STATICS

دانشگاه کردستان
University of Kurdistan
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- Vector Mechanics for Engineers: Statics, 9th edition. Ferdinand Beer- E. Russell Johnston Jr. - Phillip Cornwell.
- Engineering Mechanics-Statics, 5th Edition. J. L. Meriam, L. G. Kraige.
- Other Reference: Brain P.Self "Lectures notes on Statics"

Distributed Forces: Centroids and Centers of Gravity

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Distributed Forces: Centroids and Centers of Gravity

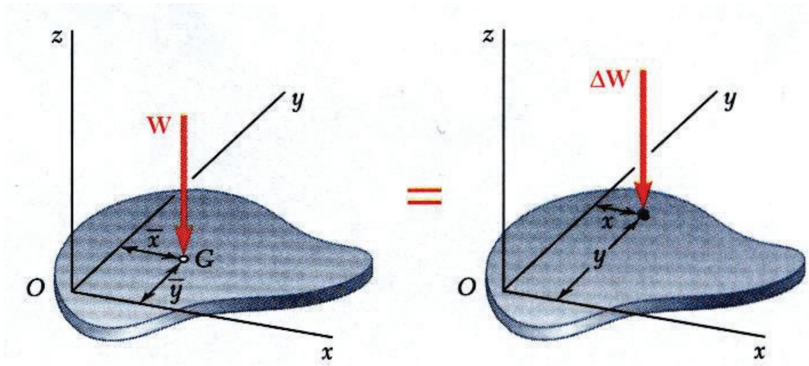
Introduction

- The earth exerts a gravitational force on each of the particles forming a body. These forces can be replace by a single equivalent force equal to the weight of the body and applied at the ***center of gravity*** for the body.
- The ***centroid*** *of an area* is analogous to the center of gravity of a body. The concept of the ***first moment of an area*** is used to locate the centroid.
- Determination of the area of a ***surface of revolution*** and the volume of a *body of revolution* are accomplished with the *Theorems of Pappus-Guldinus.*

Distributed Forces: Centroids and Centers of Gravity

❑ Center of Gravity of a 2D Body

- Center of gravity of a plate



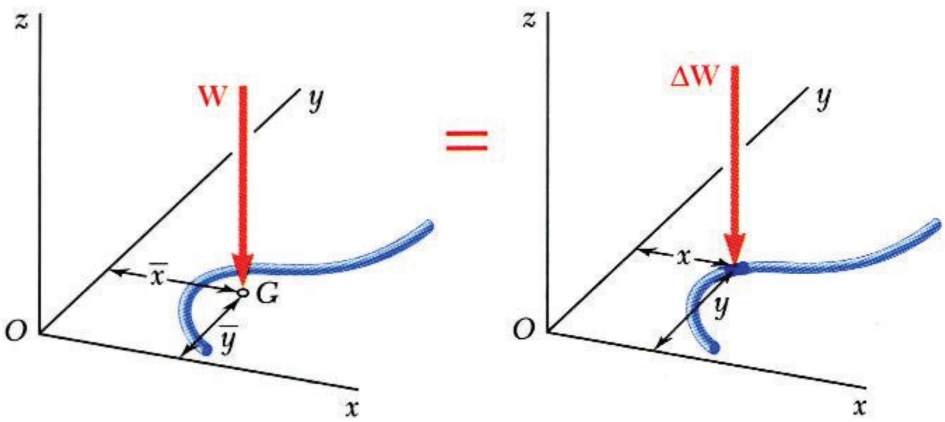
$$\sum F_z : \quad \Rightarrow \quad W = \Delta W_1 + \Delta W_1 + \cdots + \Delta W_n$$

$$\sum M_y : \quad \Rightarrow \quad \bar{x} W = \sum x \Delta W \quad \Rightarrow \quad \boxed{\bar{x} = \frac{\sum x \Delta W}{W} \quad or \quad \bar{x} = \frac{\int x\, dW}{W}}$$

$$\sum M_x : \quad \Rightarrow \quad \bar{y} W = \sum y \Delta W \quad \Rightarrow \quad \boxed{\bar{y} = \frac{\sum y \Delta W}{W} \quad or \quad \bar{y} = \frac{\int y\, dW}{W}}$$

❑ Center of Gravity of a 2D Body

- Center of gravity of a wire



$$\sum M_y : \quad \Rightarrow \quad \bar{x} W = \sum x \Delta W \quad \Rightarrow \quad \boxed{\bar{x} = \frac{\sum x \Delta W}{W} \quad or \quad \bar{x} = \frac{\int x\, dW}{W}}$$

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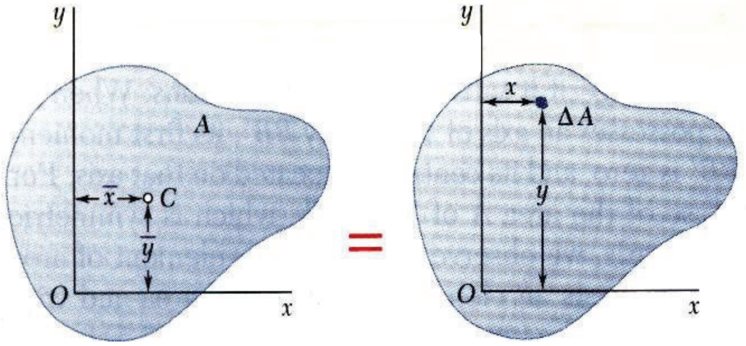
Distributed Forces: Centroids and Centers of Gravity

Centroids and First Moments of Areas

- Centroid of an area



$$\Delta W = \gamma t \Delta A \quad \Rightarrow \quad \boxed{W = \gamma t A}$$

$$\bar{x} = \frac{\int x\,dW}{W} \quad \Rightarrow \quad \bar{x} = \frac{\int x(\gamma t)dA}{(\gamma A t)} \quad \Rightarrow \quad \boxed{\bar{x} = \frac{\int x\,dA}{A}}$$

$$if \quad \boxed{Q_y = \int x\,dA \quad \Rightarrow \quad Q_y = \bar{x}A}$$

Q_y : First moment with respect to y axis

$$\bar{y} = \frac{\int y\,dW}{W} \quad \Rightarrow \quad \bar{y} = \frac{\int y(\gamma t)dA}{(\gamma A t)} \quad \Rightarrow \quad \boxed{\bar{y} = \frac{\int y\,dA}{A}}$$

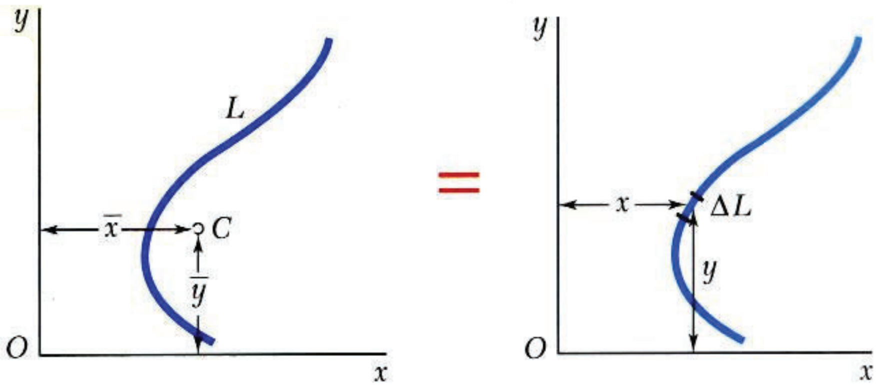
$$if \quad \boxed{Q_x = \int y\,dA \quad \Rightarrow \quad Q_x = \bar{y}A}$$

Q_x : First moment with respect to x axis

Distributed Forces: Centroids and Centers of Gravity

Centroids and First Moments of Lines

- Centroid of a line



$$\Delta W = \gamma a \Delta L \quad \Rightarrow \quad \boxed{W = \gamma a L}$$

a : Cross section area

$$\bar{x} = \frac{\int x\,dW}{W} \quad \Rightarrow \quad \bar{x} = \frac{\int x(\gamma a)\,dL}{(\gamma L a)} \quad \Rightarrow \quad \boxed{\bar{x} = \frac{\int x\;dL}{L}}$$

$$\bar{y} = \frac{\int y\,dW}{W} \quad \Rightarrow \quad \bar{y} = \frac{\int y(\gamma a)\,dL}{(\gamma L a)} \quad \Rightarrow \quad \boxed{\bar{y} = \frac{\int y\;dL}{L}}$$

$$dL = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

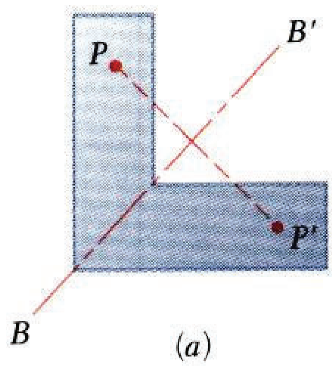
$$dL = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$

$$dL = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

Distributed Forces: Centroids and Centers of Gravity

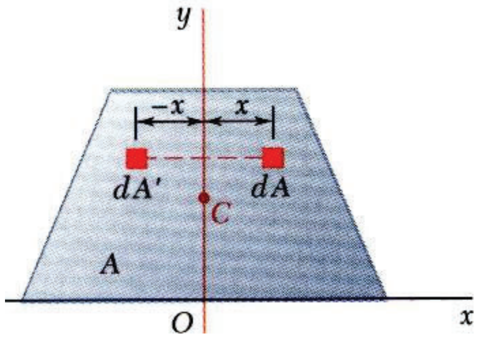
❑ First Moments of Areas and Lines

- An area is symmetric with respect to an axis *BB'* if for every point *P* there exists a point *P'* such that *PP'* is perpendicular to *BB'* and is divided into two equal parts by *BB'*.



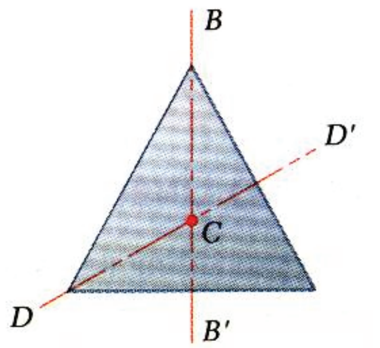
- The first moment of an area with respect to a line of symmetry is zero.

- If an area possesses a line of symmetry, its centroid lies on that axis



Distributed Forces: Centroids and Centers of Gravity

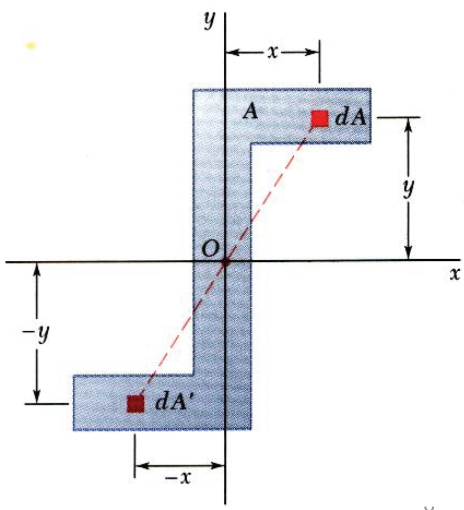
❑ First Moments of Areas and Lines



- If an area possesses two lines of symmetry, its centroid lies at their intersection.

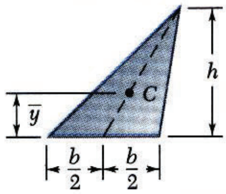
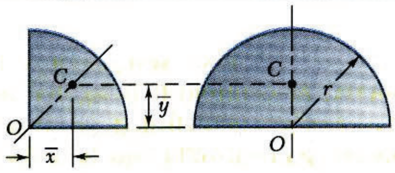
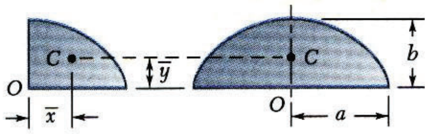
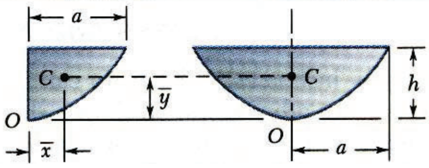
- An area is symmetric with respect to a center *O* if for every element *dA* at (*x,y*) there exists an area *dA'* of equal area at (*-x,-y*).

- The centroid of the area coincides with the center of symmetry.



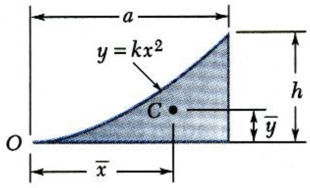
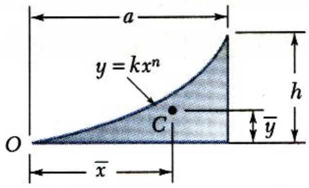
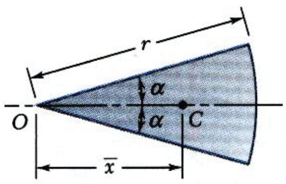
Distributed Forces: Centroids and Centers of Gravity

❑ Centroids of Common Shapes of Areas

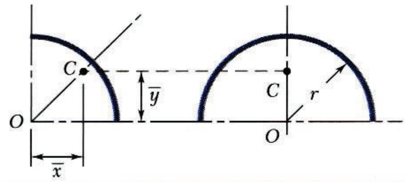
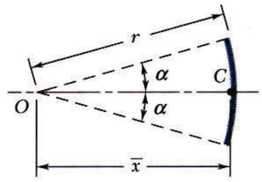
Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$

Distributed Forces: Centroids and Centers of Gravity

❑ Centroids of Common Shapes of Areas

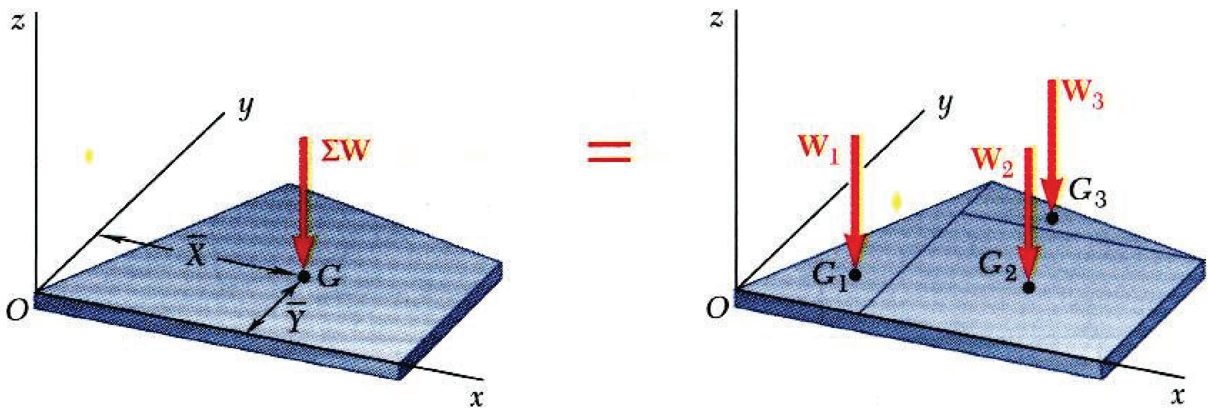
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
General spandrel		$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$	$\frac{ah}{n+1}$
Circular sector		$\frac{2r \sin \alpha}{3\alpha}$	0	αr^2

❑ Centroids of Common Shapes of Lines

Shape		$\bar{x}$	$\bar{y}$	Length
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc		0	$\frac{2r}{\pi}$	πr
Arc of circle		$\frac{r \sin \alpha}{\alpha}$	0	$2\alpha r$

❑ Composite Plates

- Composite plates

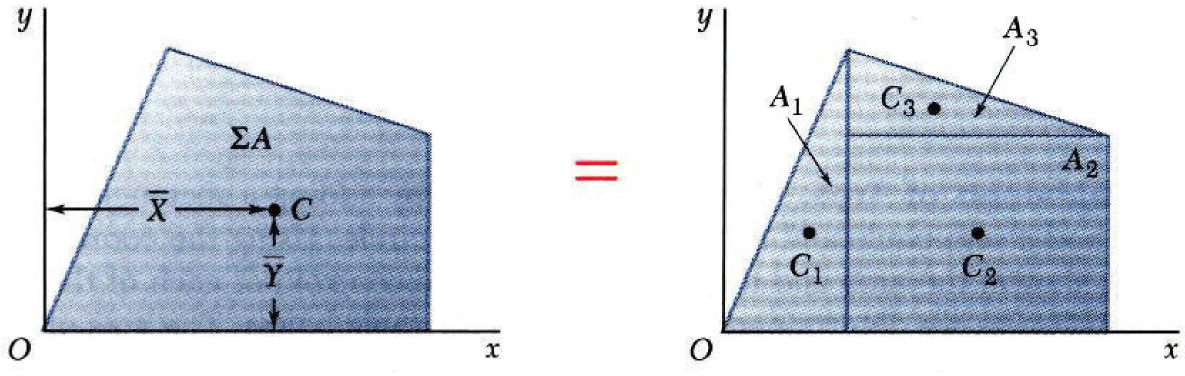


$$\bar{X} = \frac{\sum \bar{x} W}{\sum W} \qquad \bar{Y} = \frac{\sum \bar{y} W}{\sum W}$$

Distributed Forces: Centroids and Centers of Gravity

❑ Composite Areas

- Composite area

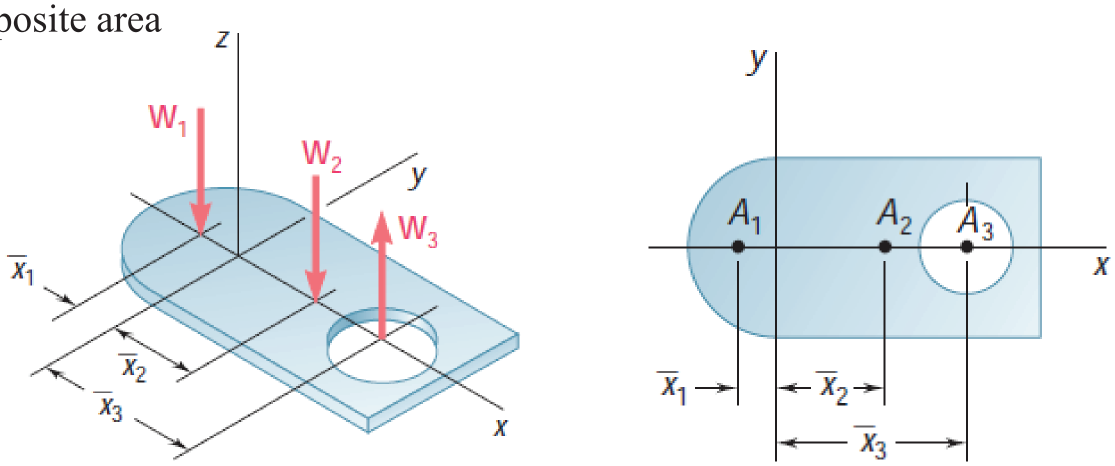


$$\overline{X} = \frac{\sum \bar{x} A}{\sum A} \qquad \overline{Y} = \frac{\sum \bar{y} A}{\sum A}$$

Distributed Forces: Centroids and Centers of Gravity

❑ Composite Areas

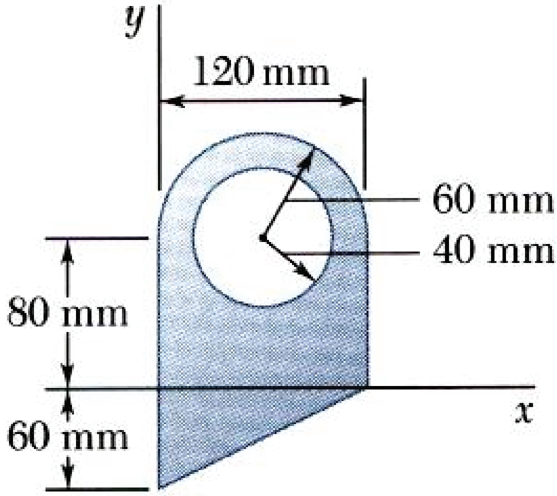
- Composite area



	$\bar{x}$	A	$\bar{x}A$
A_1 Semicircle	–	+	–
A_2 Full rectangle	+	+	+
A_3 Circular hole	+	–	–

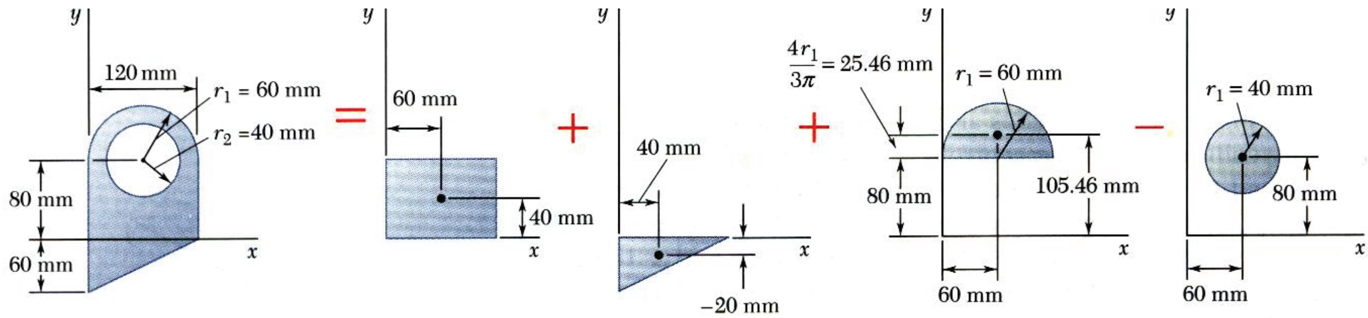
❑ Sample Problem 01

For the plane area shown, determine the first moments with respect to the x and y axes and the location of the centroid.



❑ Sample Problem 01

SOLUTION:



Component	A, mm²	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm³	$\bar{y}A$, mm³
Rectangle					
Triangle					
Semicircle					
Circle					

- Find the total area and first moments of the triangle, rectangle, and semicircle. Subtract the area and first moment of the circular cutout.

$$Q_x = +506.2\times10^3\,\text{mm}^3$$
$$Q_y = +757.7\times10^3\,\text{mm}^3$$

Distributed Forces: Centroids and Centers of Gravity

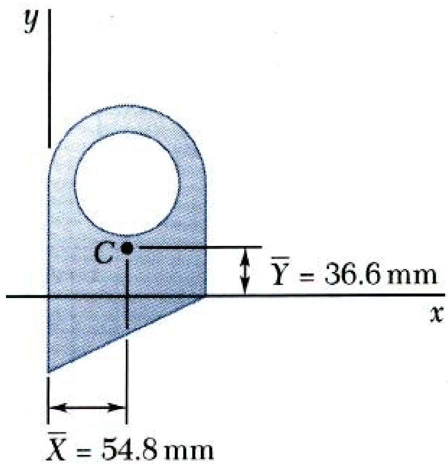
❑ Sample Problem 01

SOLUTION:

- Compute the coordinates of the area centroid by dividing the first moments by the total area.

$$\bar{X} = 54.8\,(mm)$$

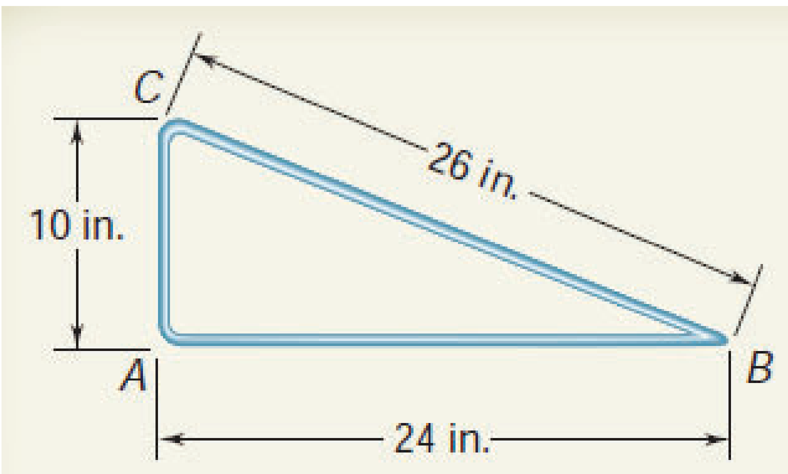
$$\bar{Y} = 36.6\,(mm)$$



Distributed Forces: Centroids and Centers of Gravity

❑ Sample Problem 02

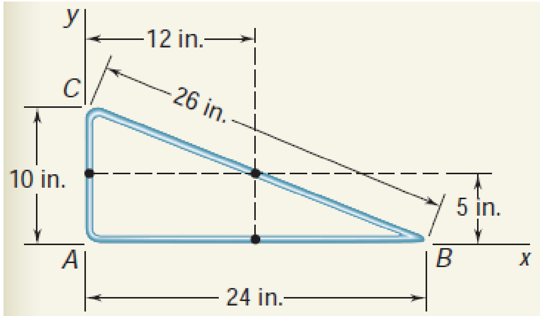
The figure shown is made from a piece of thin, homogeneoius wire Determine the location of its center of gravity.



Distributed Forces: Centroids and Centers of Gravity

❑ Sample Problem 02

SOLUTION:



Segment	L, in.	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}L$, in^2	$\bar{y}L$, in^2
AB					
BC					
CA					

$\bar{X} = 10\,(in.)$

$\bar{Y} = 3\,(in.)$

Distributed Forces: Centroids and Centers of Gravity

❑ Determination of Centroids by Integration

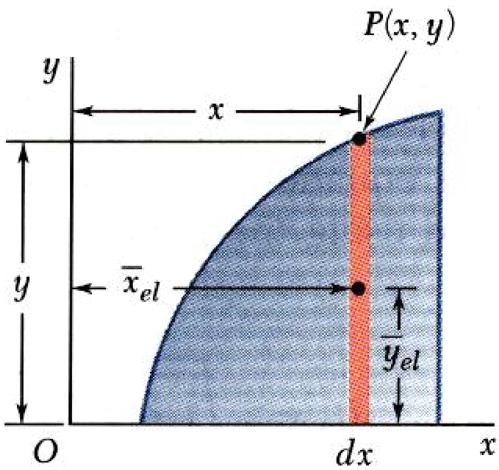
- Double integration to find the first moment may be avoided by defining dA as a thin rectangle or strip.

$$Q_y = \bar{x}A = \int x dA \Rightarrow Q_y = \iint x\,dxdy \quad or \quad Q_y = \int \bar{x}_{el}\,dA$$

$$Q_x = \bar{y}A = \int y dA \Rightarrow Q_x = \iint y\,dxdy \quad or \quad Q_x = \int \bar{y}_{el}\,dA$$

$$Q_y = \bar{x}A = \int \bar{x}_{el}\,dA \Rightarrow Q_y = \int x\,(ydx)$$

$$Q_x = \bar{y}A = \int \bar{y}_{el}\,dA \Rightarrow Q_x = \int \frac{y}{2}(ydx)$$

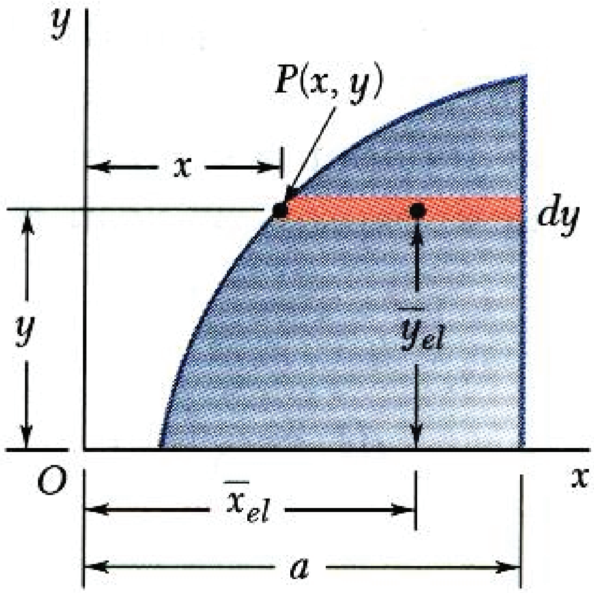


Distributed Forces: Centroids and Centers of Gravity

❑ Determination of Centroids by Integration

$$Q_y = \bar{x}A = \int \bar{x}_{el}\, dA \quad \Rightarrow \quad \boxed{Q_y = \int \frac{a+x}{2}[(a-x)\, dy]}$$

$$Q_x = \bar{y}A = \int \bar{y}_{el}\, dA \quad \Rightarrow \quad \boxed{Q_x = \int y[(a-x)\, dy]}$$

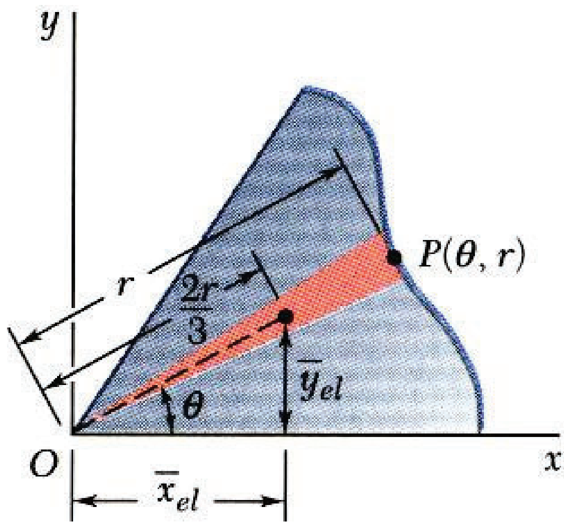


Distributed Forces: Centroids and Centers of Gravity

❑ Determination of Centroids by Integration

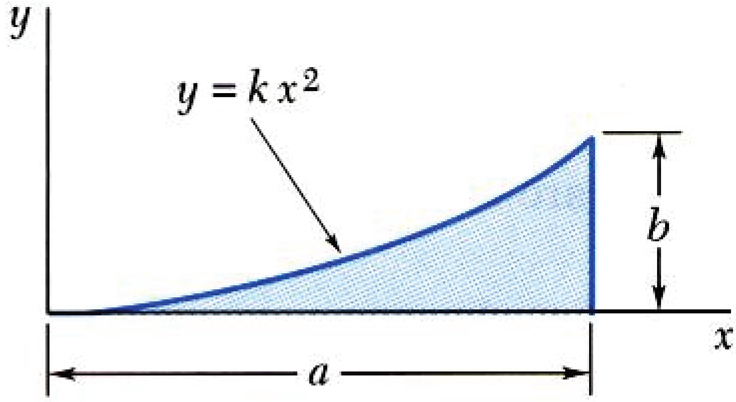
$$Q_y = \bar{x}A = \int \bar{x}_{el}\, dA \quad \Rightarrow \quad \boxed{Q_y = \int \frac{2r}{3}\cos\theta\left(\frac{1}{2}r^2 d\theta\right)}$$

$$Q_x = \bar{y}A = \int \bar{y}_{el}\, dA \quad \Rightarrow \quad \boxed{Q_x = \int \frac{2r}{3}\sin\theta\left(\frac{1}{2}r^2 d\theta\right)}$$



❑ Sample Problem 03

Determine by direct integration the location of the centroid of a parabolic spandrel.

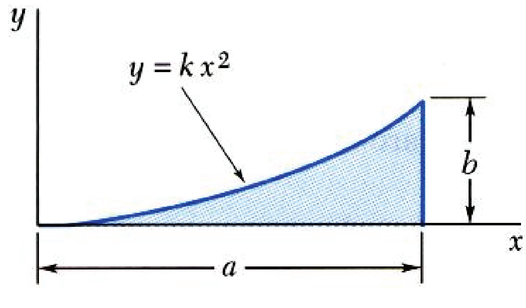


❑ Sample Problem 03

SOLUTION:

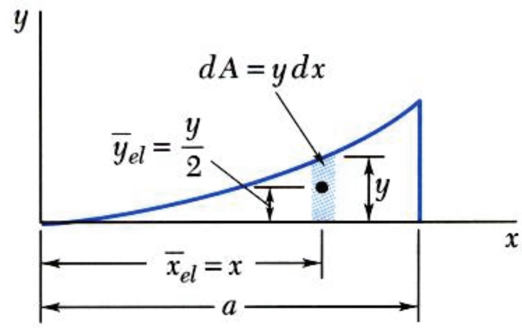
- Determine the constant k.

$$\Rightarrow \boxed{y = \frac{b}{a^2}x^2 \quad or \quad x = \frac{a}{b^{1/2}}y^{1/2}}$$



- Evaluate the total area.

$$\boxed{A = \frac{ab}{3}}$$



Distributed Forces: Centroids and Centers of Gravity

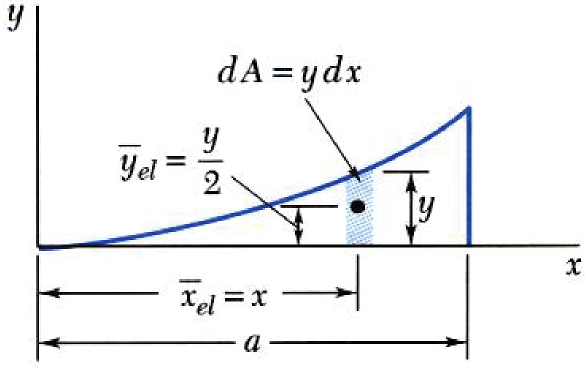
❑ Sample Problem 03

SOLUTION:

- Using vertical strips, perform a single integration to find the first moments.

$$Q_y = \frac{a^2 b}{4}$$

$$Q_x = \frac{ab^2}{10}$$



Distributed Forces: Centroids and Centers of Gravity

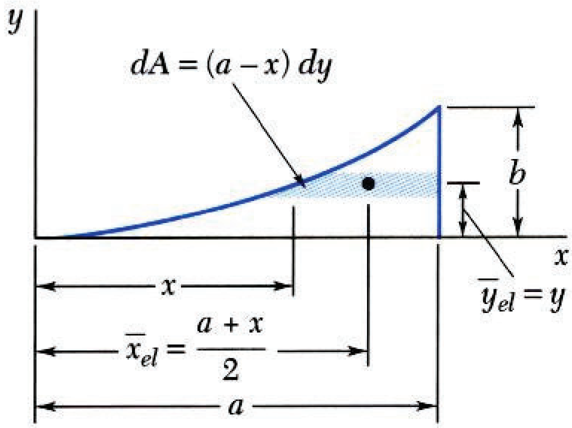
❑ Sample Problem 03

SOLUTION:

- Or, using horizontal strips, perform a single integration to find the first moments.

$$Q_y = \frac{a^2 b}{4}$$

$$Q_x = \frac{ab^2}{10}$$



Distributed Forces: Centroids and Centers of Gravity

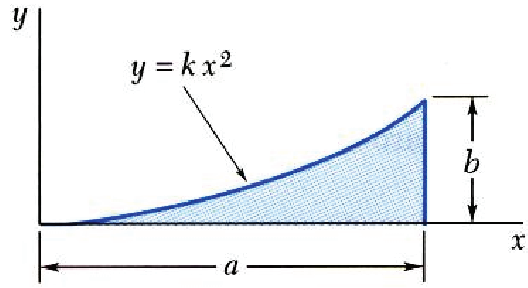
❑ Sample Problem 03

SOLUTION:

- Evaluate the centroid coordinates.

$$\bar{x} = \frac{3}{4}a$$

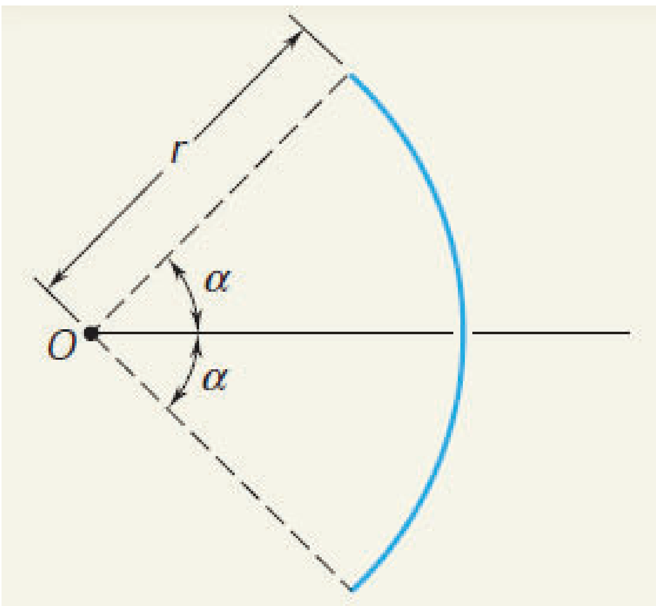
$$\bar{y} = \frac{3}{10}b$$



Distributed Forces: Centroids and Centers of Gravity

❑ Sample Problem 04

Determine the location of the centroid of the circular arc shown.



Distributed Forces: Centroids and Centers of Gravity

❑ Sample Problem 04

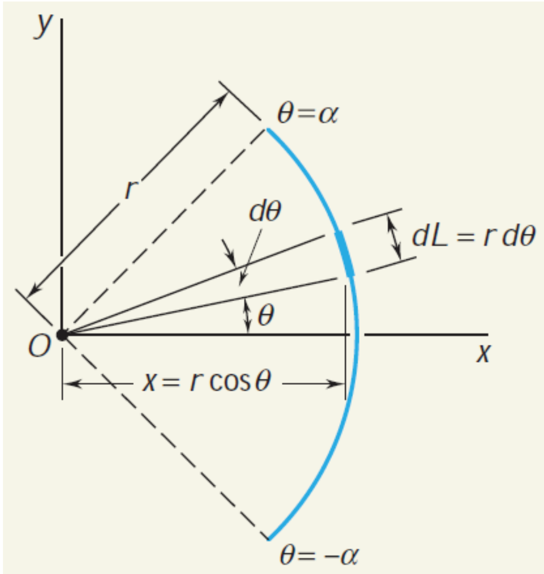
SOLUTION:

- Since the arc is symmetrical with respect to the x axis, $\bar{y}=0$. A differential element is chosen as shown, and the length or the arc L determined by integration

$$L = 2r\alpha$$

$$\Rightarrow Q_y = 2r^2 \sin\alpha$$

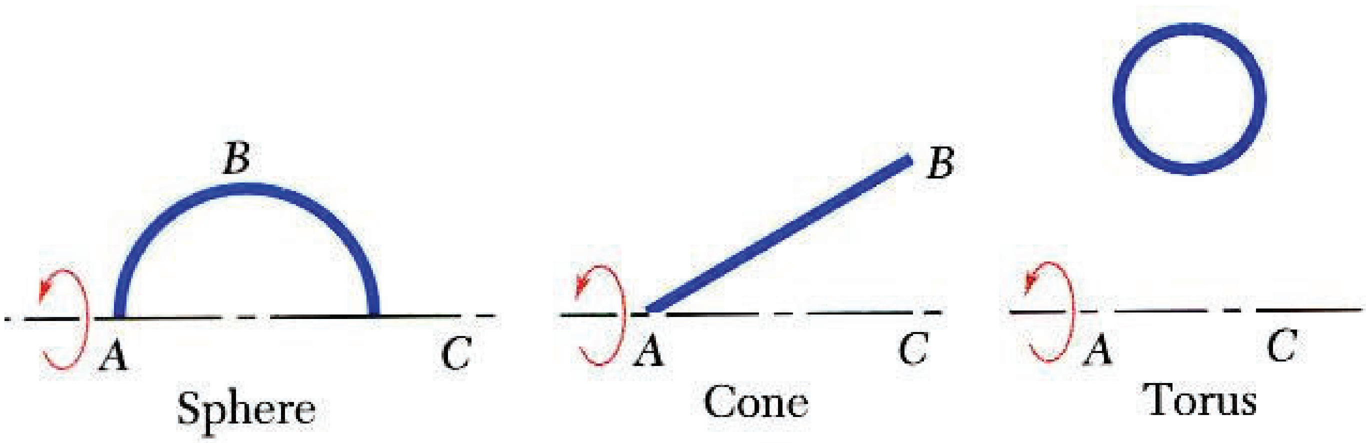
$$\bar{x} = \frac{r\sin\alpha}{\alpha}$$



Distributed Forces: Centroids and Centers of Gravity

❑ Theorems of Pappus-Guldinus

- ***Surface of revolution*** is generated by rotating a plane curve about a fixed axis.



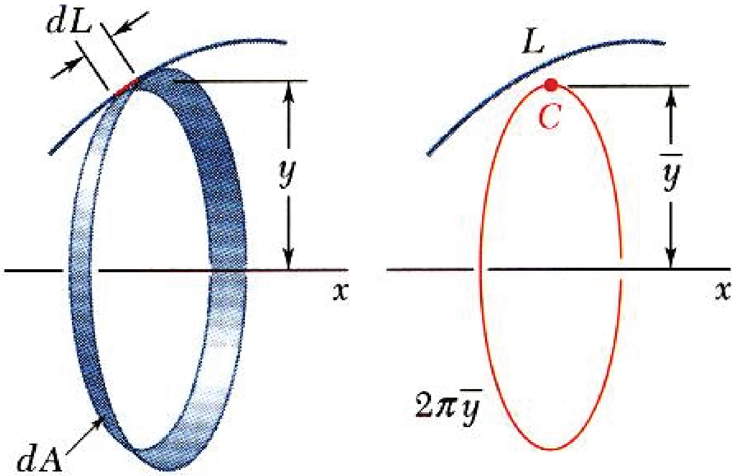
Distributed Forces: Centroids and Centers of Gravity

❑ Theorems of Pappus-Guldinus

- ***Area of a surface of revolution is equal to the length of the generating curve times the distance traveled by the centroid through the rotation.***

$$dA = 2\pi y\, dL \quad \Rightarrow \quad A = 2\pi \int y\, dL$$

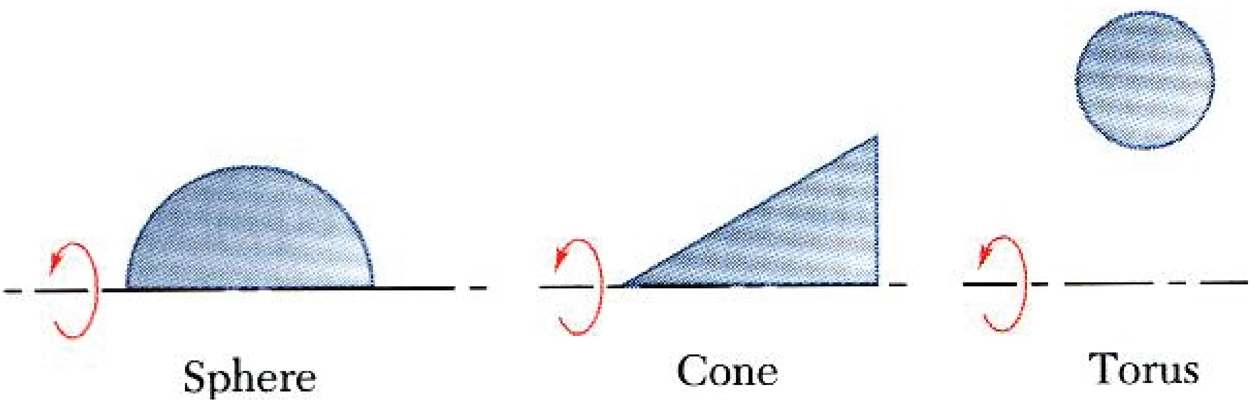
$$\Rightarrow \boxed{A = 2\pi \bar{y} L}$$



Distributed Forces: Centroids and Centers of Gravity

❑ Theorems of Pappus-Guldinus

- Body of revolution is generated by rotating a plane area about a fixed axis.



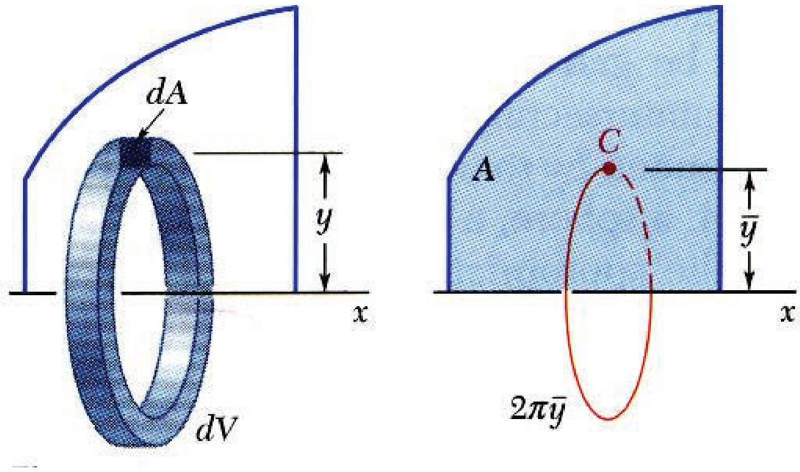
Distributed Forces: Centroids and Centers of Gravity

❑ Theorems of Pappus-Guldinus

- ***Volume of a body of revolution is equal to the generating area times the distance traveled by the centroid through the rotation.***

$$dV = 2\pi y \, dA \quad \Rightarrow \quad V = 2\pi \int y \, dA$$

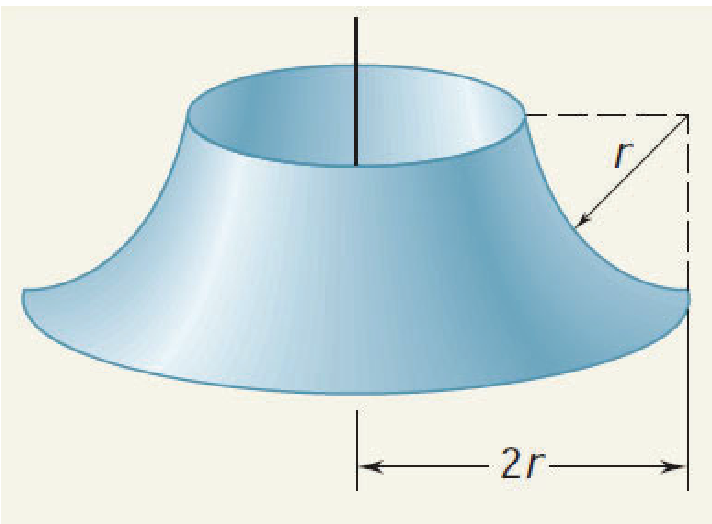
$$\Rightarrow \boxed{V = 2\pi \bar{y} A}$$



Distributed Forces: Centroids and Centers of Gravity

❑ Sample Problem 05

Determine the area of the surface of revolution shown, which is obtained by rotating a quarter-circular arc about a vertical axis.



Distributed Forces: Centroids and Centers of Gravity

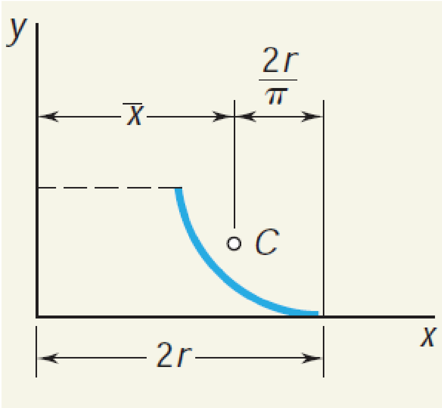
❑ Sample Problem 05

SOLUTION:

According to Theorem I of Pappu -Guldinus, the area generated is equal to the product of tlle length of the arc and Ule distance traveled by its centroid.

$$\bar{x} = 2r\left(1 - \frac{1}{\pi}\right)$$

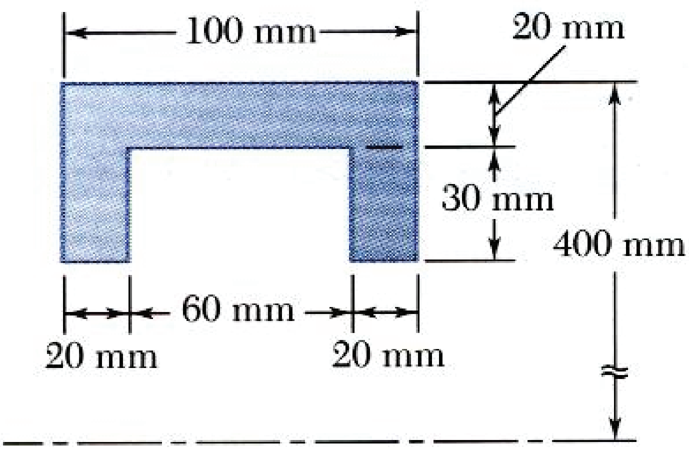
$$A = 2\pi r^2(\pi - 1)$$



Distributed Forces: Centroids and Centers of Gravity

❑ Sample Problem 06

The outside diameter of a pulley is 0.8 m, and the cross section of its rim is as shown. Knowing that the pulley is made of steel and that the density of steel is $\rho = 7.85\times10^3\ \mathrm{kg/m^3}$. determine the mass and weight of the rim.



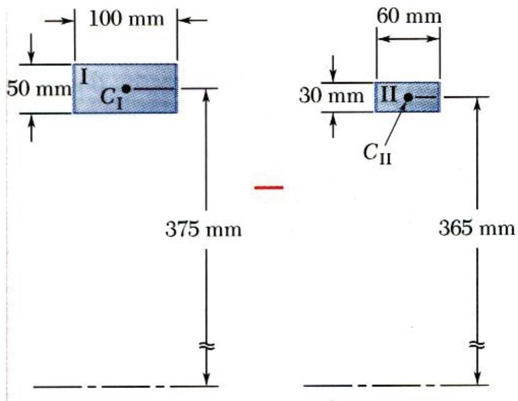
Distributed Forces: Centroids and Centers of Gravity

❑ Sample Problem 06

SOLUTION:

- Apply the theorem of Pappus-Guldinus to evaluate the volumes or revolution for the rectangular rim section and the inner cutout section.

- Multiply by density and acceleration to get the mass and weight.



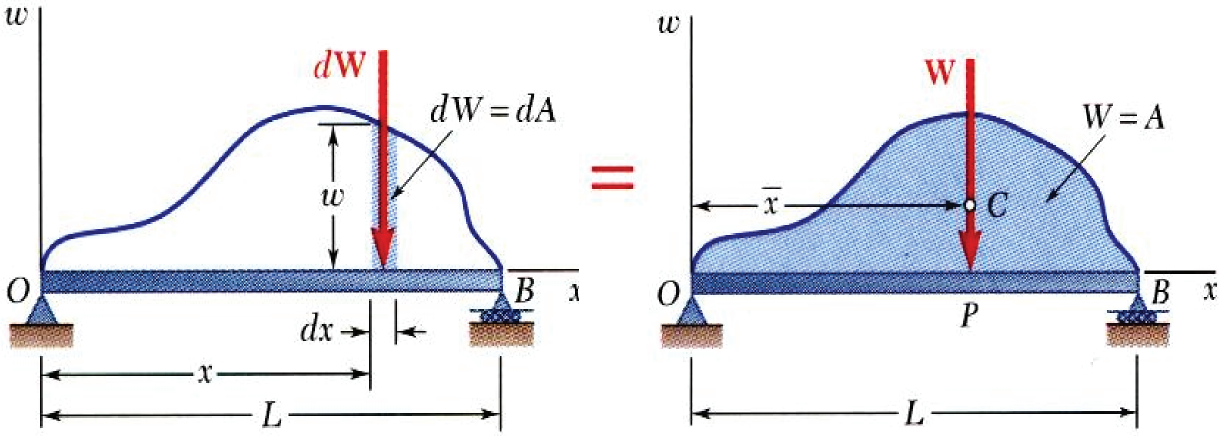
	Area, mm²	$\bar{y}$, mm	Distance Traveled by C, mm	Volume, mm³
I				
II				

$m = 60.0$ kg

$W = 589$ N

Distributed Forces: Centroids and Centers of Gravity

❑ Distributed Loads on Beams



- A distributed load is represented by plotting the load per unit length, w (N/m) . The total load is equal to the area under the load curve.

$$W = \int_0^L w\,dx = \int dA \quad \Rightarrow \quad \boxed{W = A}$$

- ***A distributed load can be replace by a concentrated load with a magnitude equal to the area under the load curve and a line of action passing through the area centroid.***

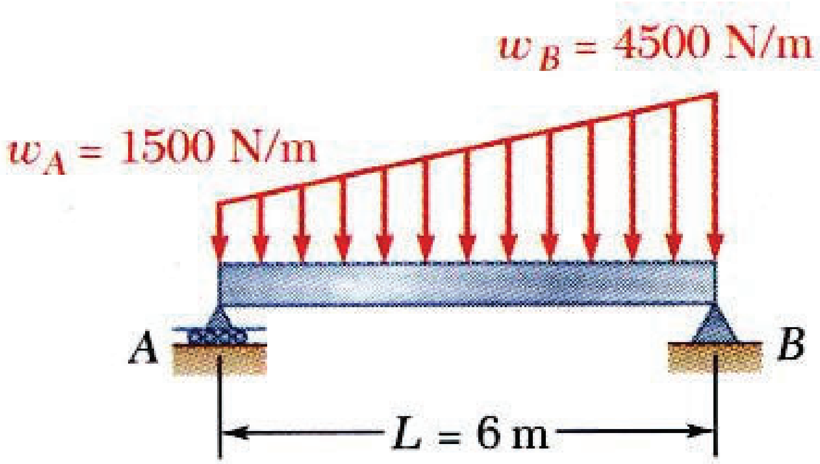
$$(OP)W = \int x\,dW \Rightarrow$$

$$(OP)\,A = \int_0^L x\,dA = \bar{x}A \quad \Rightarrow \quad \boxed{(OP) = \bar{x}}$$

Distributed Forces: Centroids and Centers of Gravity

❑ Sample Problem 07

A beam supports a distributed load as shown. Determine the equivalent concentrated load and the reactions at the supports.

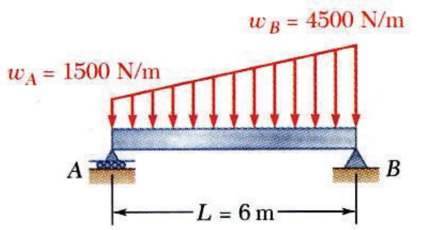


Distributed Forces: Centroids and Centers of Gravity

❑ Sample Problem 07

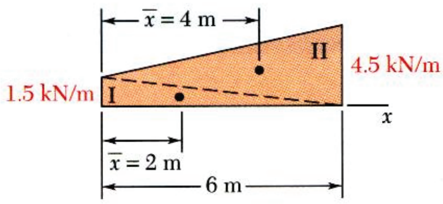
SOLUTION:

Component	A, kN	x̄, m	x̄A, kN·m
Triangle I Triangle II			



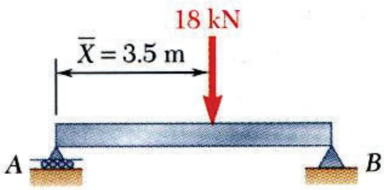
- The magnitude of the concentrated load is equal to the total load or the area under the curve.

$$F = 18.0 \text{ kN}$$



- The line of action of the concentrated load passes through the centroid of the area under the curve.

$$\bar{X} = 3.5 \text{ m}$$

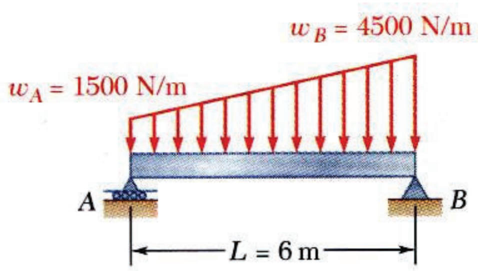


Distributed Forces: Centroids and Centers of Gravity

❑ Sample Problem 07

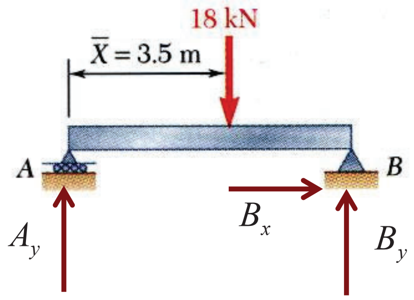
SOLUTION:

- Determine the support reactions by summing moments about the beam ends.



$$B_x = 0$$

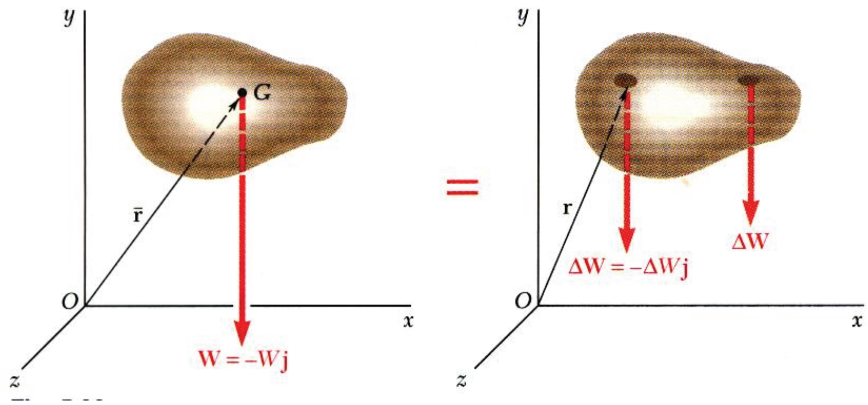
$$\Rightarrow \boxed{B_y = 10.5\ (\text{kN})}$$



$$\Rightarrow \boxed{A_y = 7.5\ (\text{kN})}$$

Distributed Forces: Centroids and Centers of Gravity

❑ Center of Gravity of a 3D Body: Centroid of a Volume



- Center of gravity G

$$-W\vec{j} = \sum(-\Delta W\vec{j})$$

$$\vec{r}_G \times (-W\vec{j}) = \sum[\vec{r} \times (-\Delta W\vec{j})]$$

$$\vec{r}_G W \times (-\vec{j}) = (\sum \vec{r}\Delta W) \times (-\vec{j})$$

$$W = \int dW \quad \Rightarrow \quad \boxed{\vec{r}_G W = \int \vec{r}\, dW}$$

- Results are independent of body orientation,

$$\boxed{\bar{x}W = \int x\,dW \quad \bar{y}W = \int y\,dW \quad \bar{z}W = \int z\,dW}$$

- For homogeneous bodies,

$$W = \gamma V \text{ and } dW = \gamma dV \quad \Rightarrow$$

$$\boxed{\bar{x}V = \int x\,dV \quad \bar{y}V = \int y\,dV \quad \bar{z}V = \int z\,dV}$$

Distributed Forces: Centroids and Centers of Gravity

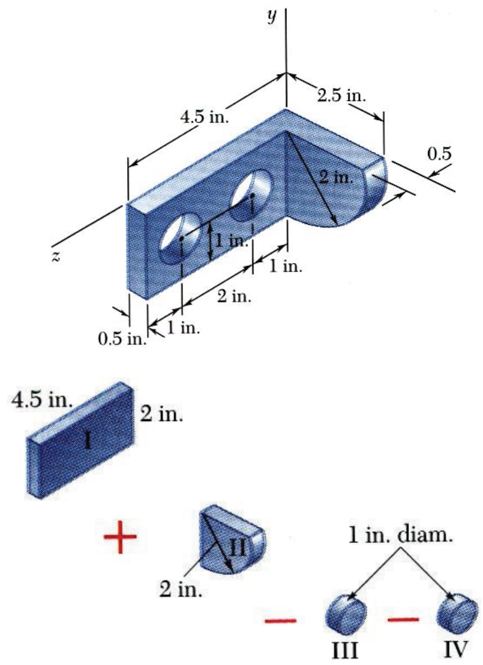
❑ Composite 3D Bodies

- Moment of the total weight concentrated at the center of gravity G is equal to the sum of the moments of the weights of the component parts.

$$\bar{X}\sum W = \sum \bar{x} W \quad \bar{Y}\sum W = \sum \bar{y} W \quad \bar{Z}\sum W = \sum \bar{z} W$$

- For homogeneous bodies,

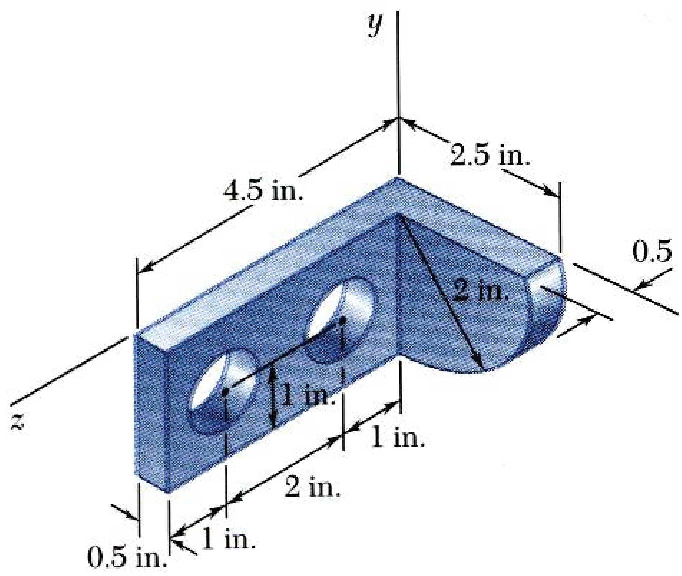
$$\bar{X}\sum V = \sum \bar{x} V \quad \bar{Y}\sum V = \sum \bar{y} V \quad \bar{Z}\sum V = \sum \bar{z} V$$



Distributed Forces: Centroids and Centers of Gravity

❑ Sample Problem 08

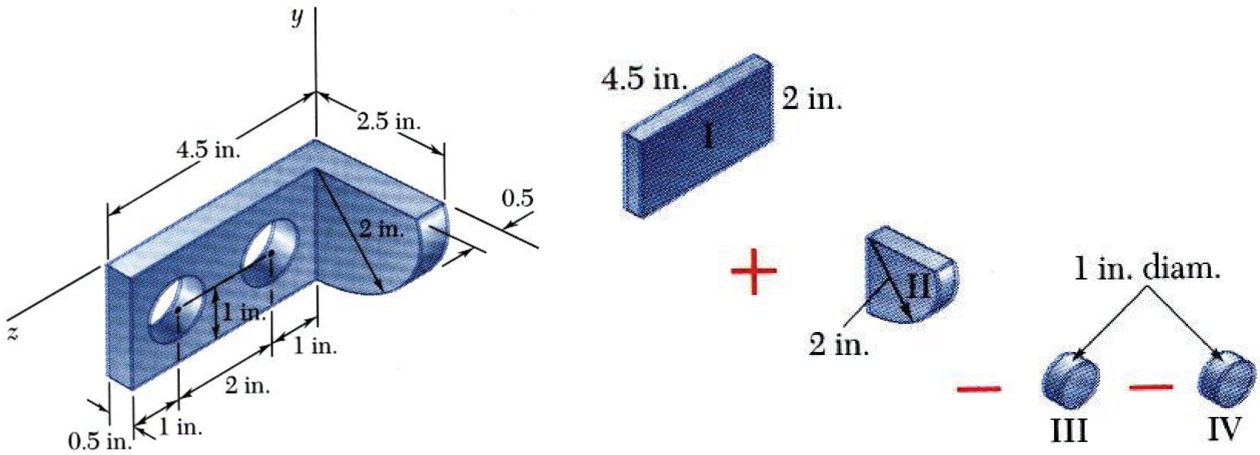
Locate the center of gravity of the steel machine element. The diameter of each hole is 1 in.



Distributed Forces: Centroids and Centers of Gravity

❑ Sample Problem 08

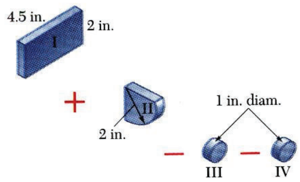
SOLUTION:

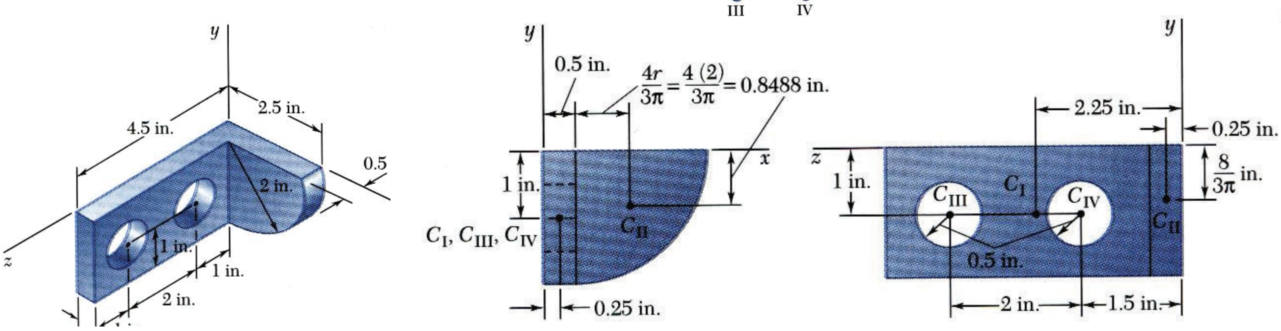


Distributed Forces: Centroids and Centers of Gravity

❑ Sample Problem 08

SOLUTION:





	V, in³	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{z}$, in.	$\bar{x}V$, in⁴	$\bar{y}V$, in⁴	$\bar{z}V$, in⁴
I II III IV							

Distributed Forces: Centroids and Centers of Gravity

❑ Sample Problem 08

SOLUTION:

	V, in^3	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{z}$, in.	$\bar{x}V$, in^4	$\bar{y}V$, in^4	$\bar{z}V$, in^4
I	$(4.5)(2)(0.5) = 4.5$	0.25	-1	2.25	1.125	-4.5	10.125
II	$\frac{1}{4}\pi(2)^2(0.5) = 1.571$	1.3488	-0.8488	0.25	2.119	-1.333	0.393
III	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	-1	3.5	-0.098	0.393	-1.374
IV	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	-1	1.5	-0.098	0.393	-0.589
	$\Sigma V = 5.286$				$\Sigma\bar{x}V = 3.048$	$\Sigma\bar{y}V = -5.047$	$\Sigma\bar{z}V = 8.555$

$\bar{X} = 0.577$ in.

$\bar{Y} = 0.577$ in.

$\bar{Z} = 0.577$ in.