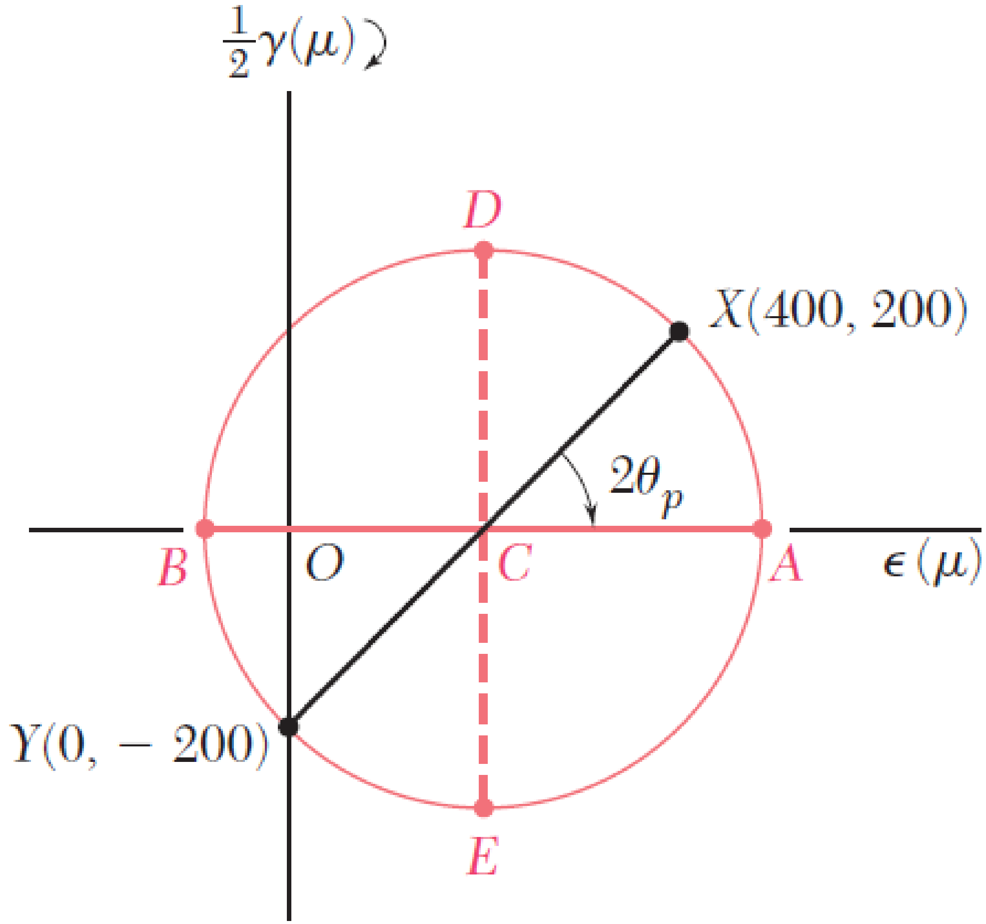
Since the side of the square associated with ε_x rotates clockwise, point X of coordinates ε_x and $|\gamma_{xy}/2|$ is plotted above the horizontal axis. Since $\varepsilon_y = 0$ and the corresponding side rotates *counterclockwise,* point Y is plotted directly *below* the origin

Mohr's Circle for Plane Strain

Example 9

Principal Axes and Principal Strains:

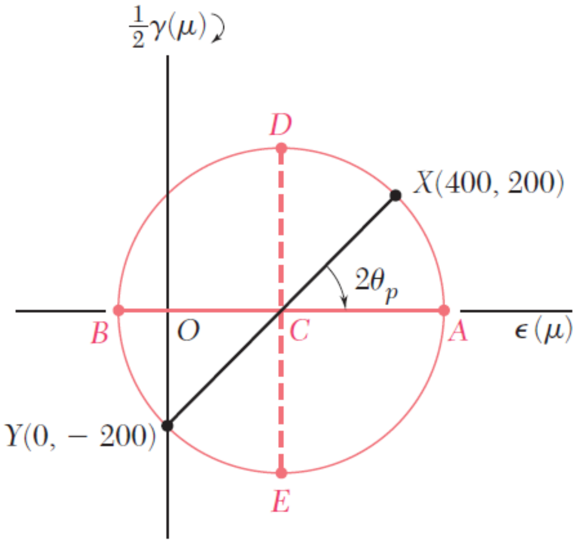
Principle Strain

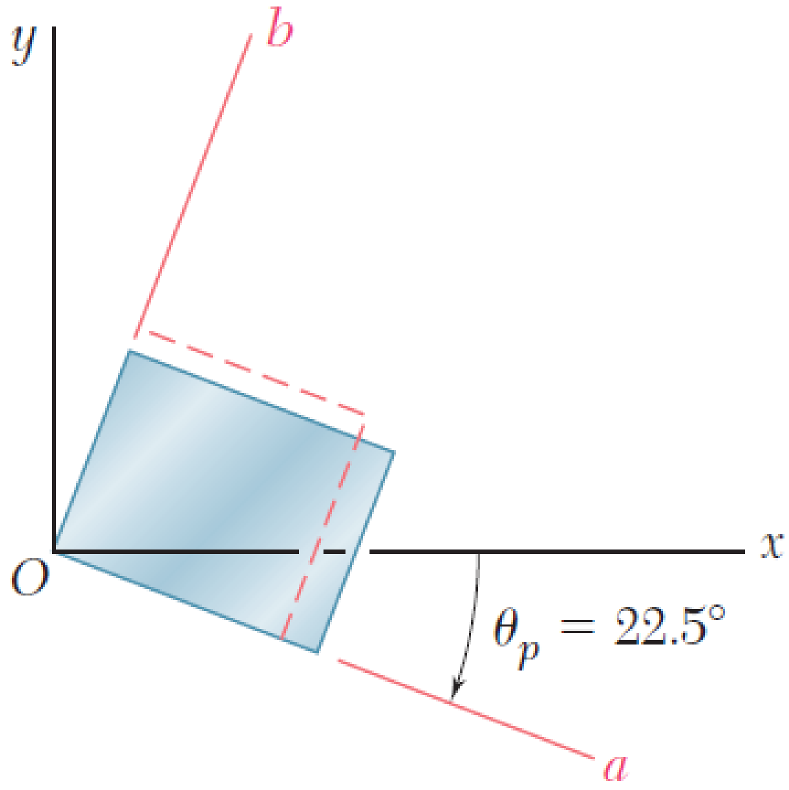


Mohr's Circle for Plane Strain

Example 9

Principal Axes and Principal Strains:

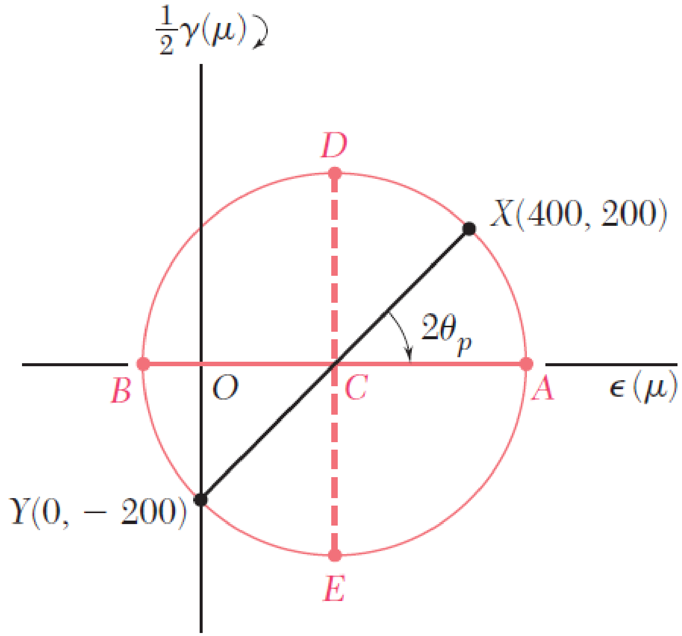




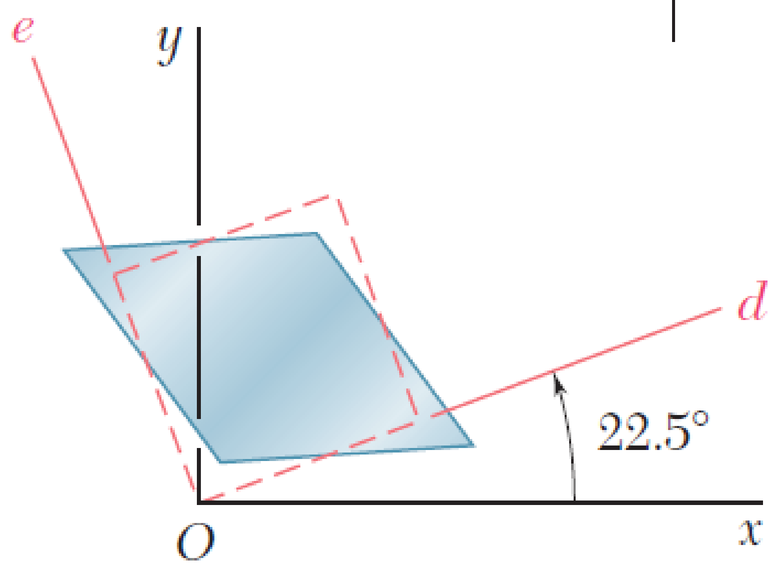
❑ Mohr's Circle for Plane Strain

Example 9

Maximum Shearing Strain



The corresponding normal strains are equal to



Three-Dimensional Analysis of Strain

Generalized Hooke's law

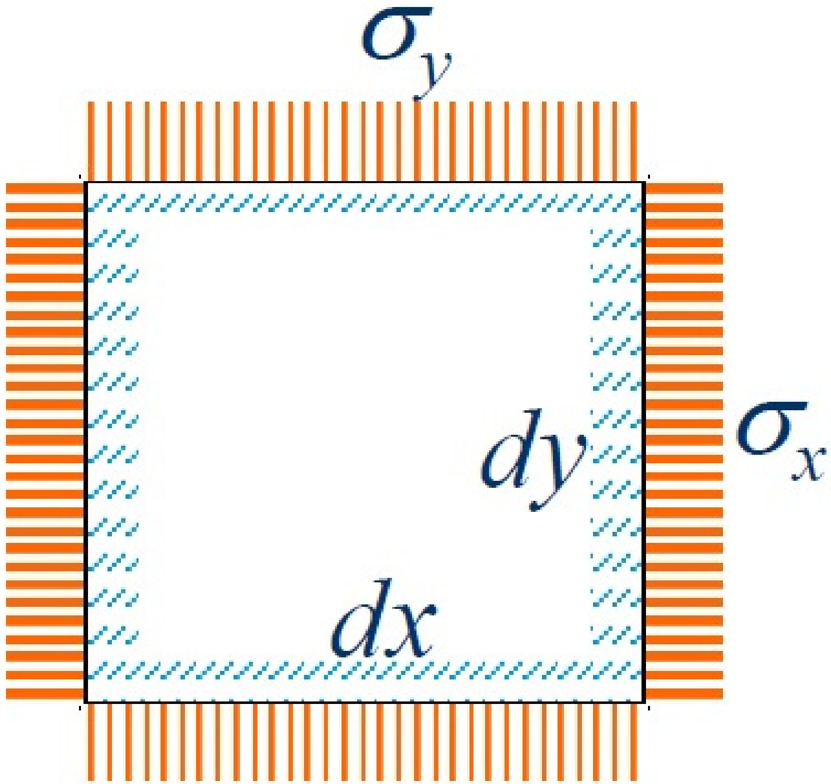
$$E = \frac{\sigma}{\varepsilon}$$

Hooke's law can be extended to include the ***biaxial*** and ***Triaxial*** states of stress that often encounter in engineering applications.

❑ Three-Dimensional Analysis of Strain

Generalized Hooke's law

Let's consider the differential element of the material subjected to biaxial state of normal stress

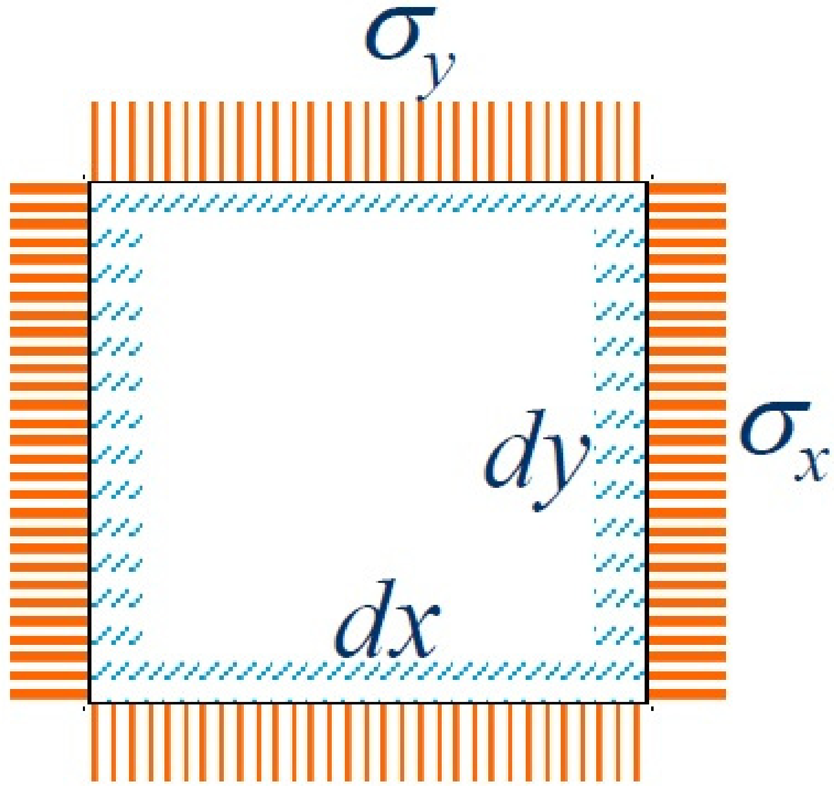


❑ Three-Dimensional Analysis of Strain

Generalized Hooke's law

– Shearing stresses have not been shown in the differential element of because they do not produce changes in the lengths of sides of the element.

– They only produce distortion of the element (angle changes), that can contributes to the angular strain.

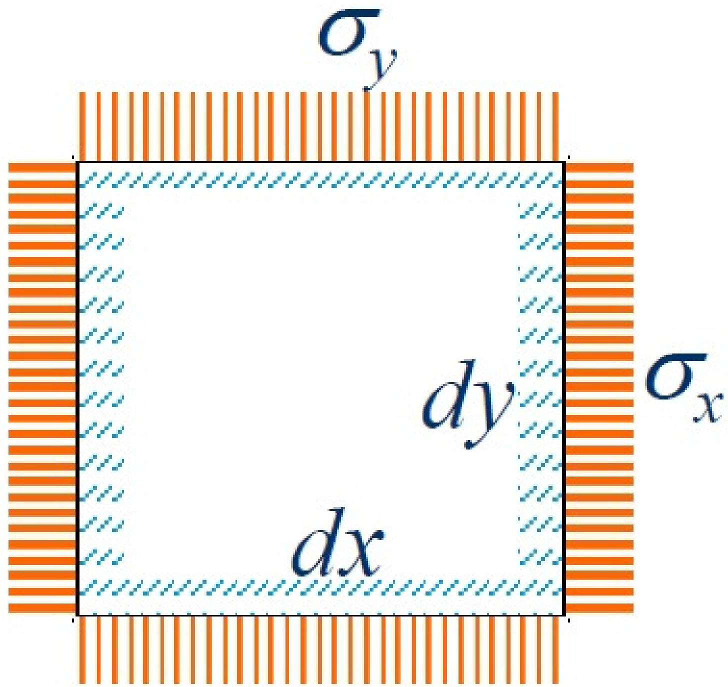


Three-Dimensional Analysis of Strain

Generalized Hooke's law

The principle of superposition.

– The ***deformation of that element in the direction*** of the normal stresses, for a combined loading, can be determined by computing the deformations resulting from the individual stresses separately and adding the values obtained algebraically.



❑ Three-Dimensional Analysis of Strain

Generalized Hooke's law

The principle of superposition.

– When applying the principle of superposition, the following conditions must be satisfied:

- Each effect is ***linearly*** related to the load that produced it.
- The effect of the first load does not scientifically change the effect of the second load.

Three-Dimensional Analysis of Strain

Generalized Hooke's law

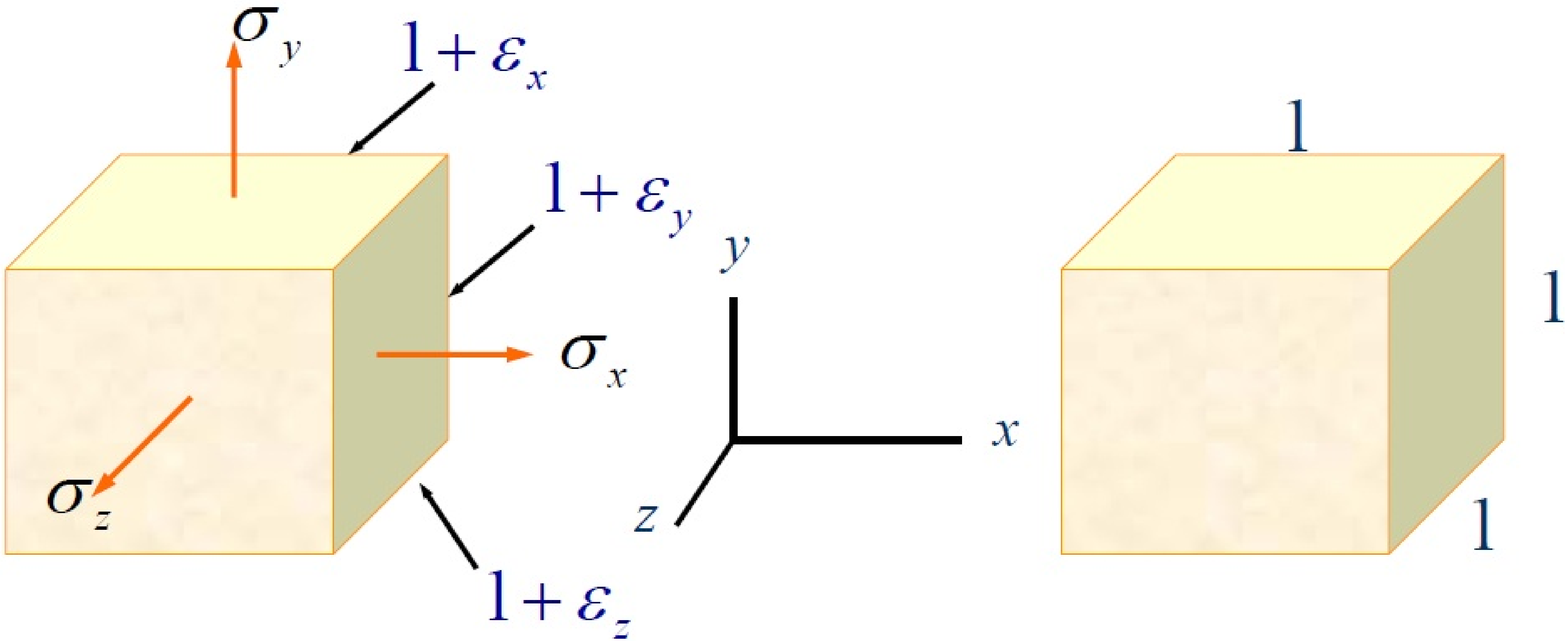
The principle of superposition.

- The first condition is satisfied if the stresses ***do not exceed the proportional limit of the material***.

- The second condition is also satisfied if the ***deformations small*** so that the small changes in the areas of the faces of the element do not produce significant changes in the stresses.

Three-Dimensional Analysis of Strain

Generalized Hooke's law



Three-Dimensional Analysis of Strain

Generalized Hooke's law

Stress	X: Direction	Y: Direction	Z: Direction
σ_x	$\varepsilon_x = \dfrac{\sigma_x}{E}$	$\varepsilon_y = -\nu \dfrac{\sigma_x}{E}$	$\varepsilon_z = -\nu \dfrac{\sigma_x}{E}$
σ_y	$\varepsilon_x = -\nu \dfrac{\sigma_y}{E}$	$\varepsilon_y = \dfrac{\sigma_y}{E}$	$\varepsilon_z = -\nu \dfrac{\sigma_y}{E}$
σ_z	$\varepsilon_x = -\nu \dfrac{\sigma_z}{E}$	$\varepsilon_y = -\nu \dfrac{\sigma_z}{E}$	$\varepsilon_z = \dfrac{\sigma_z}{E}$

Transformations of Stress and Strain

Three-Dimensional Analysis of Strain

General State of Strain

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - v(\sigma_y + \sigma_z)\right]$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - v(\sigma_x + \sigma_z)\right]$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - v(\sigma_x + \sigma_y)\right]$$

Three-Dimensional Analysis of Strain

General State of Stress

$$\sigma_x = \frac{E}{(1+v)(1-2v)}\left[(1-v)\varepsilon_x + v(\varepsilon_y + \varepsilon_z)\right]$$

$$\sigma_y = \frac{E}{(1+v)(1-2v)}\left[(1-v)\varepsilon_y + v(\varepsilon_x + \varepsilon_z)\right]$$

$$\sigma_z = \frac{E}{(1+v)(1-2v)}\left[(1-v)\varepsilon_z + v(\varepsilon_x + \varepsilon_z)\right]$$

Transformations of Stress and Strain

❑ Three-Dimensional Analysis of Strain

Plane State of Stress

$$\sigma_z = 0 \Rightarrow \frac{E}{(1+v)(1-2v)}\left[(1-v)\varepsilon_z + v(\varepsilon_x + \varepsilon_z)\right] = 0 \Rightarrow$$

$$(1-v)\varepsilon_z + v(\varepsilon_x + \varepsilon_z) = 0 \quad \Rightarrow \quad \varepsilon_z = \frac{-v}{1-v}(\varepsilon_x + \varepsilon_y) \quad \Rightarrow$$

$$\sigma_x = \frac{E}{1-v^2}\left[\varepsilon_x + v\varepsilon_y\right]$$

$$\sigma_y = \frac{E}{1-v^2}\left[\varepsilon_y + v\varepsilon_x\right]$$

Three-Dimensional Analysis of Strain

Generalized Hooke's Law for Shearing Stress and Strain in Isotropic Materials

$$G = \frac{E}{2(1+\nu)}$$

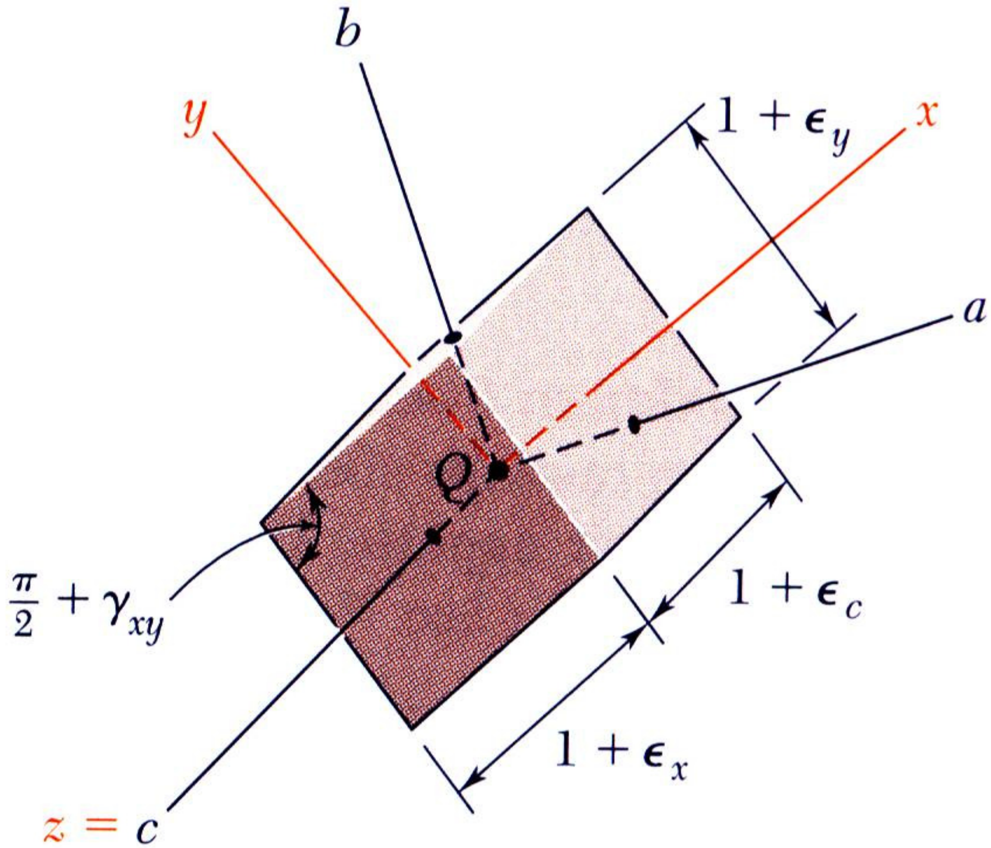
$$\tau_{xy} = G\gamma_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}$$

$$\tau_{yz} = G\gamma_{yz} = \frac{E}{2(1+\nu)}\gamma_{yz}$$

$$\tau_{xz} = G\gamma_{xz} = \frac{E}{2(1+\nu)}\gamma_{xz}$$

Three-Dimensional Analysis of Strain

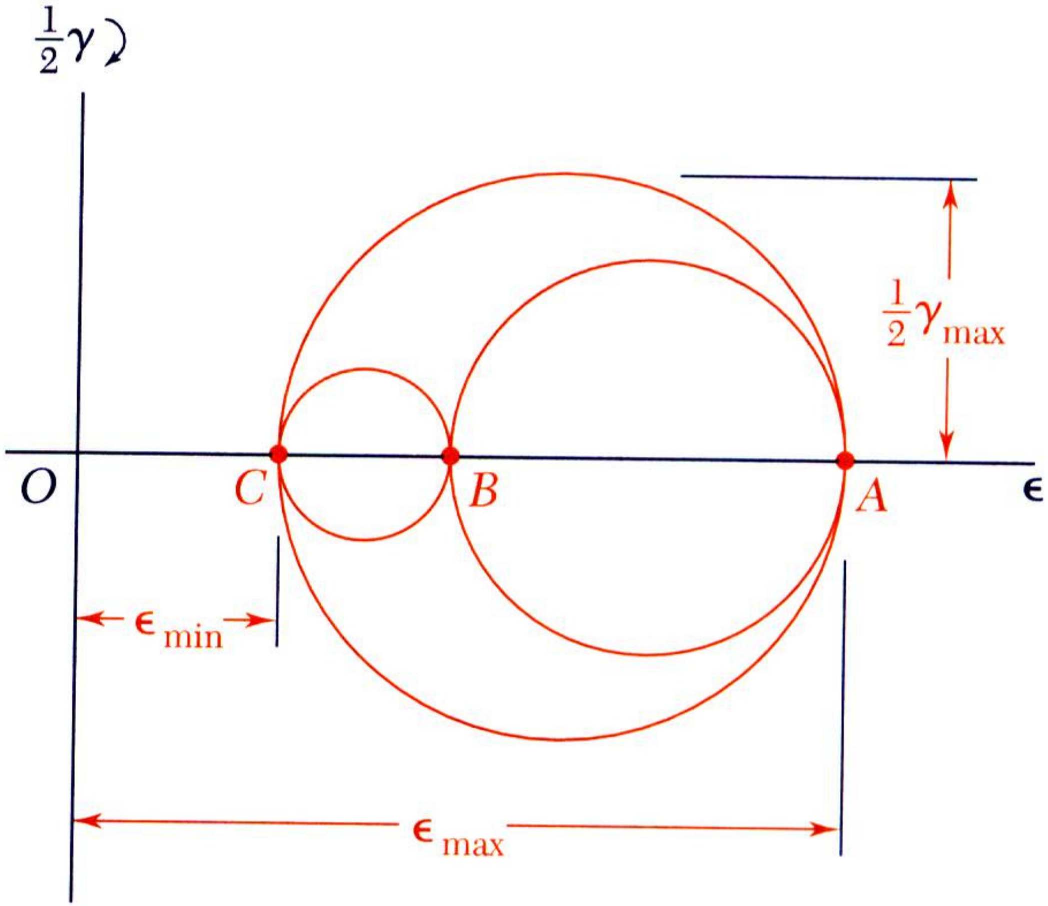
- Three principal axes exist such that the perpendicular element faces are free of shearing stresses.
- By Hooke's Law, it follows that the shearing strains are zero at the principal planes of strain.



Three-Dimensional Analysis of Strain

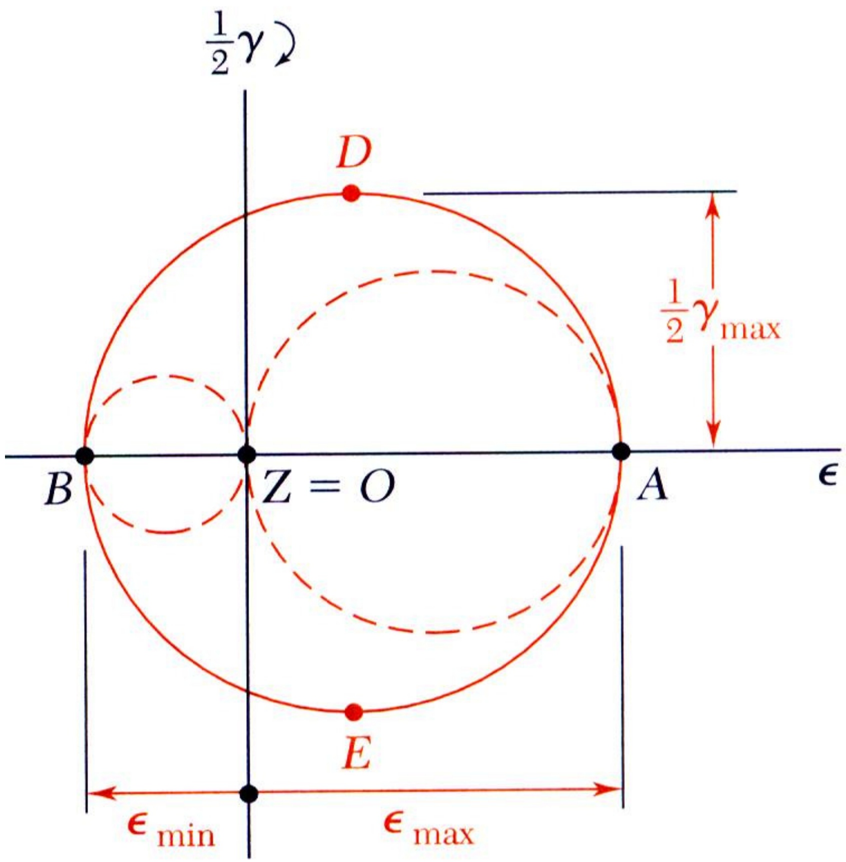
Mohr circle

Rotation about the principal axes may be represented by Mohr's circles.



❑ Three-Dimensional Analysis of Strain

Mohr circle

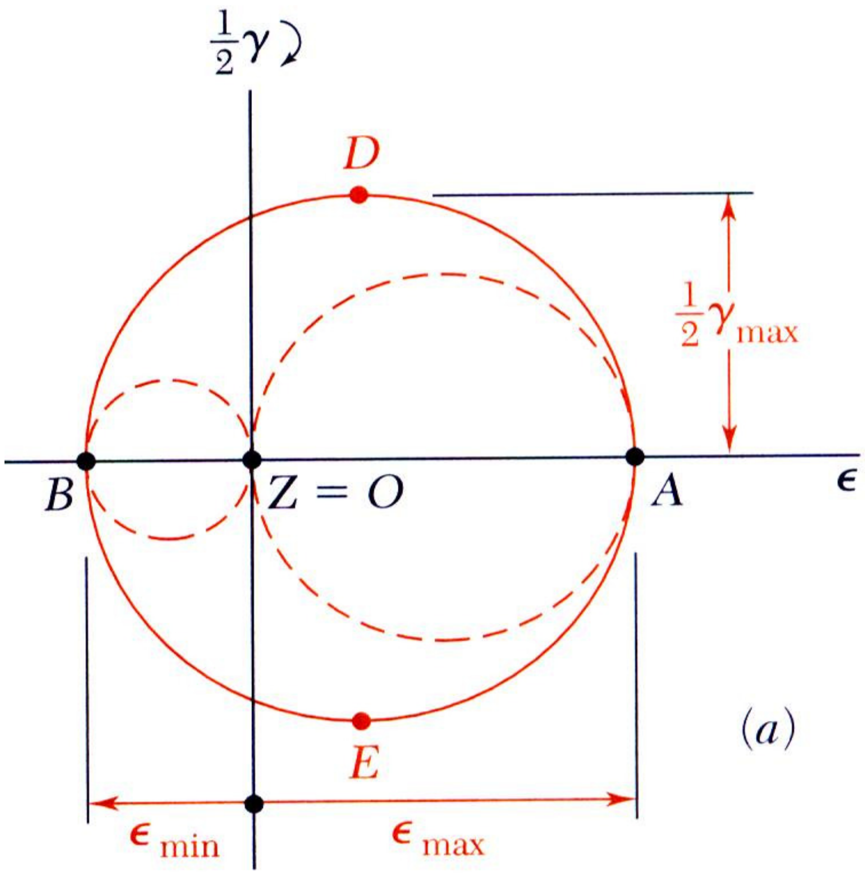


•For the case of plane strain where the *x* and *y* axes are in the plane of strain,

- -The *z* axis is also a principal axis
- -The corresponding principal normal strain is represented by the point *Z = 0* or the origin.

Three-Dimensional Analysis of Strain

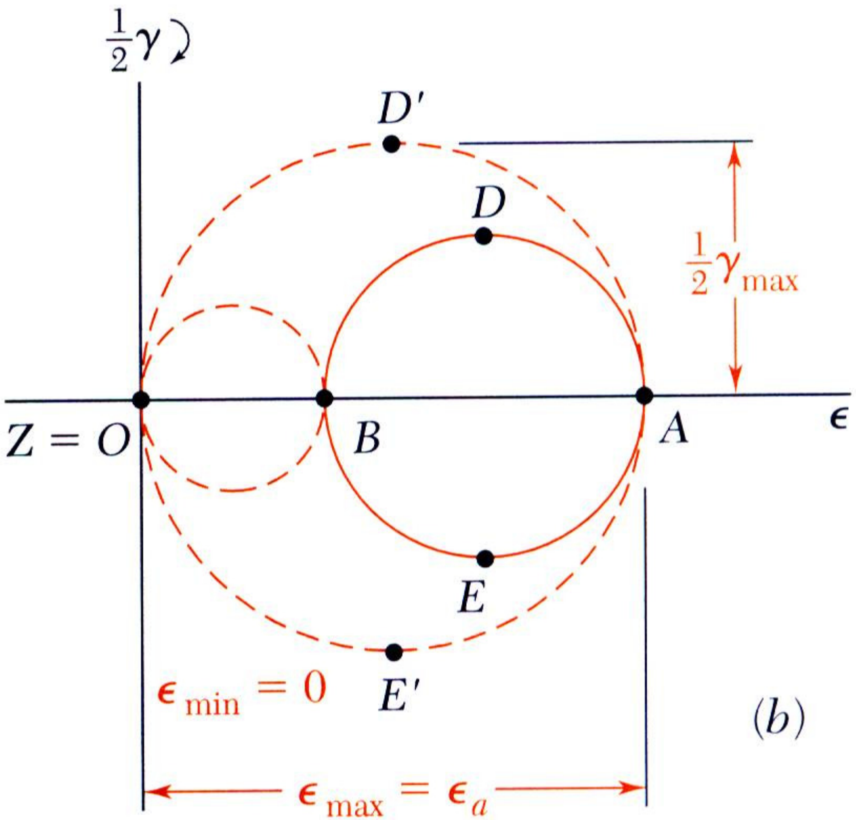
Mohr circle



- If the points *A* and *B* lie on opposite sides of the origin, the maximum shearing strain is the maximum ***in-plane shearing strain***, *D* and *E*.

❑ Three-Dimensional Analysis of Strain

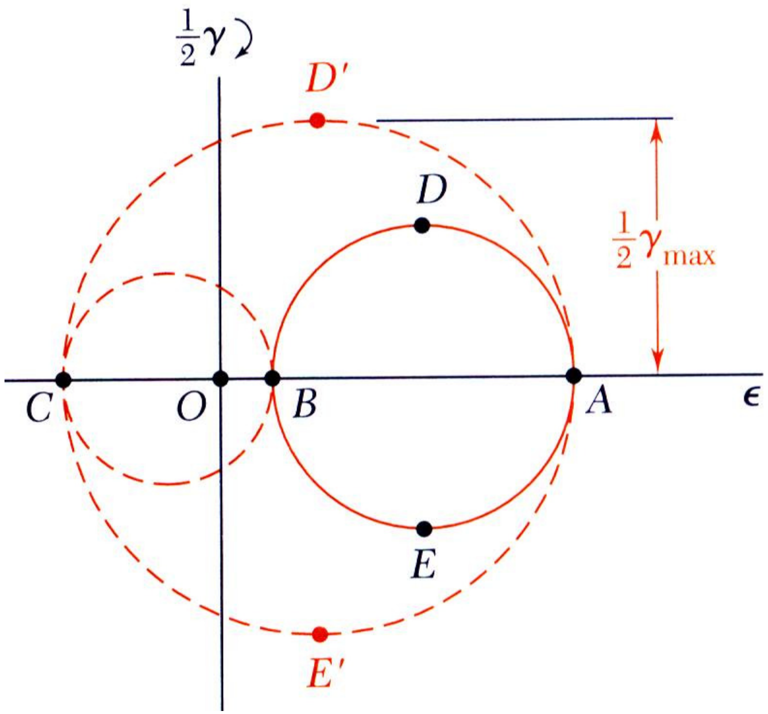
Mohr circle



•If the points A and B lie on the same side of the origin, the maximum shearing strain is ***out of the plane of strain*** and is represented by the points D' and E'.

Three-Dimensional Analysis of Strain

Mohr circle



•Consider the case of plane stress,

$$\sigma_x = \sigma_a \quad \sigma_y = \sigma_b \quad \sigma_z = 0$$

•Corresponding normal strains,

$$\varepsilon_a = \frac{\sigma_a}{E} - \frac{\nu\sigma_b}{E}$$

$$\varepsilon_b = -\frac{\nu\sigma_a}{E} + \frac{\sigma_b}{E}$$

$$\varepsilon_c = -\frac{\nu}{E}(\sigma_a + \sigma_b) = -\frac{\nu}{1-\nu}(\varepsilon_a + \varepsilon_b)$$

•Strain perpendicular to the plane of stress is not zero.

Three-Dimensional Analysis of Strain

Example 10

At a point on the surface of a structural steel machine part subjected to a biaxial state of stress, the measured strains are as follows:

$$\varepsilon_x = +750\mu m/m$$
$$\varepsilon_y = +350\mu m/m$$
$$\gamma_{xy} = -560\mu rad$$

$$E = 200Gpa$$
$$G = 76Gpa$$

Determine the Normal and shear stresses at the point.

Three-Dimensional Analysis of Strain

Example 10

Three-Dimensional Analysis of Strain

Example 11

Determine the state of strain that corresponds to the following state of stress at a point in a steel machine part.

$$\sigma_x = 15{,}000\,psi \qquad \tau_{xy} = 5500\,psi$$

$$\sigma_y = 5000\,psi \qquad \tau_{yz} = 4750\,psi$$

$$\sigma_z = 7500\,psi \qquad \tau_{zx} = 3200\,psi$$

$$E = 30{,}000ksi$$

$$\nu = 0.30$$

❑ Three-Dimensional Analysis of Strain

Example 11

❑ Three-Dimensional Analysis of Strain

Example 11

Three-Dimensional Analysis of Strain

Example 12

The principal strains on the free surface are

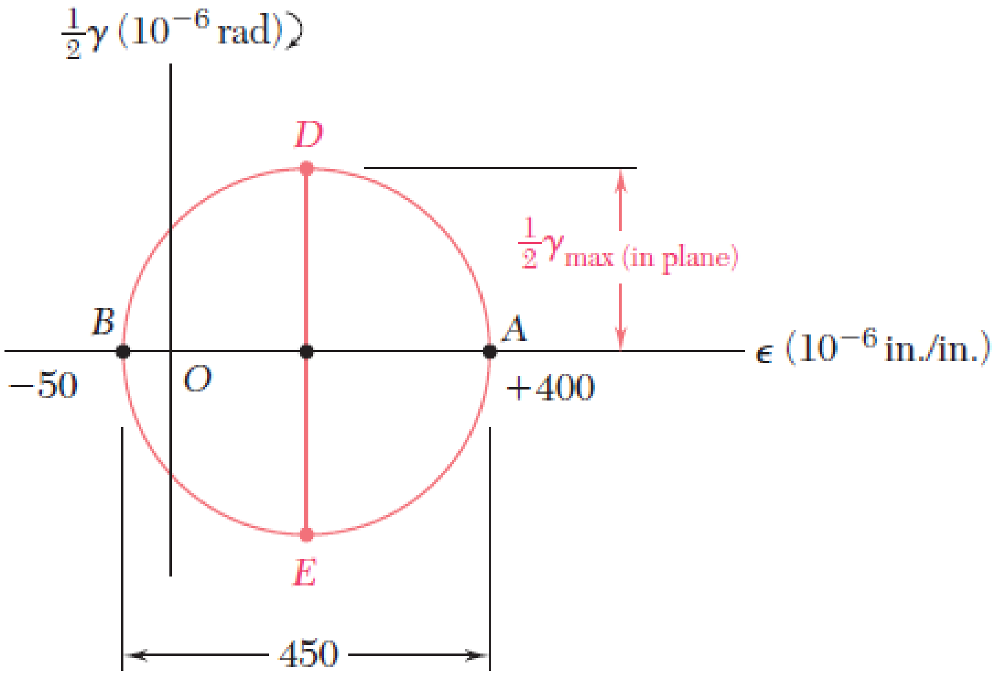
$$\varepsilon_a = 400 \times 10^{-6}$$
$$\varepsilon_b = -50 \times 10^{-6}$$

$$\nu = 0.30$$

Determine:
(a) the maximum in-plane shearing strain,
(b) the true value of the maximum shearing strain near the surface of the component.

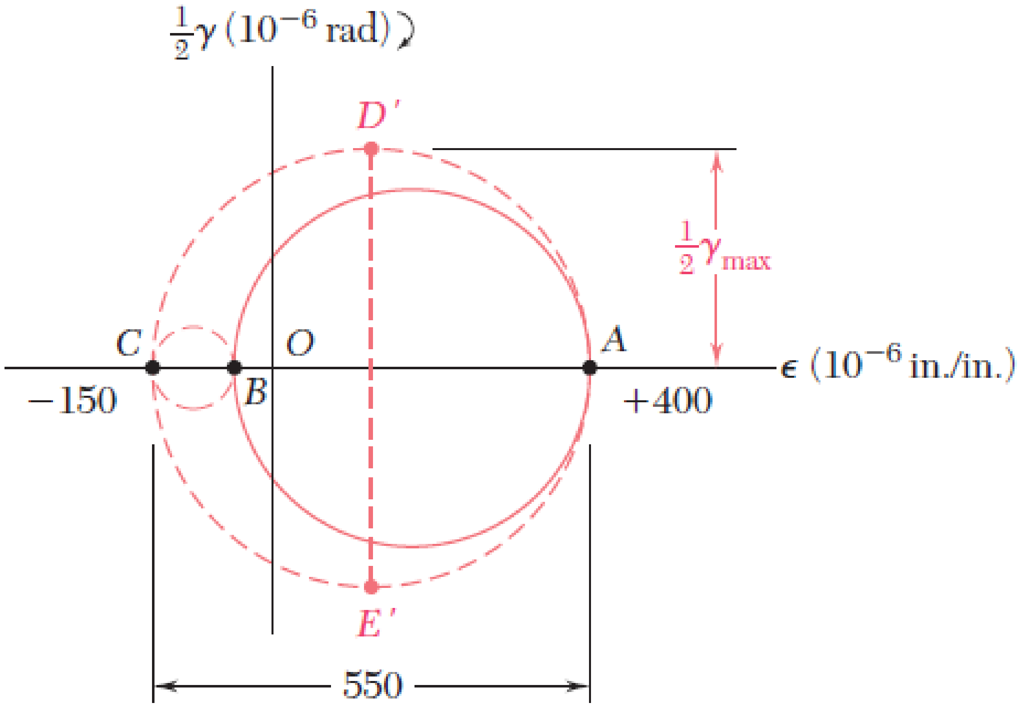
❑ Three-Dimensional Analysis of Strain

Example 12



Three-Dimensional Analysis of Strain

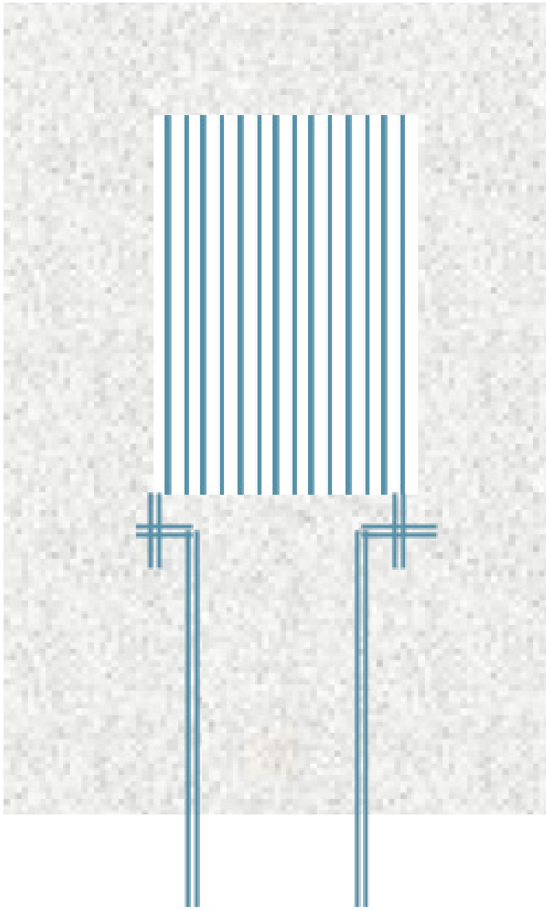
Example 12



Three-Dimensional Analysis of Strain

Strain Measurement and Rosette Analysis

- Electrical resistance strain gages provide accurate measurements of normal strain.
- The gage may consist of a length of 0.001 in-diameter wire arranged and cemented between two pieces of paper.



Three-Dimensional Analysis of Strain

Strain Measurement and Rosette Analysis

- The wire or foil gage is centered to the material for which the strain is to be determine.
- As the material is strained, the wires are lengthened or shortened.
- This lengthening and shortening will cause changes in the electrical resistance.

Three-Dimensional Analysis of Strain

Strain Measurement and Rosette Analysis

- The change in resistance can be measured and calibrated to provide normal strain

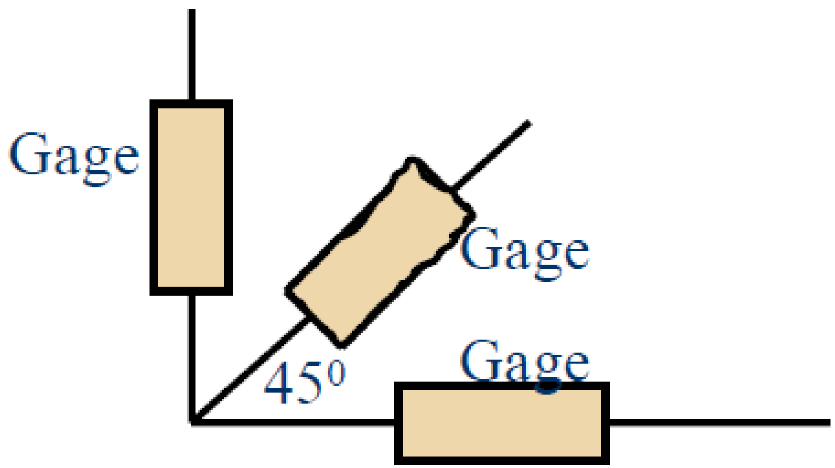
- Shearing strains are often obtained by measuring normal strains in two or three different directions.

- The shearing strains can be computed from normal strain data.

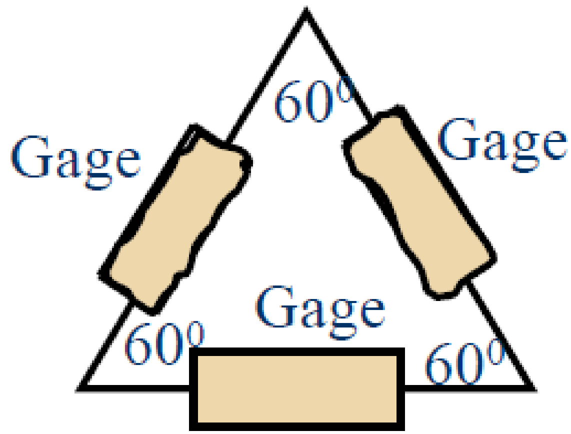
❑ Three-Dimensional Analysis of Strain

Strain Measurement and Rosette Analysis

Rosette Types



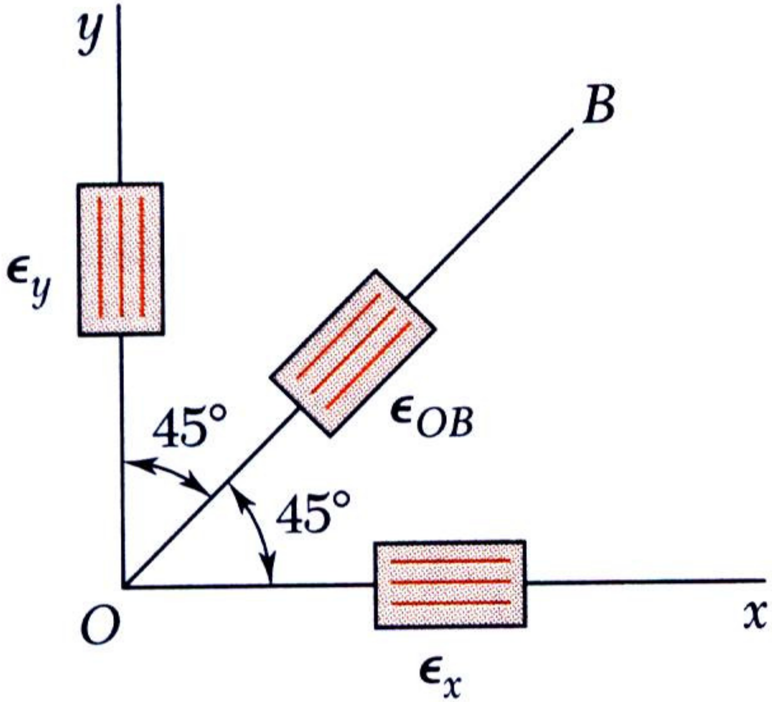
(a) 45^0 Rosette



(a) Delta Rosette

Three-Dimensional Analysis of Strain

Strain Measurement and Rosette Analysis



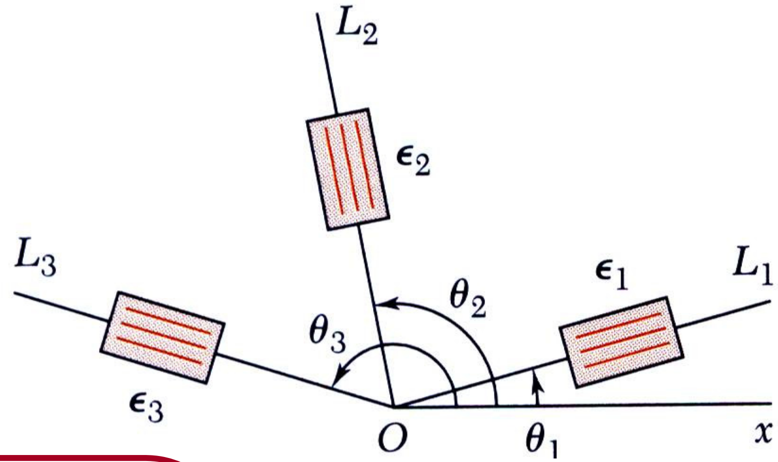
• With a 45° rosette, ε_x and ε_y are measured directly. γ_{xy} is obtained indirectly with,

$$\gamma_{xy} = 2\varepsilon_{OB} - \left(\varepsilon_x + \varepsilon_y\right)$$

Three-Dimensional Analysis of Strain

Strain Measurement and Rosette Analysis

•Normal and shearing strains may be obtained from normal strains in any three directions,



$$\varepsilon_1 = \varepsilon_x \cos^2\theta_1 + \varepsilon_y \sin^2\theta_1 + \gamma_{xy} \sin\theta_1 \cos\theta_1$$

$$\varepsilon_2 = \varepsilon_x \cos^2\theta_2 + \varepsilon_y \sin^2\theta_2 + \gamma_{xy} \sin\theta_2 \cos\theta_2$$

$$\varepsilon_3 = \varepsilon_x \cos^2\theta_3 + \varepsilon_y \sin^2\theta_3 + \gamma_{xy} \sin\theta_3 \cos\theta_3$$

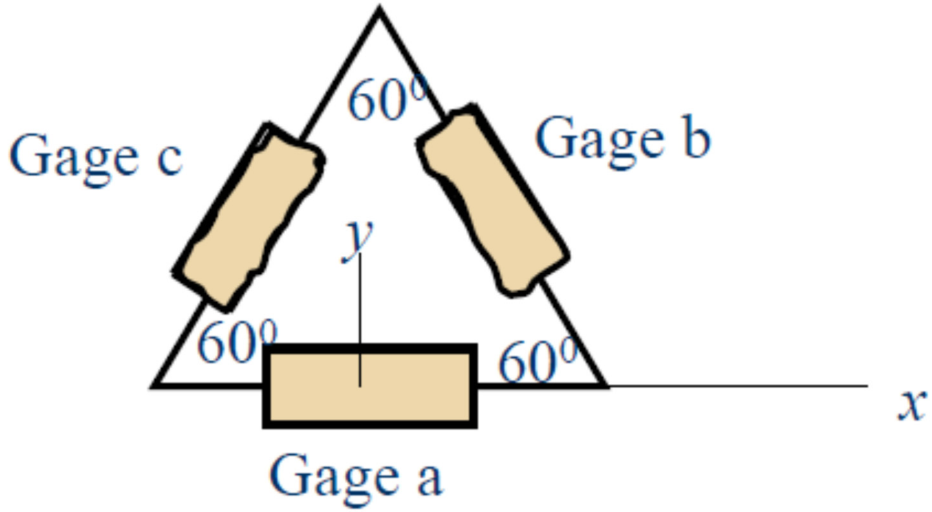
Three-Dimensional Analysis of Strain

Example 13

At a point on the free surface of an aluminum alloy machine part, the strain rosette shown in following was used to obtain this normal strain data:

$$\varepsilon_a = +780\mu$$
$$\varepsilon_b = +345\mu$$
$$\varepsilon_c = -332\mu$$

$$E = 73\,Gpa$$
$$\nu = 0.33$$



Determine:

(a) the strain components.

(b) the principal strains and maximum shearing strain.

❑ Three-Dimensional Analysis of Strain

Example 13

Three-Dimensional Analysis of Strain

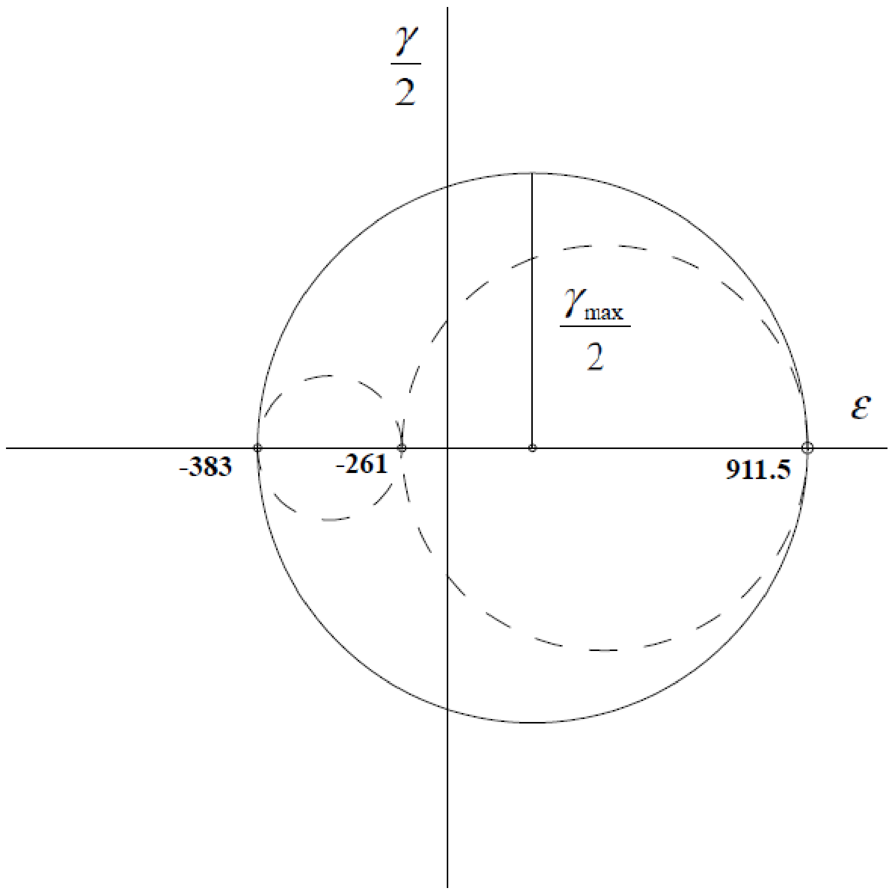
Example 13

Solving these equations yields

Principle Strains

Three-Dimensional Analysis of Strain

Example 13



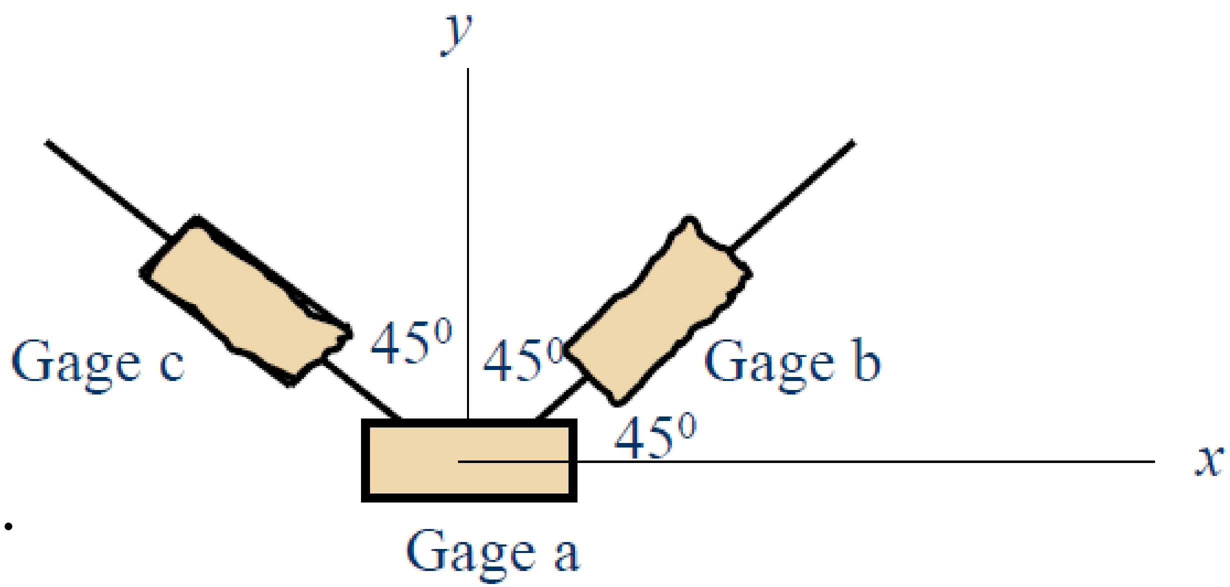
❑ Three-Dimensional Analysis of Strain

Example 14

The strain rosette was used to obtain normal strain data at a point on the free surface of aluminum alloy structural component. rosette was used to obtain this normal strain data:

$$\varepsilon_a = +525\mu$$
$$\varepsilon_b = +450\mu$$
$$\varepsilon_c = +1425\mu$$

$$E = 73\,Gpa$$
$$G = 28\,Gpa$$



Determine:

(a) The strain components.

(b) The stress components.

(c) The principal stresses and maximum shearing stress at the point

❑ Three-Dimensional Analysis of Strain

Example 14

Three-Dimensional Analysis of Strain

Example 14

Solving these equations yields

Principle Strains

❑ Three-Dimensional Analysis of Strain

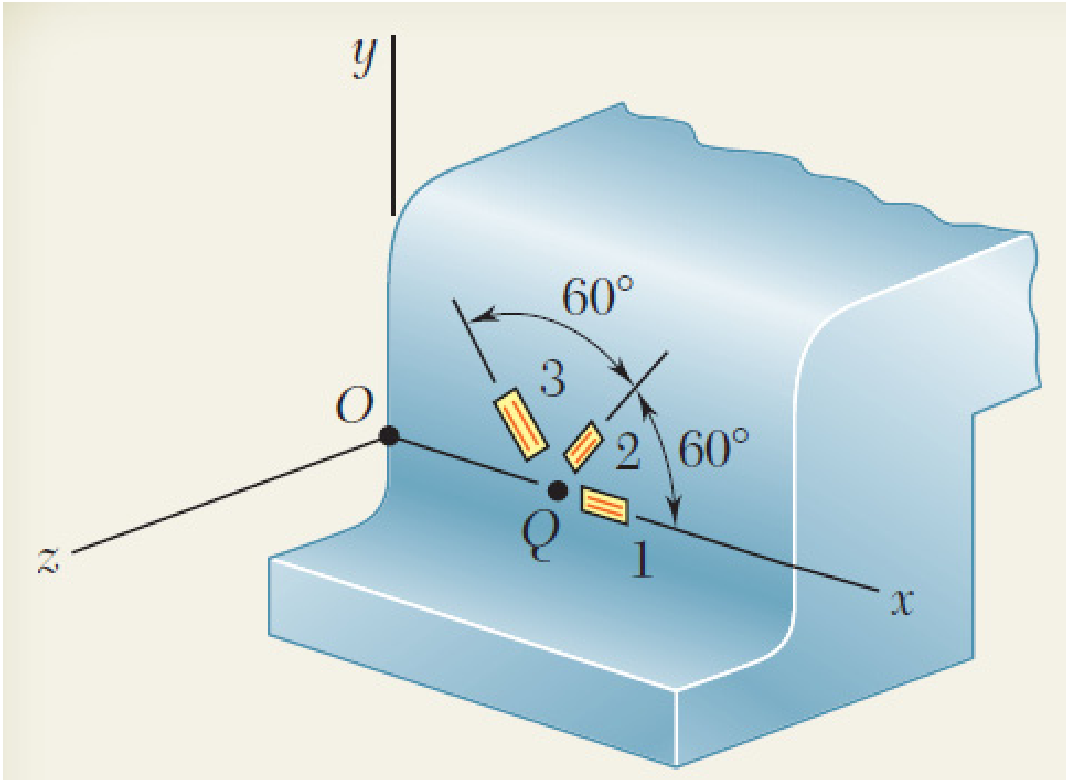
Example 14

Three-Dimensional Analysis of Strain

Example 14

Three-Dimensional Analysis of Strain

Example 15



$$\varepsilon_1 = 40\mu$$
$$\varepsilon_2 = 980\mu$$
$$\varepsilon_3 = 330\mu$$

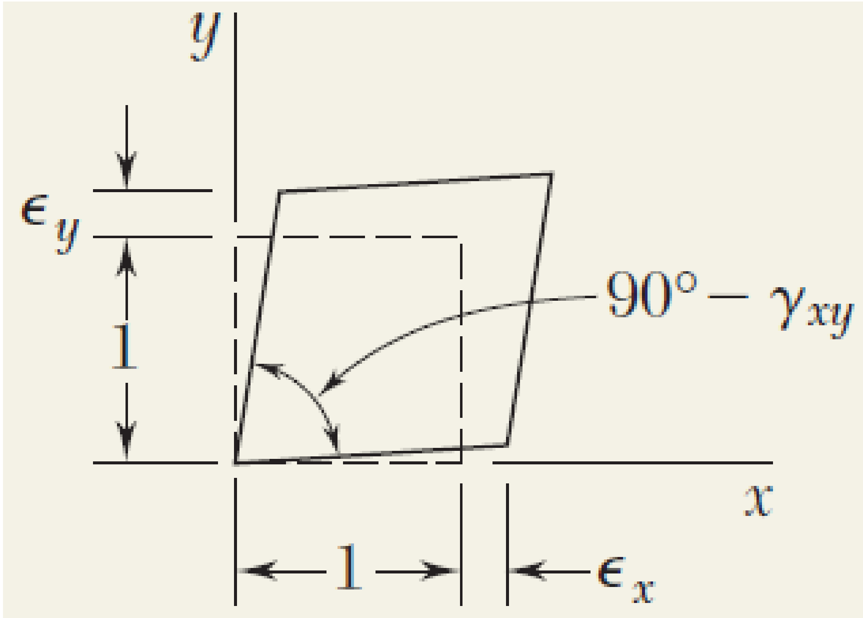
Determine:
(a) The strain components.
(b) The principle strains.
(c) Maximum shearing strain.

❑ Three-Dimensional Analysis of Strain

Example 15

❑ Three-Dimensional Analysis of Strain

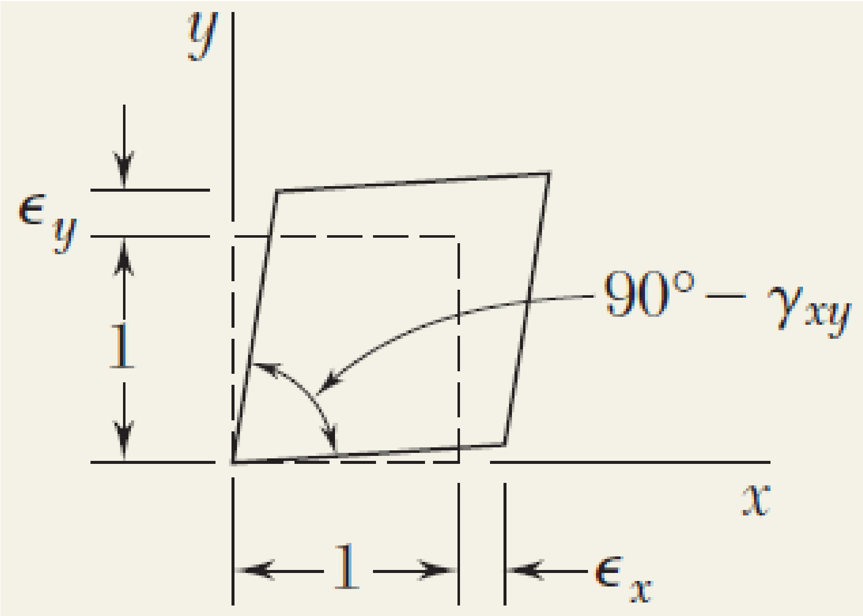
Example 15

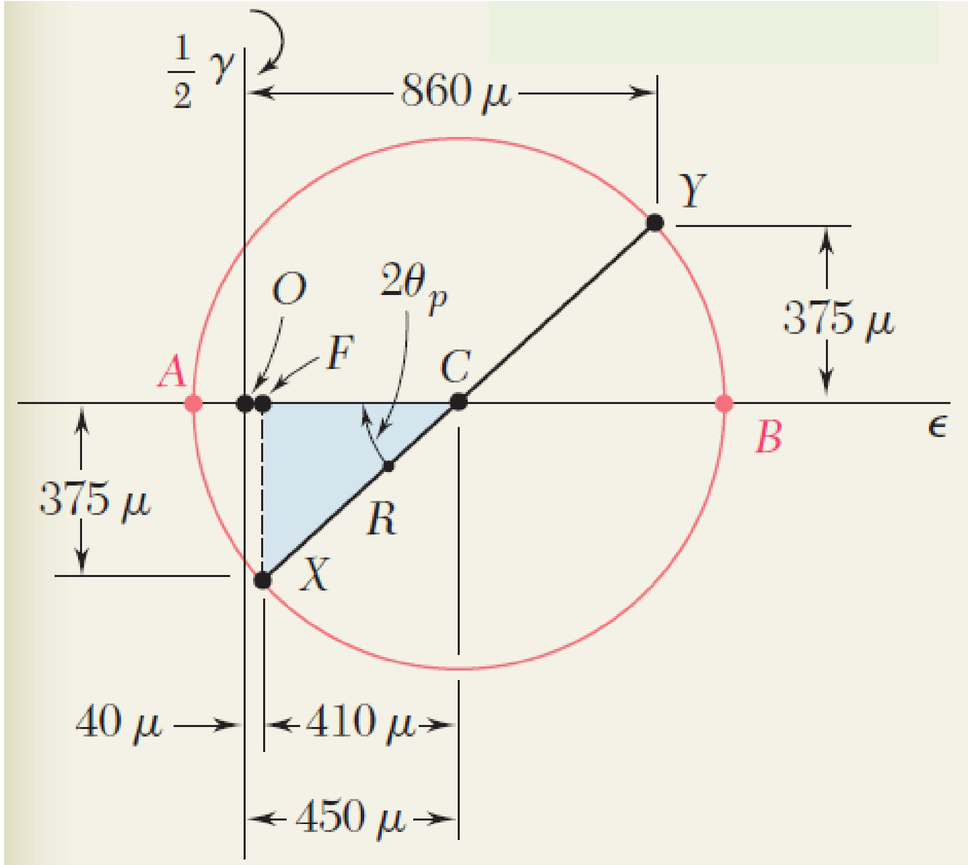


Solving these equations yields

❑ Three-Dimensional Analysis of Strain

Example 15

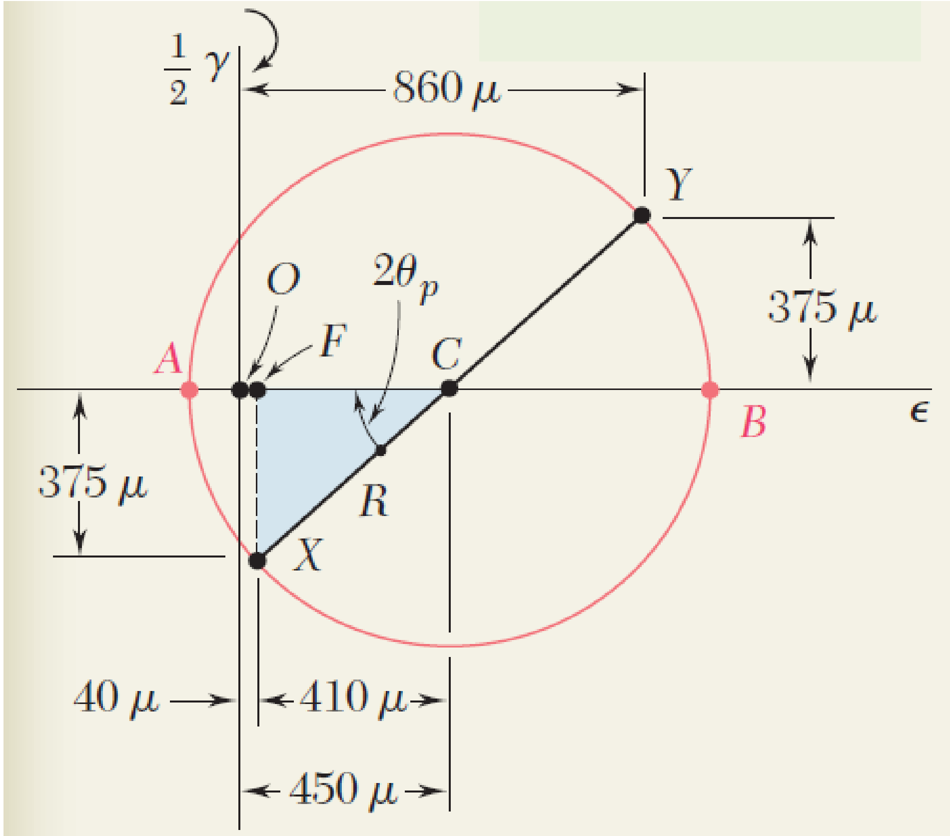




Counterclockwise **Clockwise**

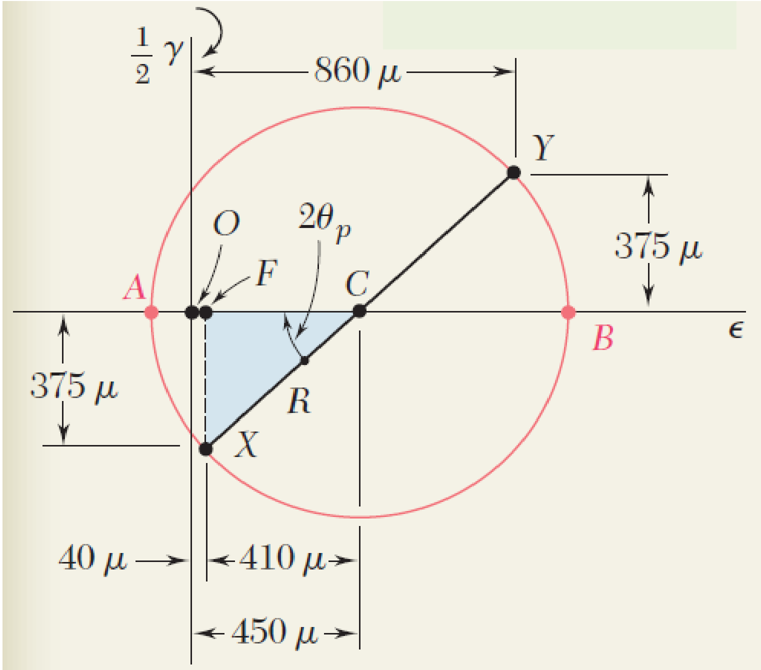
Three-Dimensional Analysis of Strain

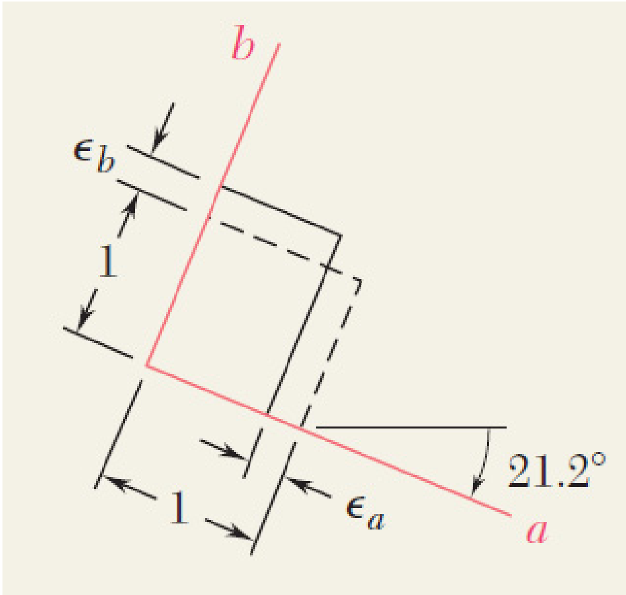
Example 15



❑ Three-Dimensional Analysis of Strain

Example 15





❑ Three-Dimensional Analysis of Strain

Example 15