Mechanics of Materials

دانشگاه کردستان
University of Kurdistan
زانکۆی کوردستان

Ferdinand P.Beer, E.Russel Johnston, Jr., John T.Dewolf

Other Reference:
J.Wat Oler "Lectures notes on Mechanics od Materials"
Ibrahim A.Assakkaf "Lectures notes on Mechanics od Materials"

Buckling of Columns

By: Kaveh Karami
Associate Prof. of Structural Engineering
https://prof.uok.ac.ir/Ka.Karami

Buckling of Columns

- **Introduction**

- In discussing the analysis and design of various structures in the previous chapters, we had two primary concerns:
  - the strength of the structure, i.e. its ability to support a specified load without experiencing excessive stresses;
  - the ability of the structure to support a specified load without undergoing unacceptable deformations.

❑ Introduction

- Now we shall be concerned with stability of the structure,
  - with its ability to support a given load without experiencing a sudden change in its configuration.
- Our discussion will relate mainly to columns,
  - the analysis and design of vertical prismatic members supporting axial loads.

❑ Introduction

- Structures may fail in a variety of ways, depending on the :
  - Type of structure
  - Conditions of support
  - Kinds of loads
  - Material used

- **Introduction**

- Failure is prevented by designing structures so that the maximum stresses and maximum displacements remain within tolerable limits.
- Strength and stiffness are important factors in design as we have already discussed
- Another type of failure is buckling

- **Introduction**

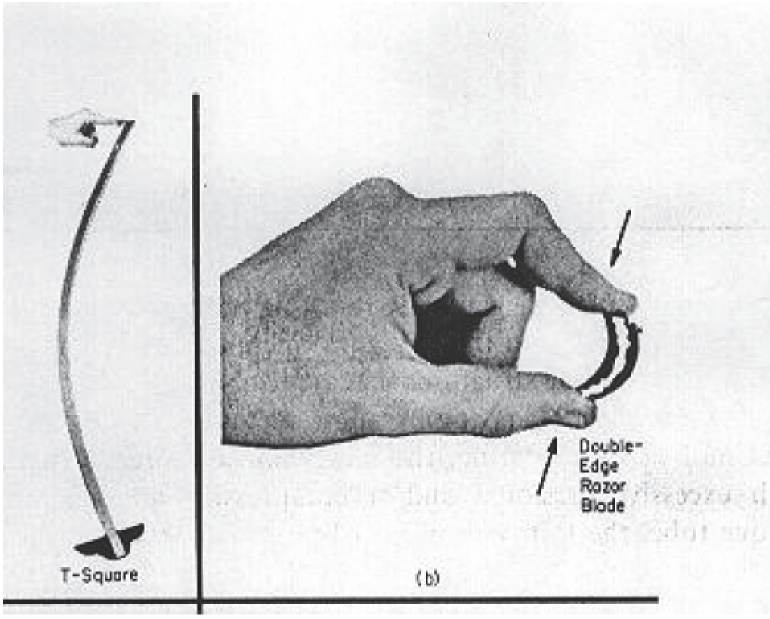
Buckling

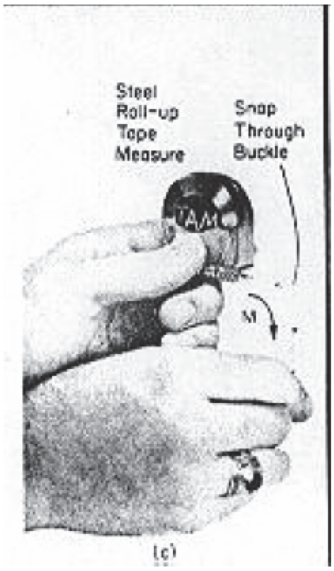
– Buckling is a mode of failure generally resulting from structural instability due to ***compressive action*** on the structural member or element involved.

Buckling of Columns

❑ Introduction

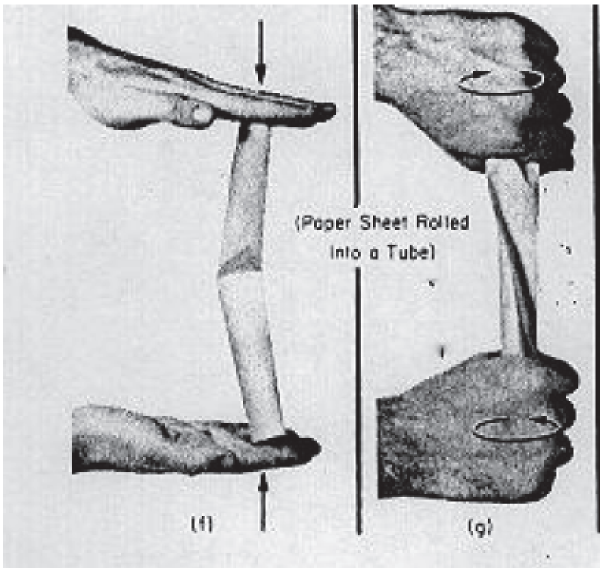
Buckling is not limited to columns.





Buckling of Columns

❑ Introduction



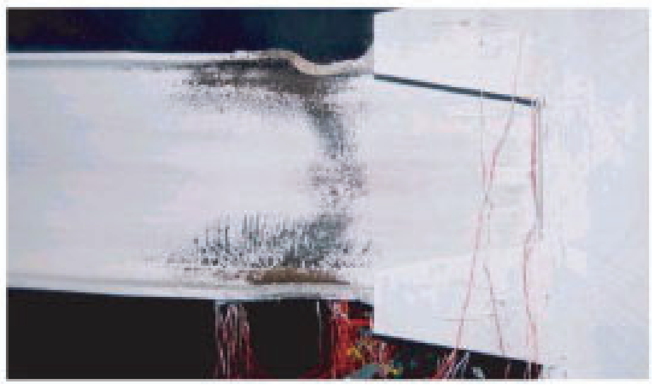
Any thin-walled torque tube

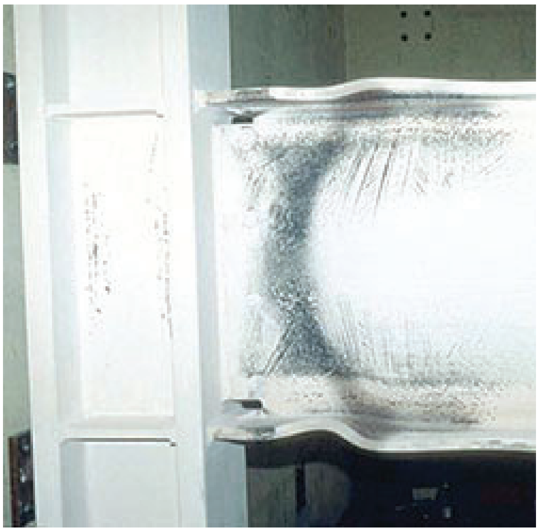
Step on empty aluminum can

Buckling of Columns

❑ Introduction

A thin flange of an I-beam subjected to excessive compressive bending effects.





Buckling of Columns

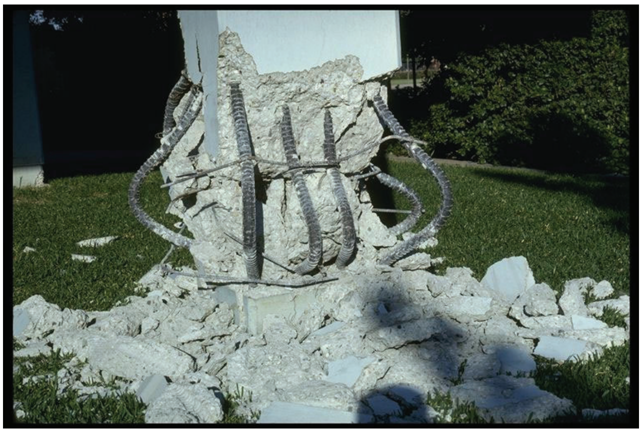
❑ Introduction

The thin web of an I-beam with excessive shear load

Buckling of Columns

❑ Introduction

The distinctive feature of buckling is the ***catastrophic*** and often spectacular nature of failure.



Buckling of Columns

❑ The Nature of Buckling

– In the previous chapters, we related load to stress and load to deformation.

– For these non-buckling cases of axial, torsional, bending, and combined loading, the stress or deformation was the significant quantity in failure.

– Buckling of a member is uniquely different in that the quantity significant in failure is the buckling load itself.

Buckling of Columns

❑ The Nature of Buckling

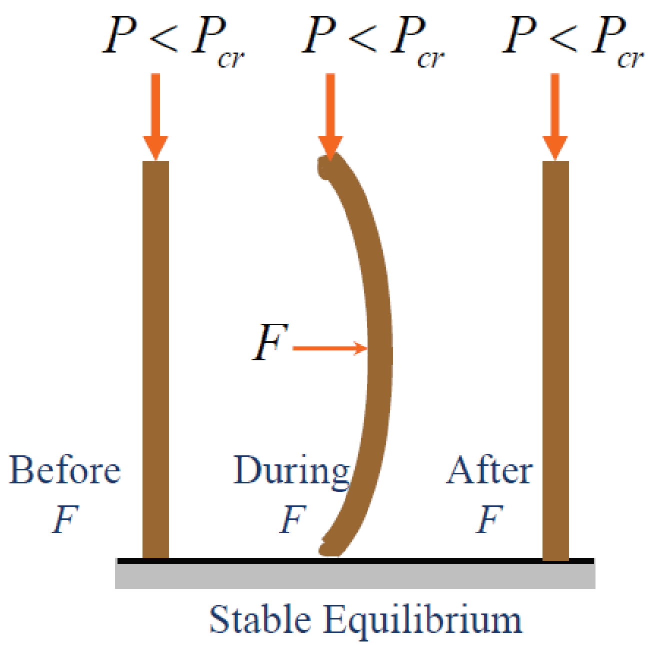
– The failure (buckling) load bears no unique relationship to the stress and deformation at failure.

– Our usual approach of deriving a load stress and load-deformation relations cannot be used here, instead, the approach to find an expression for the buckling load *Pcr*.

Buckling of Columns

❑ Mechanism of Buckling

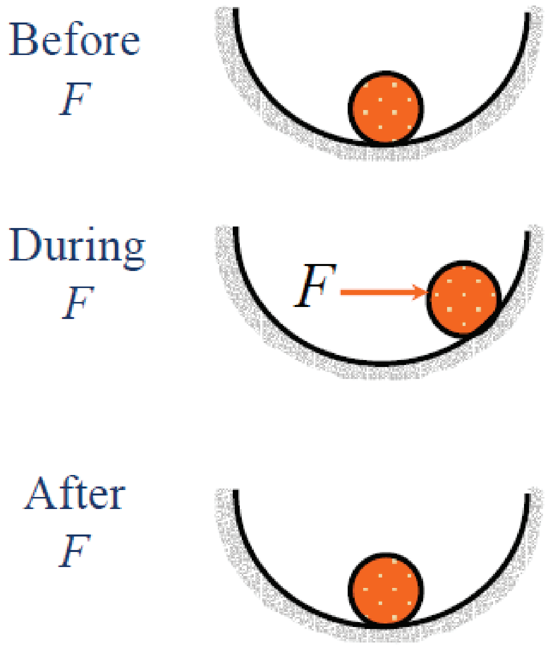
• In Figure, some axial load P is applied to the column.
• The column is then given a small deflection by applying the small lateral force F.
• If the load P is sufficiently small, when the force F is removed, the column will go back to its original straight condition.



Buckling of Columns

❑ Mechanism of Buckling

• The column will go back to its original straight condition just as the ball returns to the bottom of the curved container.
• In Figure of the ball and the curved container, gravity tends to restore the ball to its original position, while for the column the elasticity of the column itself acts as restoring force.
• This action constitutes stable equilibrium.

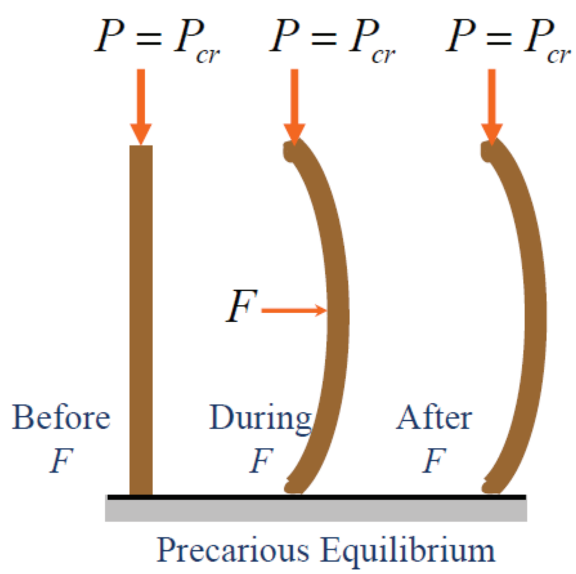


Buckling of Columns

❑ Mechanism of Buckling

• The same procedure can be repeated for increased value of the load P until some critical value Pcr is reached.

• When the column carries this load, and a lateral force F is applied and removed, the column will remain in the slightly deflected position. The elastic restoring force of the column is not sufficient to return the column to its original straight position but is sufficient to prevent excessive deflection of the column.



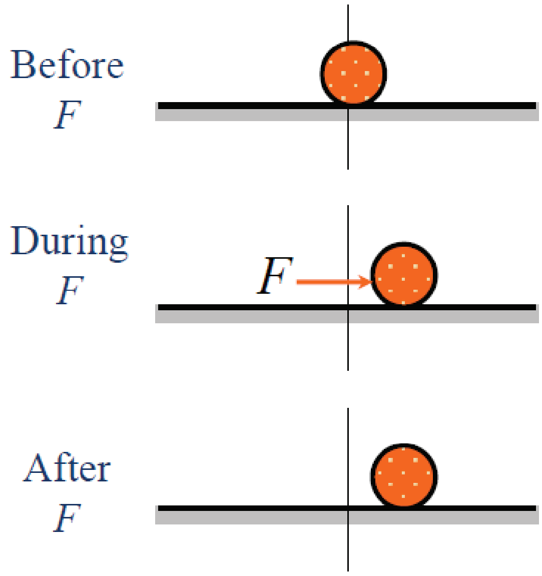
Buckling of Columns

❑ Mechanism of Buckling

• In Figure of the ball and the flat surface, the amount of deflection will depend on the magnitude of the lateral force F.

• Hence, the column can be in equilibrium in an infinite number of slightly bent positions.

• This action constitutes neutral or precarious equilibrium.

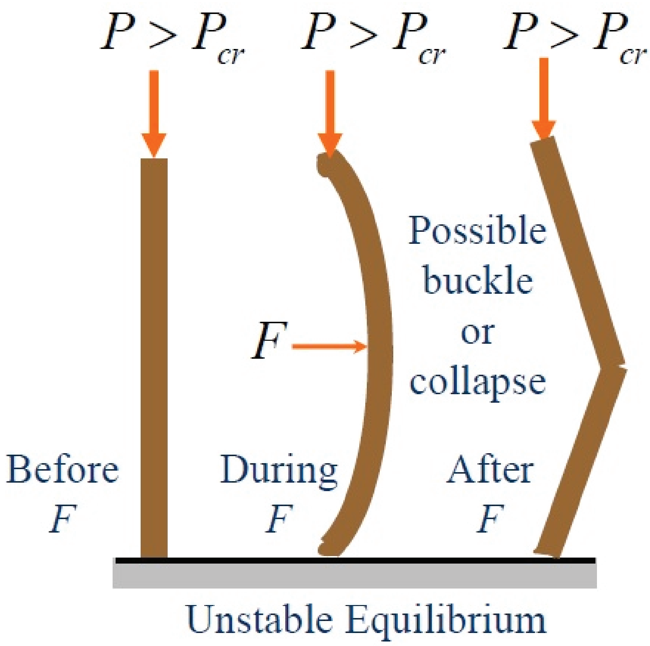


Buckling of Columns

❑ Mechanism of Buckling

• If the column is subjected to an axial compressive load P that exceeds Pcr, as shown in Figure, and a lateral force F is applied and removed, the column will bend considerably.

• That is, the elastic restoring force of the column is not sufficient to prevent a small disturbance from growing into an excessively large deflection.

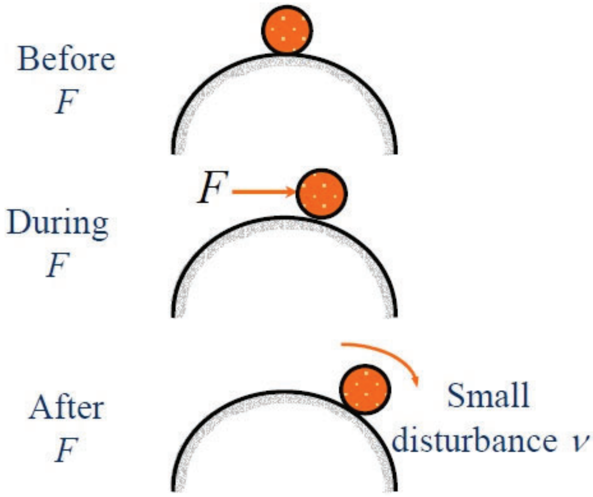


Buckling of Columns

❑ Mechanism of Buckling

• Depending on the magnitude of P, the column either will remain in the bent position or will completely collapse and fracture, just as the ball will roll off the curved surface in Figure.

• This type of behavior indicates that for axial loads greater than Pcr, the straight position of a column is one of unstable equilibrium in that a small disturbance will tend to grow into an excessive deformation.



Buckling of Columns

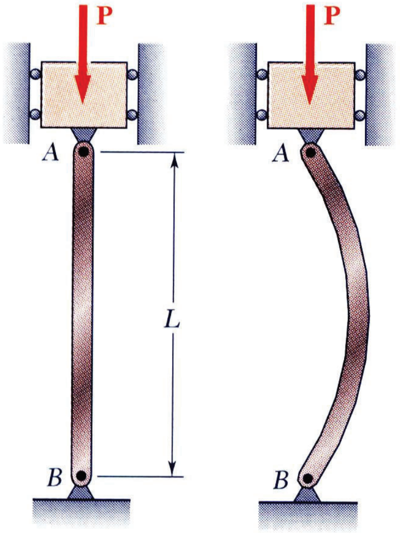
❑ The Nature of Buckling

Definition

"Buckling can be defined as the sudden large deformation of structure due to a slight increase of an existing load under which the structure had exhibited little, if any, deformation before the load was increased."

Buckling of Columns

❑ Stability of Structures



- In the design of columns, cross-sectional area is selected such that
  - allowable stress is not exceeded

$$\sigma = \frac{P}{A} \le \sigma_{all}$$

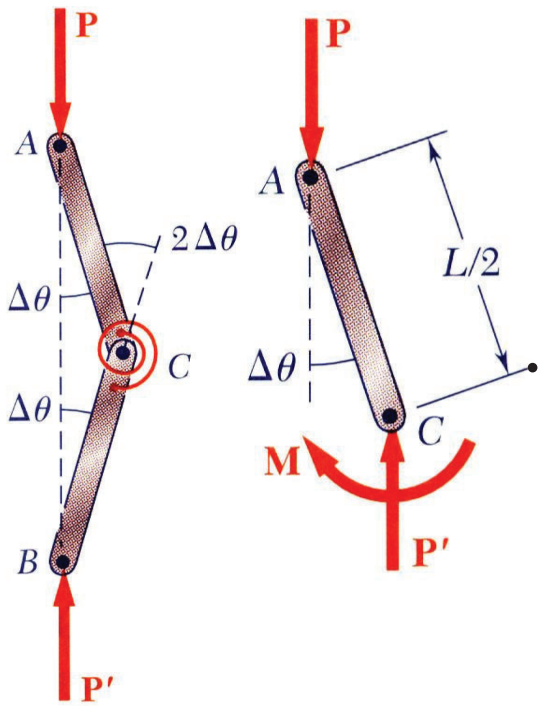
  - deformation falls within specifications

$$\delta = \frac{PL}{AE} \le \delta_{spec}$$

- After these design calculations, may discover that the column is unstable under loading and that it suddenly becomes sharply curved or buckles.

Buckling of Columns

❑ Stability of Structures



- Consider model with two rods and torsional spring. After a small perturbation,

$$K(2\Delta\theta) = \text{restoring moment}$$

$$P\frac{L}{2}\sin\Delta\theta = P\frac{L}{2}\Delta\theta = \text{destabilizing moment}$$

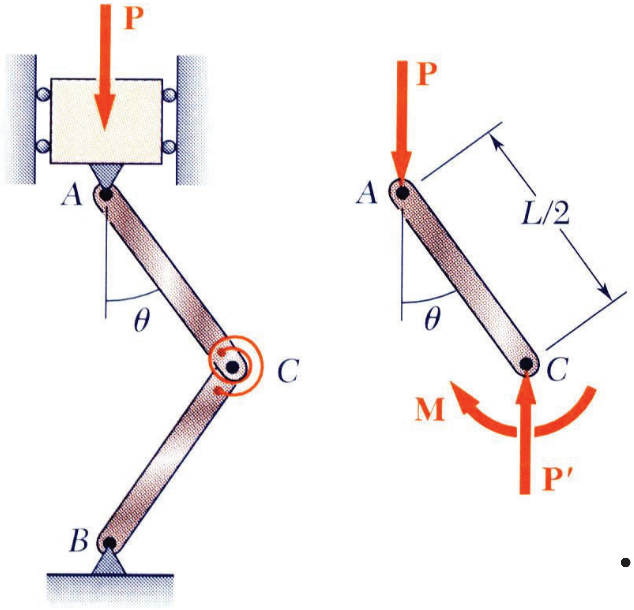
- Column is stable (tends to return to aligned orientation) if

$$P\frac{L}{2}\Delta\theta < K(2\Delta\theta)$$

$$P < P_{cr} = \frac{4K}{L}$$

Buckling of Columns

❑ Stability of Structures



- Assume that a load P is applied. After a perturbation, the system settles to a new equilibrium configuration at a finite deflection angle.

$$P\frac{L}{2}\sin\theta = K(2\theta)$$

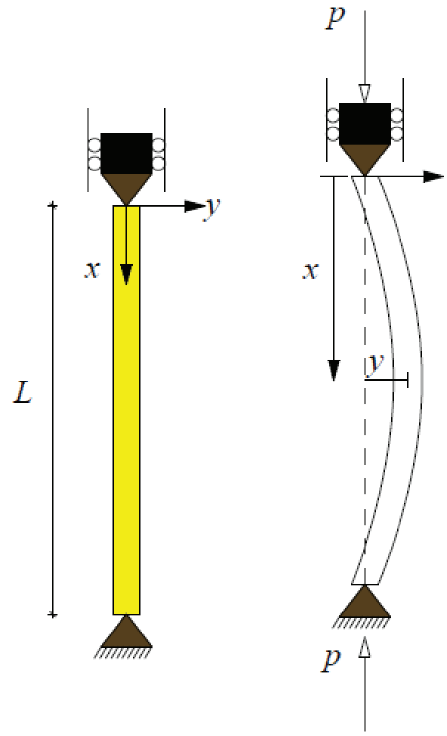
$$\frac{PL}{4K} = \frac{P}{P_{cr}} = \frac{\theta}{\sin\theta}$$

- Noting that $\sin\theta < \theta$, the assumed configuration is only possible if $P > P_{cr}$

Buckling of Columns

❑ Critical Buckling Load

– The purpose of this analysis is to determine the minimum axial compressive load for which a column will experience lateral deflection.



– Governing Differential Equation:

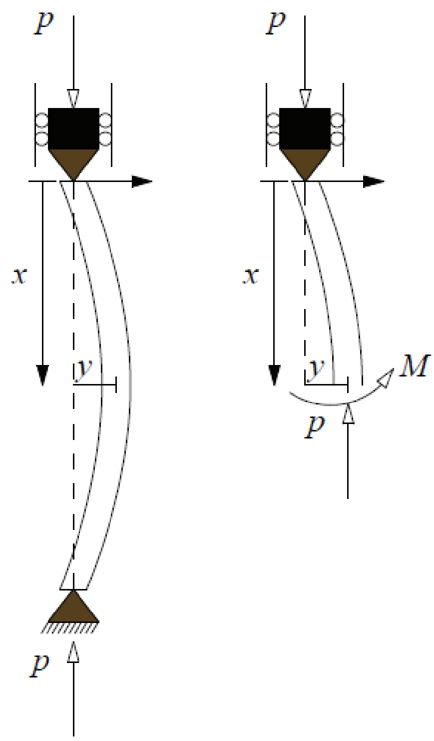
• Consider a buckled simply-supported column of length L under an external axial compression force P. The transverse displacement of the buckled column is represented by y.

Buckling of Columns

❑ Critical Buckling Load

– Governing Differential Equation:

The figure shows the forces and moments acting on a cross-section in the buckled column. Moment equilibrium on the lower free body yields a solution for the internal bending moment M,



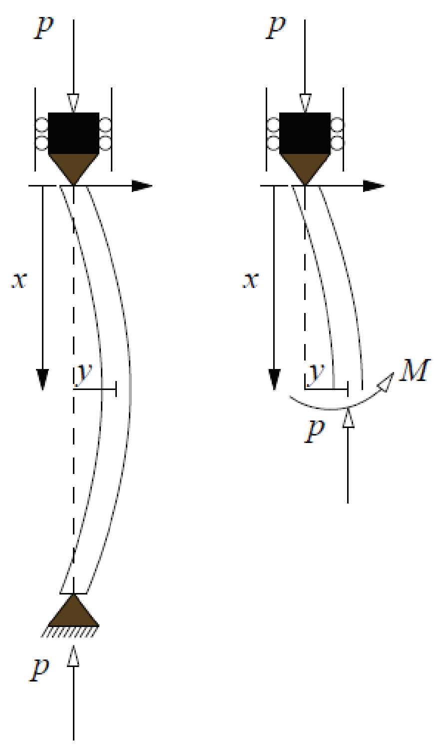
$$\sum M = 0 \quad \Rightarrow$$

$$Py + M = 0$$

Buckling of Columns

❑ Critical Buckling Load

– Governing Differential Equation:



Recall the relationship between the moment M and the transverse displacement y for the elastic curve,

$$EI\frac{d^2y}{dx^2} = M$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + \frac{P}{EI}y = 0$$

The governing equation is a second order homogeneous ordinary differential equation with constant coefficients.

Buckling of Columns

❑ Critical Buckling Load

– Governing Differential Equation:

The solution is found to be,

$$y(x) = A\cos(\alpha x) + B\sin(\alpha x)$$

where $$\alpha^2 = \frac{P}{EI}$$

The coefficients A and B are constants, which can be determined using the column's kinematic boundary conditions.

Buckling of Columns

❑ Critical Buckling Load

– Governing Differential Equation:

Kinematic Boundary Conditions

$$y(x) = A\cos(\alpha x) + B\sin(\alpha x)$$

$$at\ x = 0, y = 0 \quad \Rightarrow \quad 0 = A + 0, \quad giving\ that\ A = 0$$

$$at\ x = L, y = 0 \quad \Rightarrow \quad 0 = B\sin(\alpha L)$$

If B = 0, No bending moment exists, so the only logical solution is for sin(αL) = 0 and the only way that this can happen is if :

$$\alpha L = n\pi$$

where n = 1,2,3, . . .

Buckling of Columns

❑ Critical Buckling Load

– Governing Differential Equation:

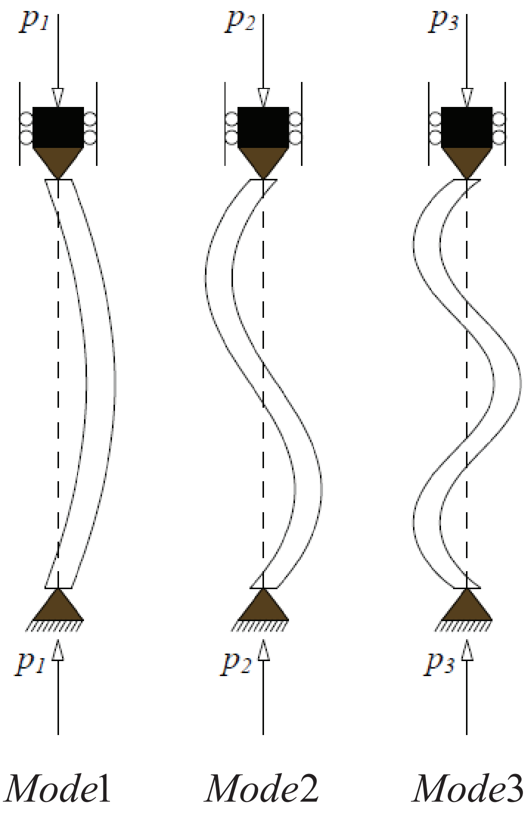
Kinematic Boundary Conditions

$$\alpha^2 = \frac{P}{EI} = \left(\frac{n\pi}{L}\right)^2 \Rightarrow \boxed{P = n^2 \frac{\pi^2 EI}{L^2}}$$

$$Mode1 \quad n = 1 \quad \Rightarrow \quad \boxed{P_1 = \frac{\pi^2 EI}{L^2}}$$

$$Mode2 \quad n = 2 \quad \Rightarrow \quad \boxed{P_2 = \frac{4\pi^2 EI}{L^2}}$$

$$Mode3 \quad n = 3 \quad \Rightarrow \quad \boxed{P_2 = \frac{9\pi^2 EI}{L^2}}$$



Buckling of Columns

❑ Critical Buckling Load

The lowest load that causes buckling is called critical load ($n = 1$). The critical buckling load (***Euler Buckling***) for a long column is given by

$$\boxed{P_{cr} = \frac{\pi^2 EI}{L^2}}$$

where
E = modulus of elasticity of the material
I = moment of inertia of the cross section
L = length of column

Buckling of Columns

❑ Critical Buckling Load

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

- Equation 9 is usually called Euler's formula. Although Leonard Euler did publish the governing equation in 1744.
- J. L. Lagrange is considered the first to show that a non-trivial solution exists only when n is an integer.
- Thomas Young then suggested the critical load (n = 1) and pointed out the solution was valid when the column is slender in his 1807 book.
- The "slender" column idea was not quantitatively developed until A.Considère performed a series of 32 tests in 1889.

Buckling of Columns

❑ Critical Buckling Stress

$$\left.\begin{aligned} P_{cr} &= \frac{\pi^2 EI}{L^2} \\ r &= \sqrt{\frac{I}{A}} \Rightarrow A = \frac{I}{r^2} \end{aligned}\right\} \Rightarrow \sigma_{cr} = \frac{P_{cr}}{A} = \frac{\frac{\pi^2 EI}{L^2}}{\frac{I}{r^2}} \Rightarrow \sigma_{cr} = \frac{\pi^2 E}{\left(\frac{L}{r}\right)^2}$$

where

r = radius of gyration

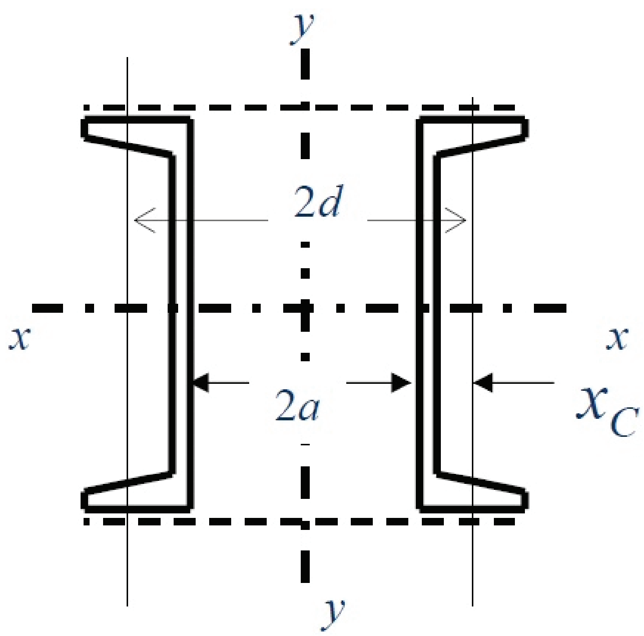
(L/r) = slenderness ratio of column

When calculating the critical buckling for columns, I (or r) should be obtained about the weak axis.

❑ Critical Buckling Stress

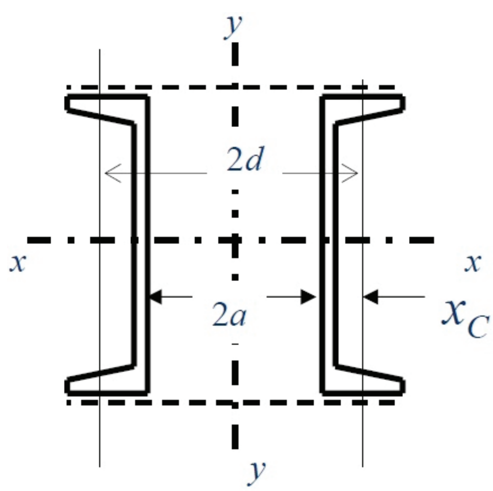
Example 01

Determine the Gyration Radius of the shown section.



❑ Critical Buckling Stress

Example 01

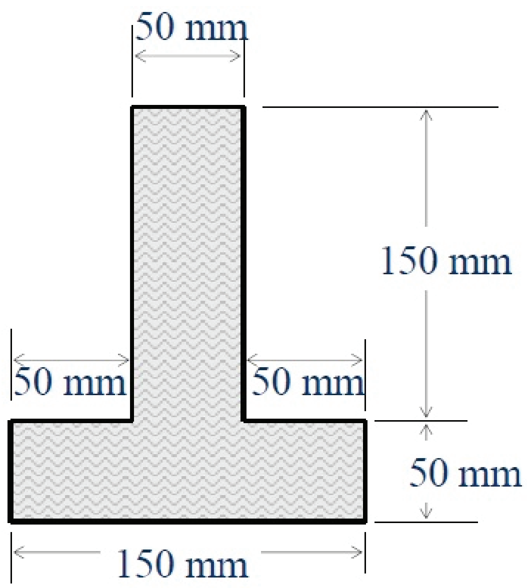


Buckling of Columns

❑ Critical Buckling Stress

Example 02

A 3-m column with the cross section shown in Figure is constructed from two pieces of timber. The timbers are nailed together so that they act as a unit.
Determine (a) the slenderness ratio, (b) the Euler buckling load (E = 13 GPa for timber), and (c) the axial stress in the column when Euler load is applied.

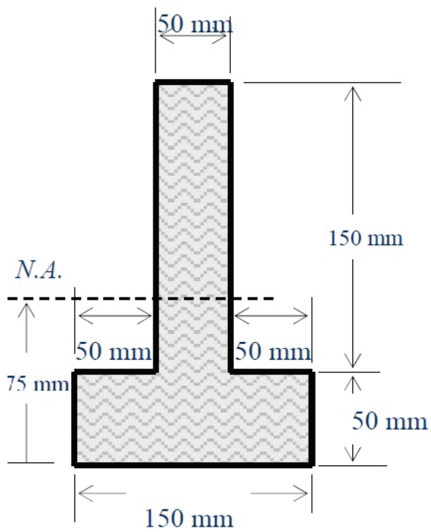


Buckling of Columns

❑ Critical Buckling Stress

Example 02

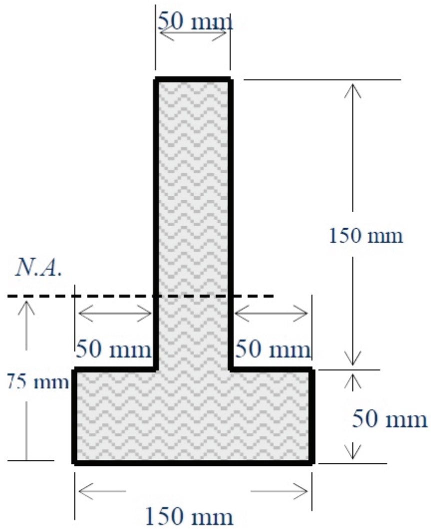
Properties of the cross section:



Buckling of Columns

❑ Critical Buckling Stress

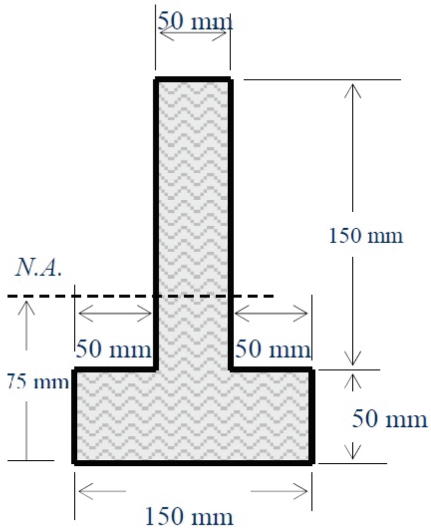
Example 02



Buckling of Columns

❑ Critical Buckling Stress

Example 02



(c) Axial Stress:

❑ Critical Buckling Stress

Example 03

A WT6 × 36 structural steel section is used for an 18-ft column. Determine
(a) The slenderness ratio.
(b) The Euler buckling load. Use $E = 29\times10^3\ ksi$.
(c) The axial stress in the column when Euler load is applied.

$A = 10.6\ in^2$
$I_x = 23.2\ in^4$
$I_y = 97.5\ in^4$
$r_x = 1.48\ in$
$r_y = 3.04\ in$

❑ Critical Buckling Stress

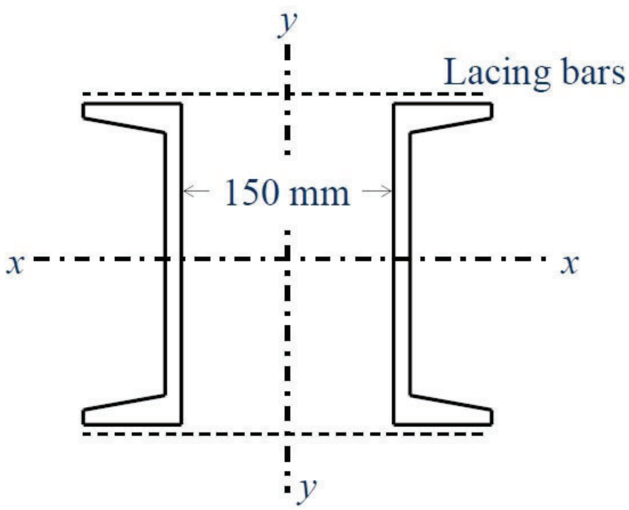
Example 03

Buckling of Columns

❑ Critical Buckling Stress

Example 04

Two C229 × 30 structural steel channels are used for a column that is 12 m long. Determine the total compressive load required to buckle the two members if
(a) They act independently of each other. Use E = 200 GPa.
(b) They are laced 150 mm back to back as shown in Figure.



C229 × 30:

$$A = 3795\ mm^2$$
$$I_x = 25.3\times10^6\ mm^4$$
$$I_y = 1.01\times10^6\ mm^4$$
$$r_x = 81.8\ mm$$
$$r_y = 16.3\ mm$$
$$x_c = 14.8\ mm$$

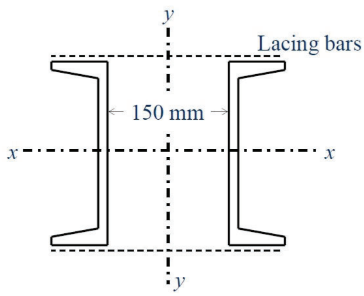
Buckling of Columns

❑ Critical Buckling Stress

Example 04

C229 × 30:

$$A = 3795\ mm^2$$
$$I_x = 25.3\times10^6\ mm^4$$
$$I_y = 1.01\times10^6\ mm^4$$
$$r_x = 81.8\ mm$$
$$r_y = 16.3\ mm$$
$$x_c = 14.8\ mm$$



(a) They act independently of each other

Buckling of Columns

❑ Critical Buckling Stress

Example 04

C229 × 30:

(b) They are laced 150 mm back to back

$A = 3795\ mm^2$

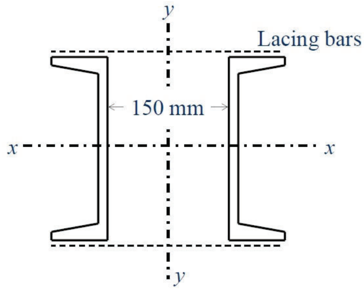
$I_x = 25.3\times10^6\ mm^4$

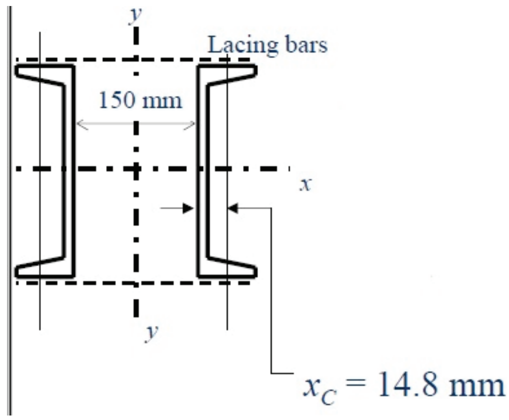
$I_y = 1.01\times10^6\ mm^4$

$r_x = 81.8\ mm$

$r_y = 16.3\ mm$

$x_c = 14.8\ mm$



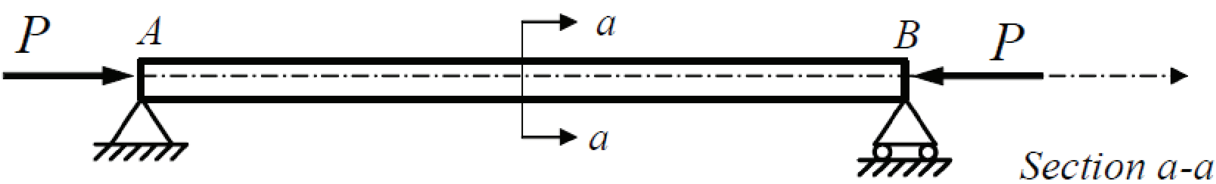


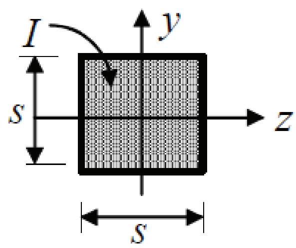
Buckling of Columns

❑ Critical Buckling Stress

Example 05

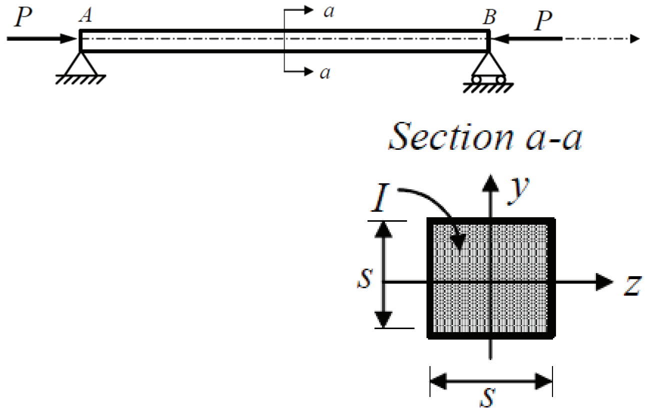
A 2m long pin ended column of square cross section. Assuming E=12.5GPa, σ_{allow}=12MPa for compression parallel to the grain, and using a factor of safety of 2.5 in computing Euler's critical load for buckling, determining the size of the cross section if the column is to safely support (a) a P = 100kN load and (b) a P = 200kN load.





Buckling of Columns

❑ Critical Buckling Stress

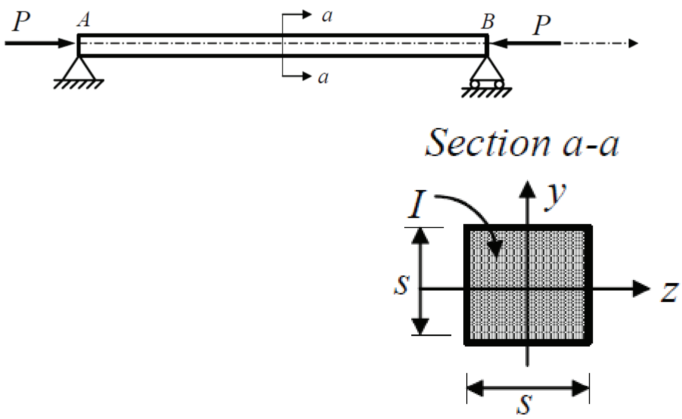


Example 05

Second moment of area

Buckling of Columns

❑ Critical Buckling Stress



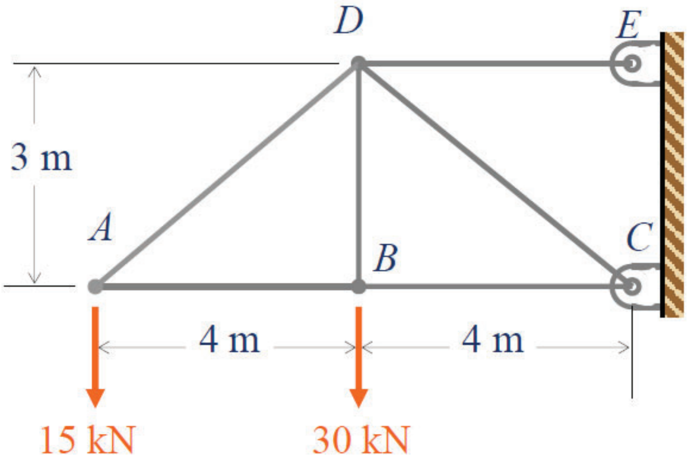
Example 05

Buckling of Columns

❑ Critical Buckling Stress

Example 06

A simple pin-connected truss is loaded and supported as shown in Figure. All members of the truss are WT 102 × 43 sections made of structural steel with a modulus of elasticity of 200 GPa and a yield strength of 250 MPa. Determine (a) the factor of safety with respect to failure by slip, and (b) the factor of safety with respect to failure by buckling.



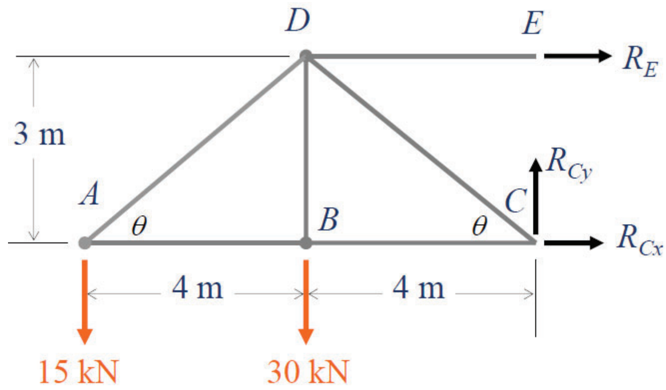
WT 102 × 43: $A = 5515\ mm^2$, $r_{\min} = 26.2\ mm$

Buckling of Columns

❑ Critical Buckling Stress

Example 06

Free-body diagram:

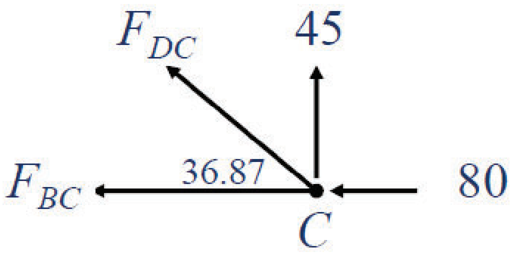


Buckling of Columns

❑ Critical Buckling Stress

Example 06

At pin *C*:

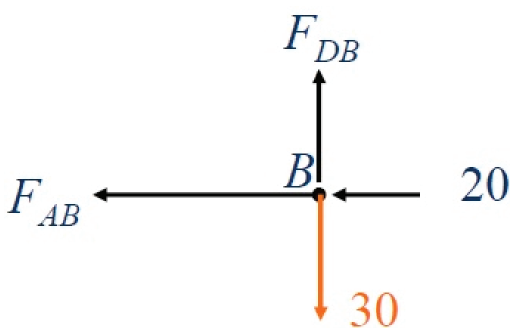


Buckling of Columns

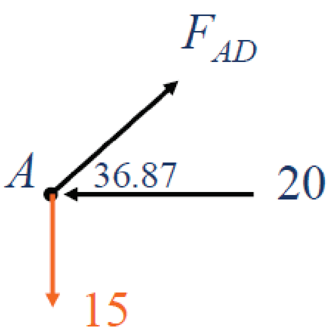
❑ Critical Buckling Stress

Example 06

At pin *B*:



At pin *A*:

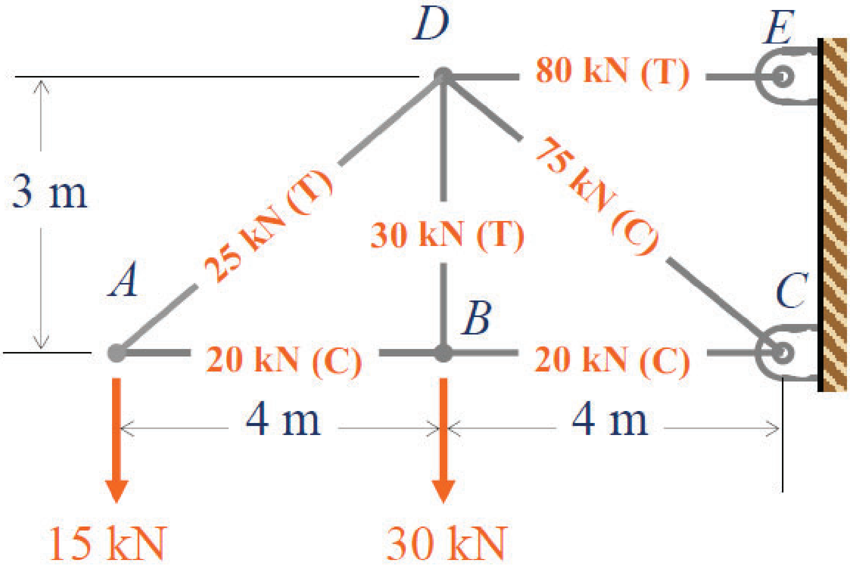


Buckling of Columns

❑ Critical Buckling Stress

Example 06

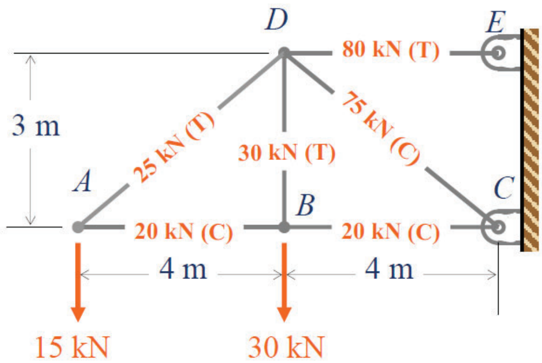
Thus, the forces in the truss are as follows:



Buckling of Columns

❑ Critical Buckling Stress

Example 06

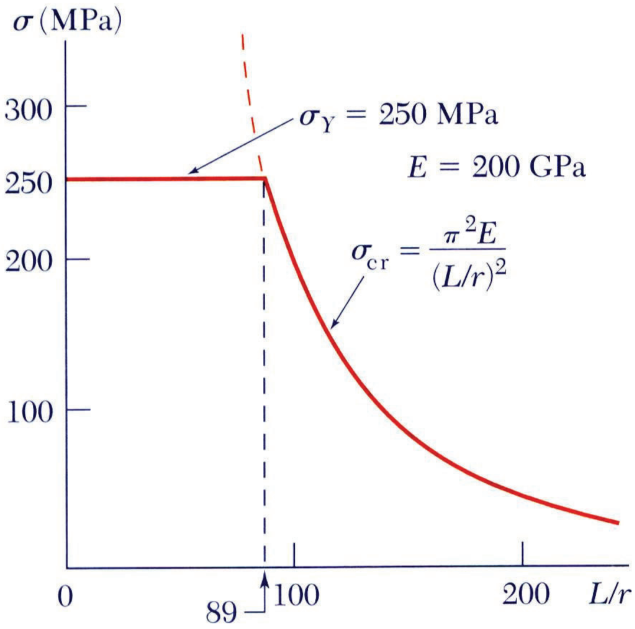


(a) Factor of safety with respect to slip:

(b) Factor of safety with respect to failure by buckling:

Buckling of Columns

❑ Euler's Formula for Pin-Ended Beams



- The value of stress corresponding to the critical load,

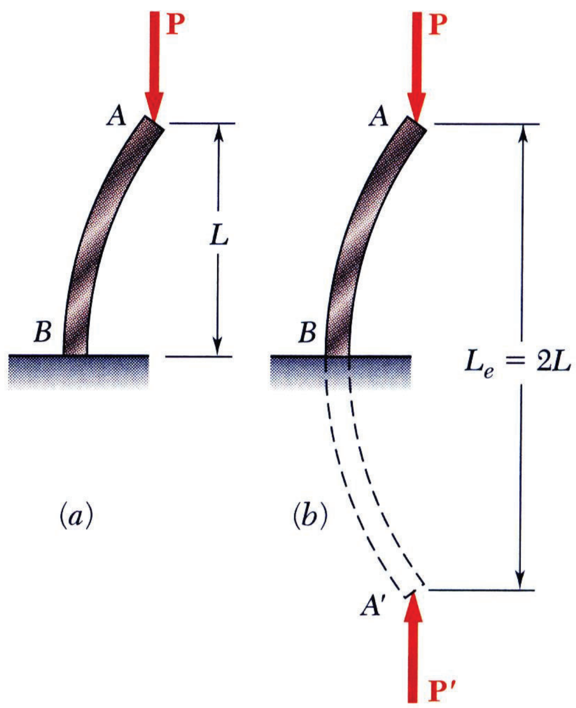
$$\sigma < \sigma_{cr} = \frac{\pi^2 E}{(L/r)^2}$$

$$\frac{L}{r} = slenderness\ ratio$$

If this equation is plotted for steel it gives For a column not to fail by either yielding or buckling, its stress must remain underneath this diagram

Buckling of Columns

❑ Extension of Euler's Formula



- A column with one fixed and one free end, will behave as the upper-half of a pin-connected column.

- The critical loading is calculated from Euler's formula,

$$P_{cr} = \frac{\pi^2 EI}{(2L)^2}$$

$$\sigma_{cr} = \frac{\pi^2 E}{(2L/r)^2}$$

$$2L = \text{equivalent length}$$

Buckling of Columns

❑ Extension of Euler's Formula

The Effective Length Concept

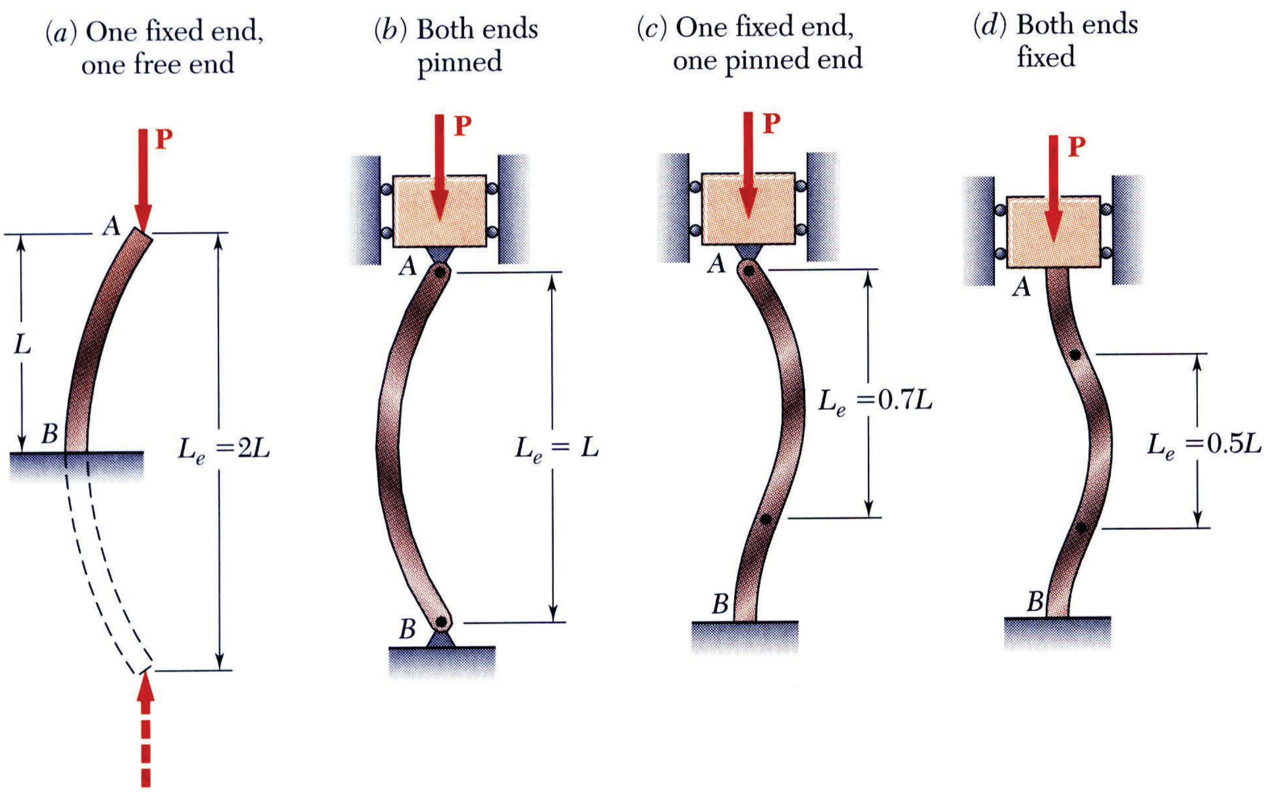
Definition:

The effective length (Le) of a column is defined as the distance between successive inflection points or points of zero moment.

$$\sigma_{cr} = \frac{\pi^2 E}{(L_e/r)^2} = \frac{\pi^2 E}{(KL/r)^2}$$

Buckling of Columns

❑ Extension of Euler's Formula



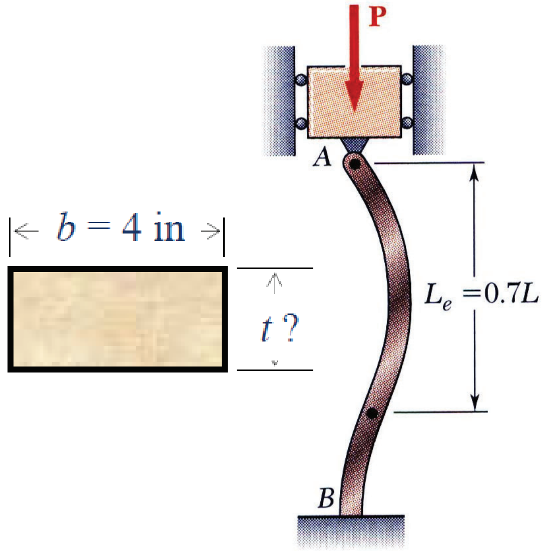
Buckling of Columns

❑ Critical Buckling Stress

Example 07

What is the least thickness a rectangular wood plank 4 in. wide can have, if it is used for a 20-ft column with one end fixed and one end pivoted, and must support an axial load of 1000 lb? Use a factor of safety (FS) of 5.

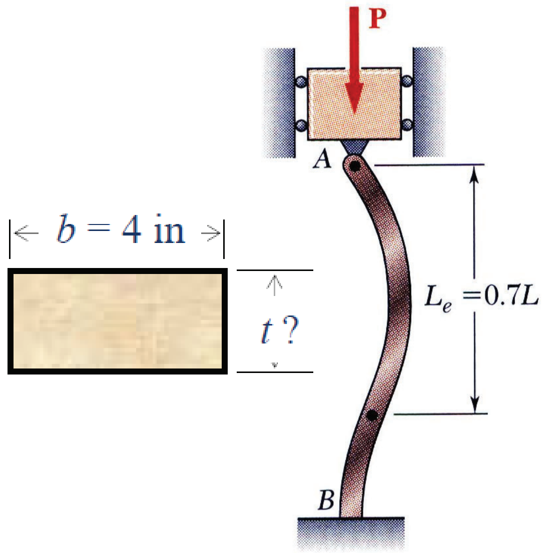
$E = 1.5 \times 10^6$ psi



Buckling of Columns

❑ Critical Buckling Stress

Example 07



Buckling of Columns

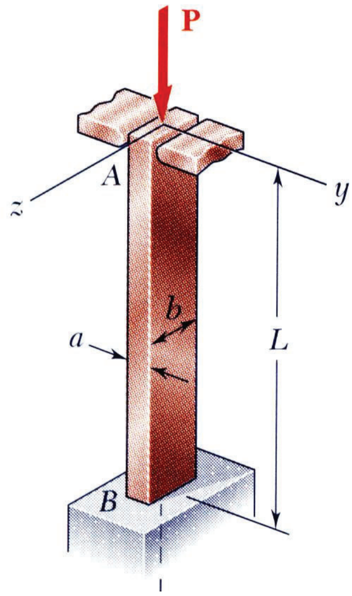
❑ Critical Buckling Stress

Example 08

An aluminum column of length L and rectangular cross-section has a fixed end at B and supports a centric load at A. Two smooth and rounded fixed plates restrain end A from moving in one of the vertical planes of symmetry but allow it to move in the other plane.

a) Determine the ratio a/b of the two sides of the cross-section corresponding to the most efficient design against buckling.

b) Design the most efficient cross-section for the column.



$L = 20$ in.

$E = 10.1 \text{ x } 10^6$ psi

$P = 5$ kips

$FS = 2.5$

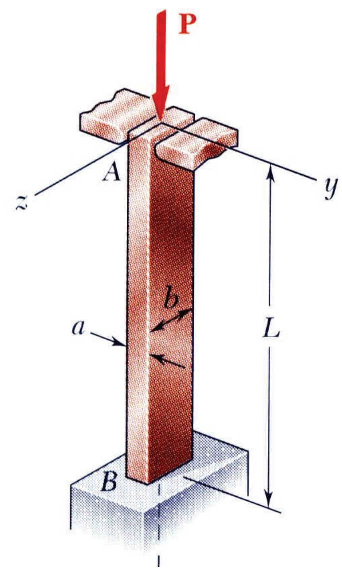
Buckling of Columns

❑ Critical Buckling Stress

Example 08

SOLUTION:

The most efficient design occurs when the resistance to buckling is equal in both planes of symmetry. This occurs when the slenderness ratios are equal.



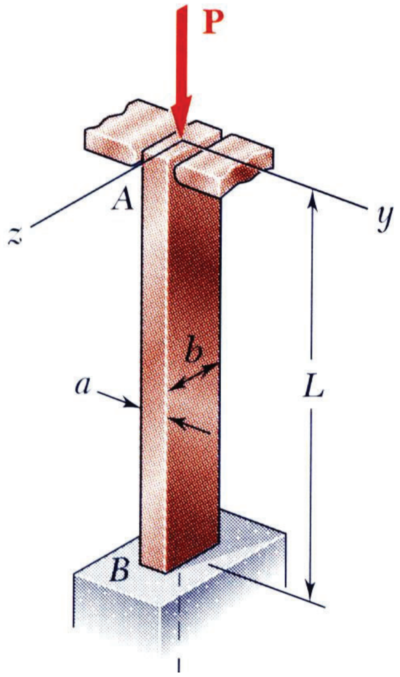
- Buckling in xy Plane:

- Buckling in xz Plane:

Buckling of Columns

❑ Critical Buckling Stress

Example 08

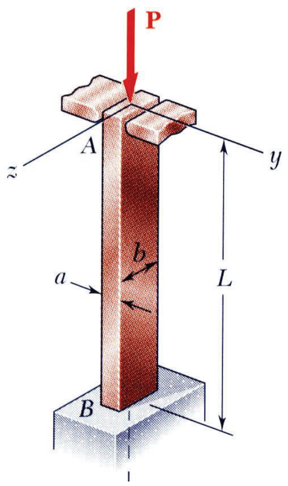


- Most efficient design:

Buckling of Columns

❑ Critical Buckling Stress

Example 08



$L = 20$ in. $E = 10.1 \times 10^6$ psi

$P = 5$ kips $FS = 2.5$ & $a/b = 0.35$

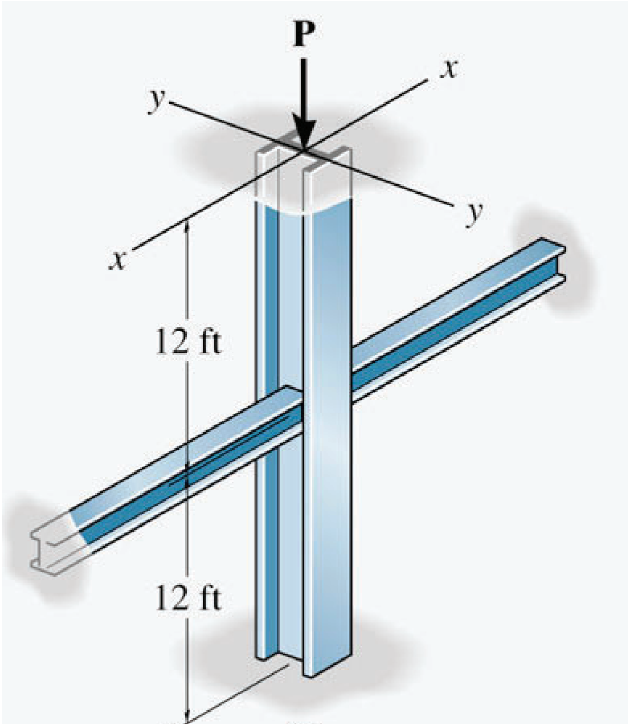
- Design:

Buckling of Columns

❑ Critical Buckling Stress

Example 09

A W 6x15 steel column is 24 ft long and is fixed at its ends as shown in Figure. Its load-carrying capacity is increased by bracing it about the y - y (weak) axis using struts that are assumed to be pin-connected to its midheight. Determine the load it can support so that the column does not buckle nor the material exceed the yield stress.



$E_{st} = 29\times10^3\, ksi \qquad \sigma_Y = 70\, ksi$

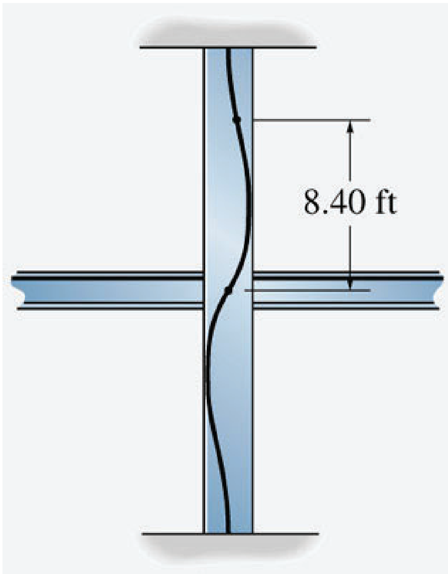
$I_y = 29.1\, in^4 \qquad I_x = 9.32\, in^4$

$r_y = 2.56\, in \qquad r_x = 1.46\, in \qquad A = 4.43\, in^2$

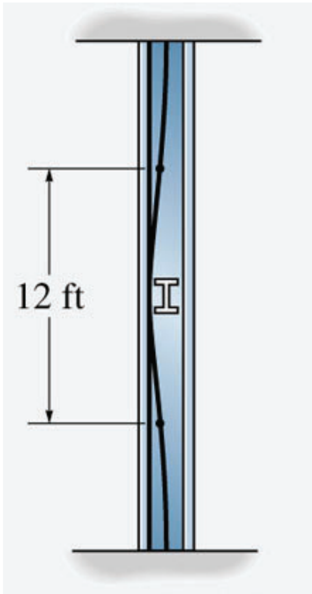
Buckling of Columns

❑ Critical Buckling Stress

Example 09



y–y axis buckling

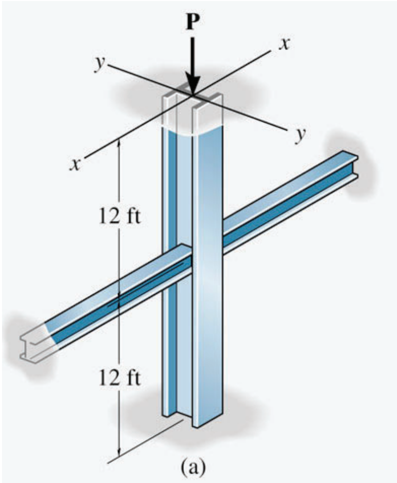


x–x axis buckling

Buckling of Columns

❑ Critical Buckling Stress

Example 09

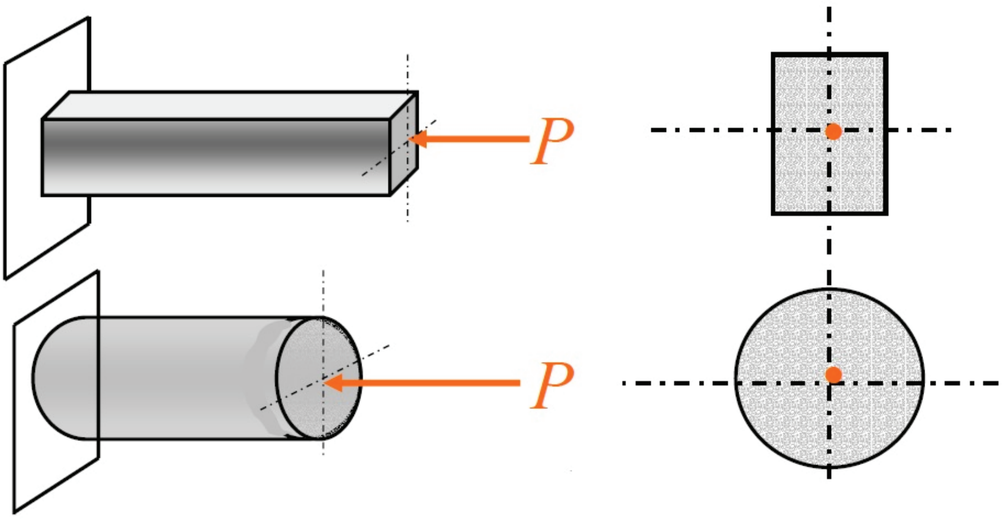


Since this stress is less than the yield stress, buckling will occur before the material yields

Buckling of Columns

❑ Buckling: Eccentric Loading

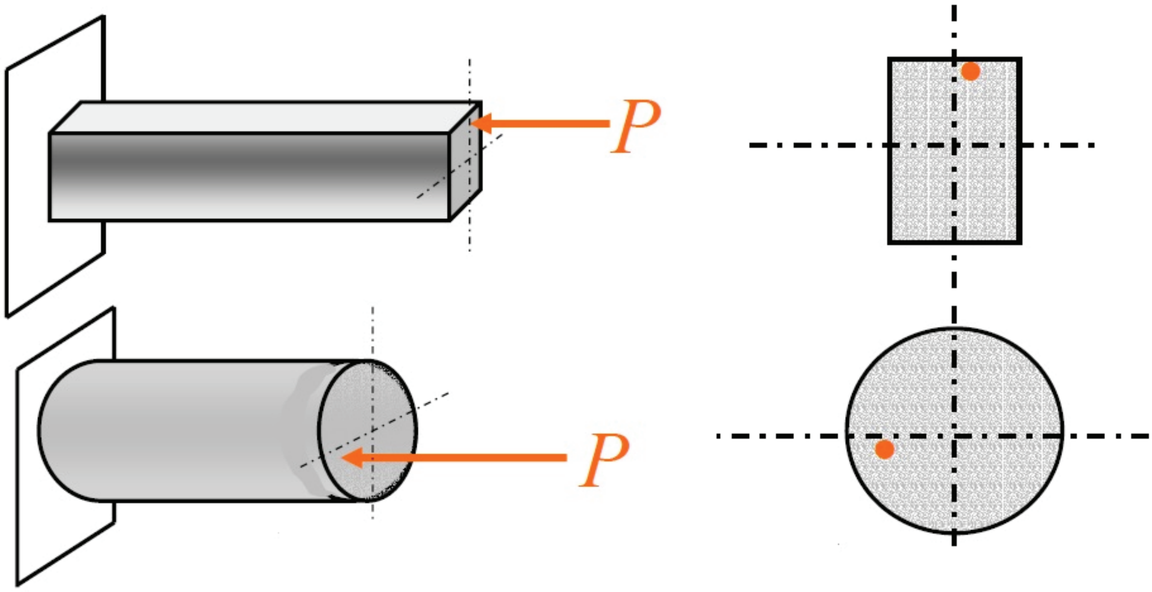
The Euler formula that was developed earlier was based on the assumption that the concentrated compressive load *P* on the column acts though the centroid of the cross section of the column



Buckling of Columns

❑ Buckling: Eccentric Loading

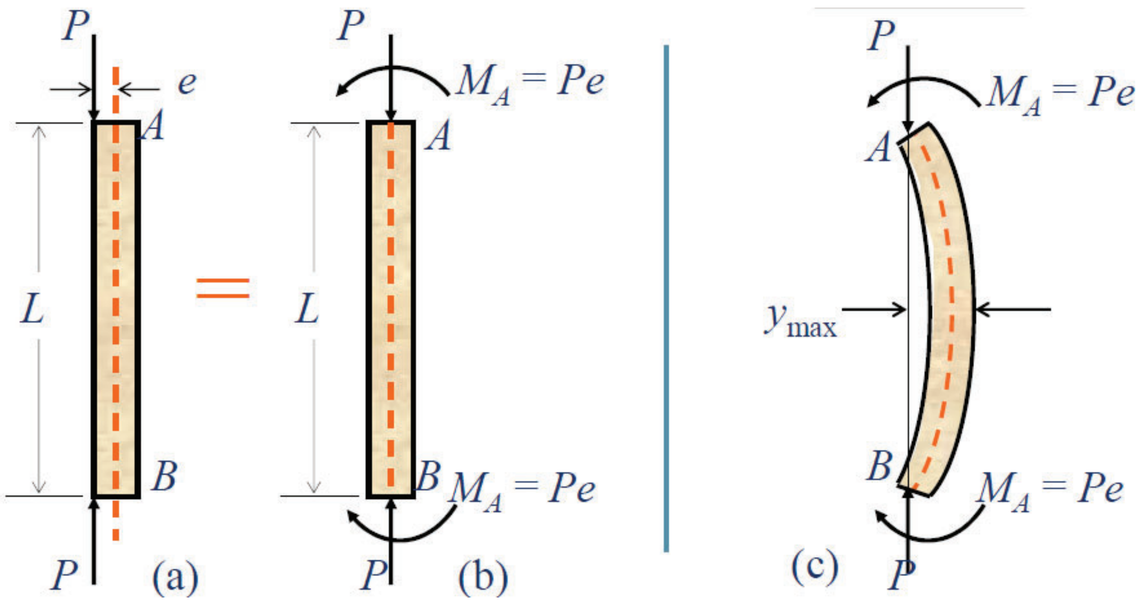
In many realistic situations, however, this is not the case. The load P applied to a column is never perfectly centric.



Buckling of Columns

❑ Eccentric Loading; The Secant Formula

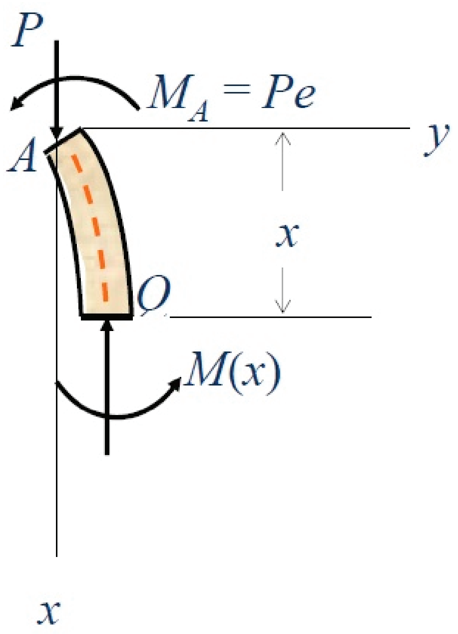
As the eccentric load is increased, both the couple MA and the axial force P increase, and both cause the column to bend further.

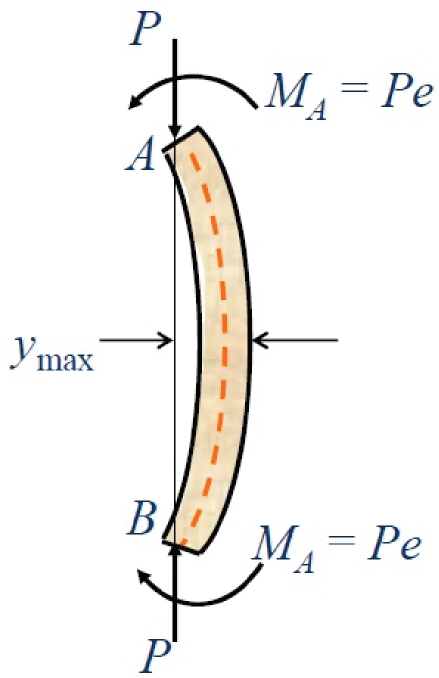


Buckling of Columns

❑ Eccentric Loading; The Secant Formula

Derivation of the formula





Buckling of Columns

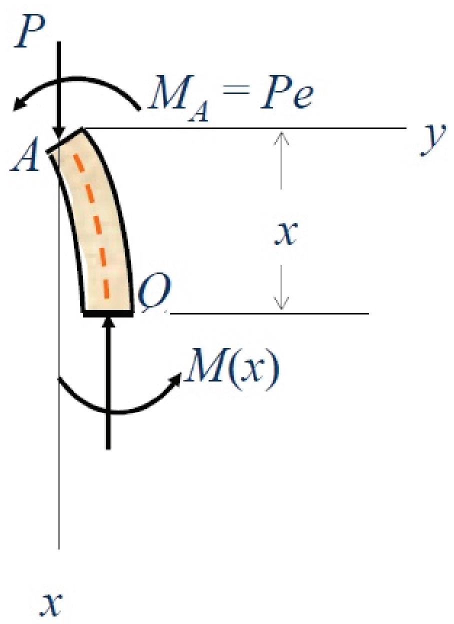
❑ Eccentric Loading; The Secant Formula

Derivation of the formula

$$M(x) = -Py - M_A = -Py - Pe$$

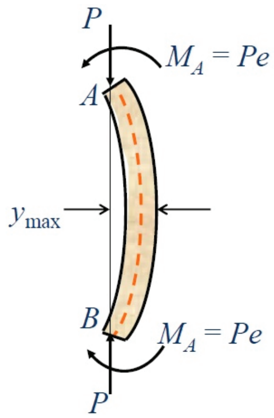
$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI} \quad \Rightarrow \quad \boxed{\frac{d^2y}{dx^2} = -\frac{P}{EI}y - \frac{Pe}{EI}}$$

$$If \quad p^2 = \frac{P}{EI} \quad \Rightarrow \quad \boxed{\frac{d^2y}{dx^2} + p^2 y = -p^2 e}$$



Buckling of Columns

❑ Eccentric Loading; The Secant Formula



Derivation of the formula

The general solution of the differential equation $\dfrac{d^2y}{dx^2} + p^2 y = -p^2 e$

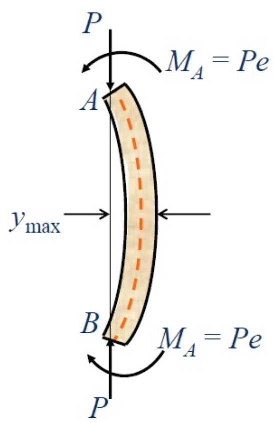
$$y = A\sin(px) + B\cos(px) - e$$

Using the boundary condition $y = 0$, at $x = 0$, gives $B = e$

Using the other boundary condition at the other end: y = 0, at x = L, gives $A\sin(pL) = e[1-\cos(pL)]$

Buckling of Columns

❑ Eccentric Loading; The Secant Formula



Recalling that
$$\sin(pL) = 2\sin\left(\frac{pL}{2}\right)\cos\left(\frac{pL}{2}\right)$$
$$1-\cos(pL) = 2\sin^2\left(\frac{pL}{2}\right)$$

$$A\sin(pL) = e[1-\cos(pL)] \quad \Rightarrow \quad A\left[2\sin\left(\frac{pL}{2}\right)\cos\left(\frac{pL}{2}\right)\right] = e\left[2\sin^2\left(\frac{pL}{2}\right)\right]$$

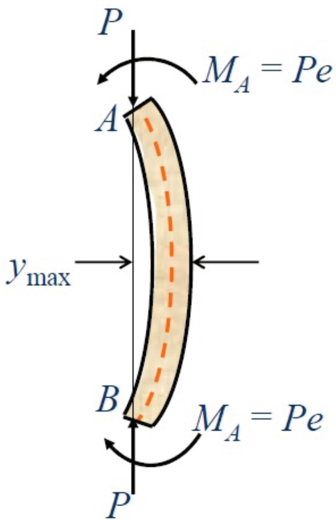
$$\Rightarrow \quad A = e\tan\left(\frac{pL}{2}\right)$$

$$B = e$$

$$\Rightarrow \quad y = e\left[\tan\left(\frac{pL}{2}\right)\sin(px) + \cos(px) - 1\right]$$

Buckling of Columns

❑ Eccentric Loading; The Secant Formula



The maximum deflection is obtained by setting $x = L/2$

$$y_{max} = e\left[\sec\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right) - 1\right]$$

If $\sqrt{\frac{P}{EI}}\frac{L}{2} = \frac{\pi}{2} \Rightarrow \sec\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right) \to \infty \Rightarrow y_{max} \to \infty$

$$\sqrt{\frac{P}{EI}}\frac{L}{2} = \frac{\pi}{2} \Rightarrow P_{cr} = \frac{\pi^2 EI}{L^2} \Rightarrow EI = \frac{P_{cr}L^2}{\pi^2}$$

Buckling of Columns

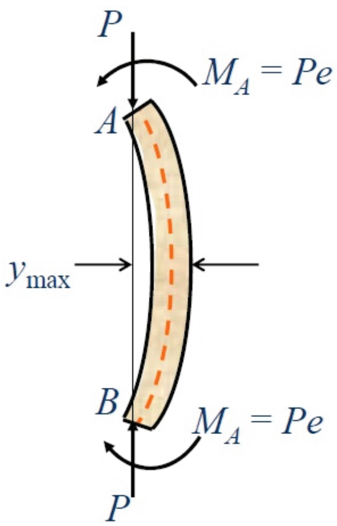
❑ Eccentric Loading; The Secant Formula

The maximum deflection is obtained by setting $x = L/2$

$$y_{max} = e\left[\sec\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right) - 1\right]$$

$$EI = \frac{P_{cr}L^2}{\pi^2}$$

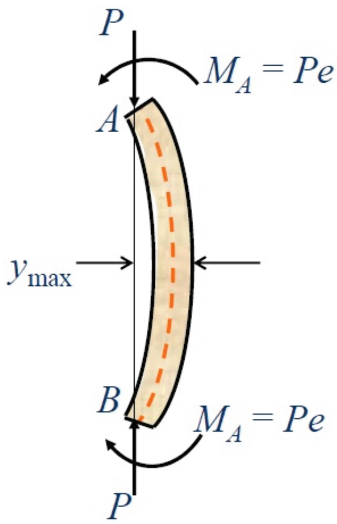
$$\Rightarrow y_{max} = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right]$$



Buckling of Columns

❑ Eccentric Loading; The Secant Formula

The maximum stress occur at midspan of the column (at x = L/2), and can computed from



$$\sigma_{max} = \frac{P}{A} + \frac{M_{max}c}{I}$$

$$M_{max} = Py_{max} + Pe = P(y_{max} + e)$$

$$\Rightarrow \sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\sqrt{\frac{P}{EI}}\frac{KL}{2}\right)\right] \quad or \quad \sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{1}{2}\sqrt{\frac{P}{EA}}\frac{KL}{r}\right)\right]$$

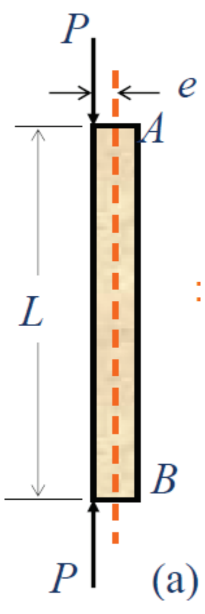
Buckling of Columns

❑ Eccentric Loading; The Secant Formula

The ***secant formula*** for a column subjected to eccentric compressive load *P* is given by

$$\frac{P}{A} = \frac{\sigma_{max}}{\left[1 + \frac{ec}{r^2}\sec\left(\frac{1}{2}\sqrt{\frac{P}{EA}}\frac{KL}{r}\right)\right]}$$

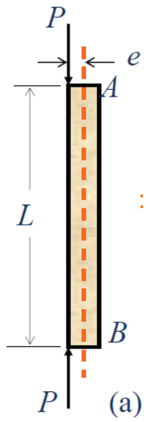
$$P = \frac{\sigma_{max}A}{\left[1 + \frac{ec}{r^2}\sec\left(\frac{1}{2}\sqrt{\frac{P}{EA}}\frac{KL}{r}\right)\right]}$$



Buckling of Columns

❑ Eccentric Loading; The Secant Formula

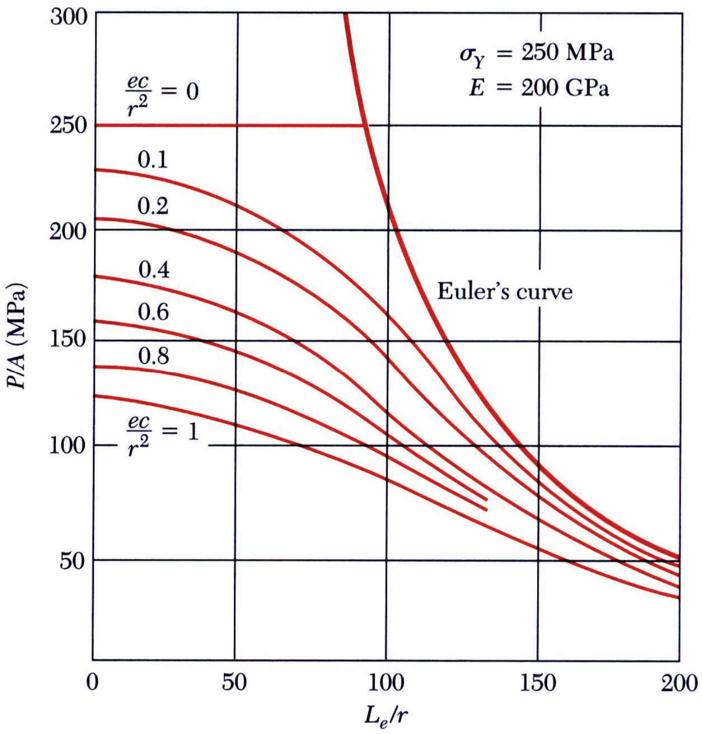
The formula is referred to as the secant formula; it defines the force per unit area, P/A, which causes a specified maximum stress σ_{max} in a column of given effective slenderness ratio, KL/r, for a given value of the ratio ec/r^2 , where e is the eccentricity of the applied load.



$$\frac{P}{A} = \frac{\sigma_{max}}{\left[1 + \frac{ec}{r^2}\sec\left(\frac{1}{2}\sqrt{\frac{P}{EA}}\frac{KL}{r}\right)\right]}$$

Buckling of Columns

❑ Eccentric Loading; The Secant Formula

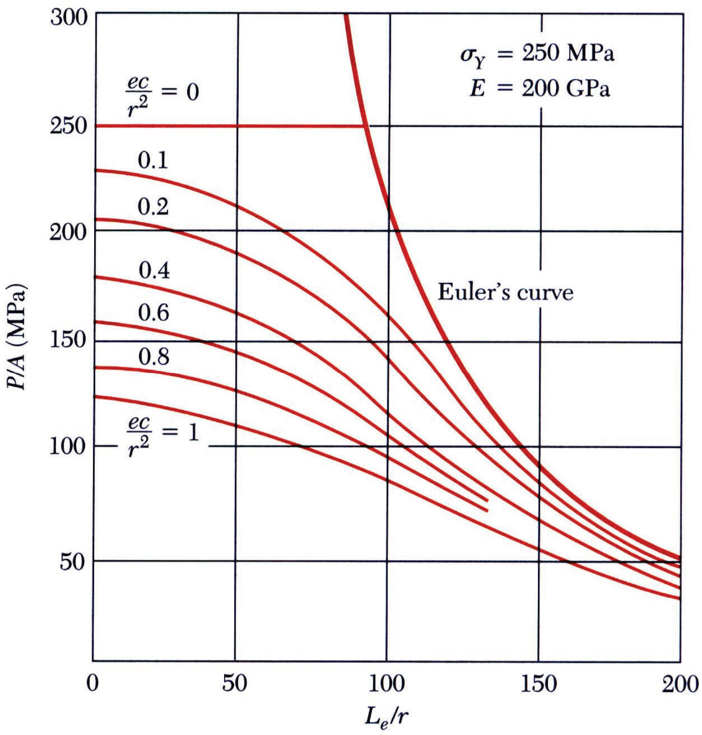


Load per unit area, P/A, causing ***yield*** in column

$$\frac{P}{A} = \frac{\sigma_Y}{\left[1 + \frac{ec}{r^2}\sec\left(\frac{1}{2}\sqrt{\frac{P}{EA}}\frac{KL}{r}\right)\right]}$$

Buckling of Columns

❑ Eccentric Loading; The Secant Formula



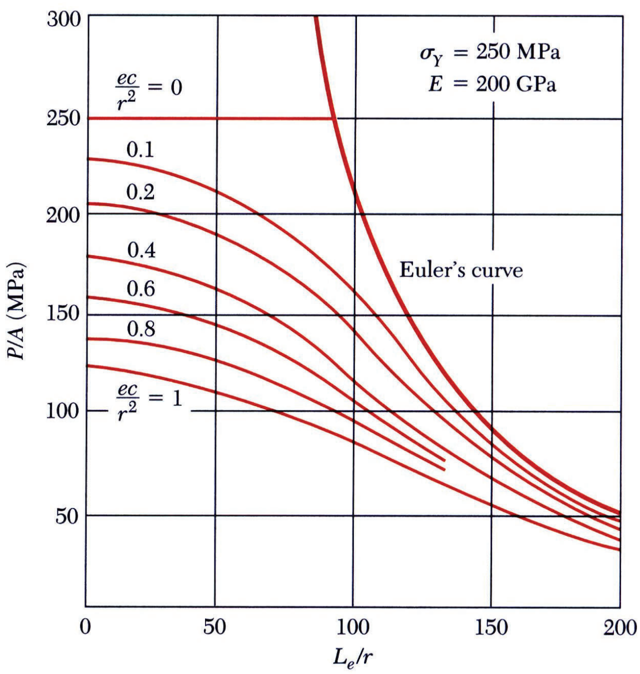
It should be noted that **for small values of KL/r**, the secant is almost equal to unity and P/A (or P) may be assumed equal to

$$\frac{P}{A} = \frac{\sigma_{max}}{1 + \frac{ec}{r^2}}$$

$$P = \frac{\sigma_{max} A}{1 + \frac{ec}{r^2}}$$

Buckling of Columns

❑ Eccentric Loading; The Secant Formula



- For large value of KL/*r*, the curves corresponding to the various values of the ratio ec/r^2 get very close to Euler's curve, and thus that the effect of the eccentricity of the loading on the value of *P/A* becomes negligible.
- The secant formula is mainly useful for intermediate values of KL/*r*.

Buckling of Columns

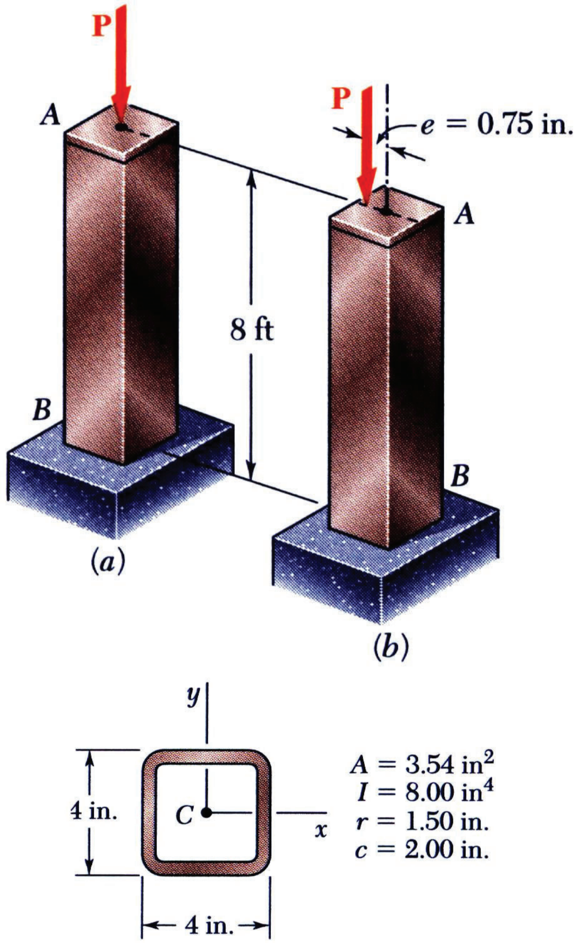
❑ Critical Buckling Stress

Example 10

The uniform column consists of an 8-ft section of structural tubing having the cross-section shown.

a) Using Euler's formula and a factor of safety of two, determine the allowable centric load for the column and the corresponding normal stress.

b) Assuming that the allowable load, found in part *a*, is applied at a point 0.75 in. from the geometric axis of the column, determine the horizontal deflection of the top of the column and the maximum normal stress in the column.



Buckling of Columns

❑ Critical Buckling Stress

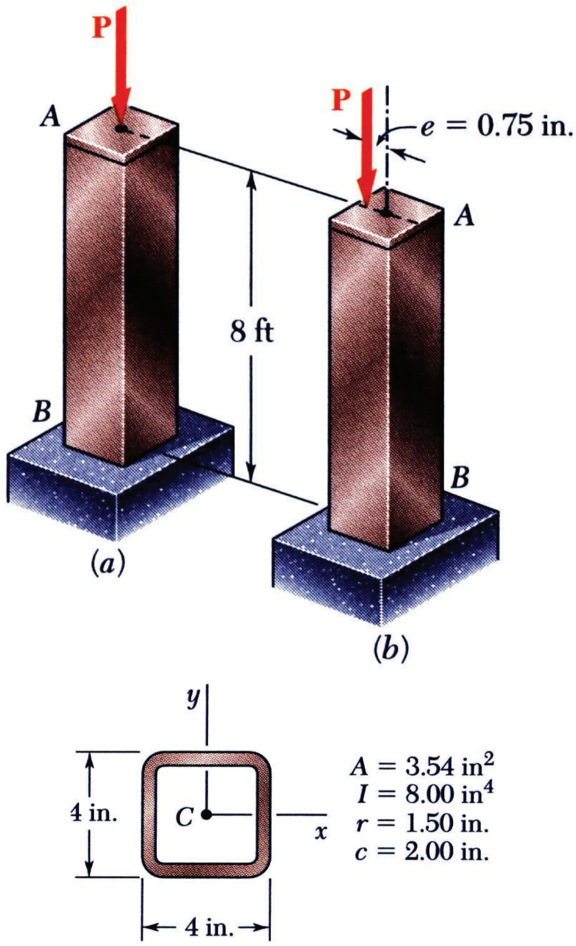
Example 10

SOLUTION:

- **Maximum allowable centric load:**

  - Effective length,

- Critical load,



Buckling of Columns

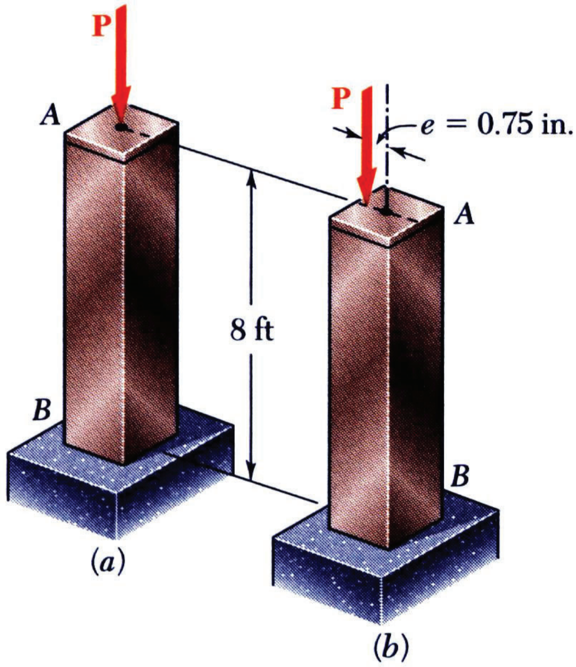
❑ Critical Buckling Stress

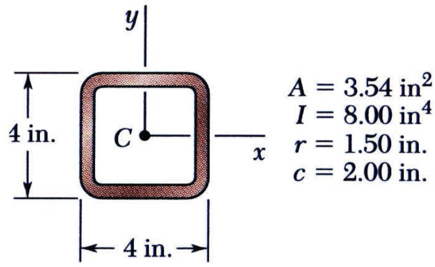
Example 10

SOLUTION:

- **Maximum allowable centric load:**

  - Allowable load,





Buckling of Columns

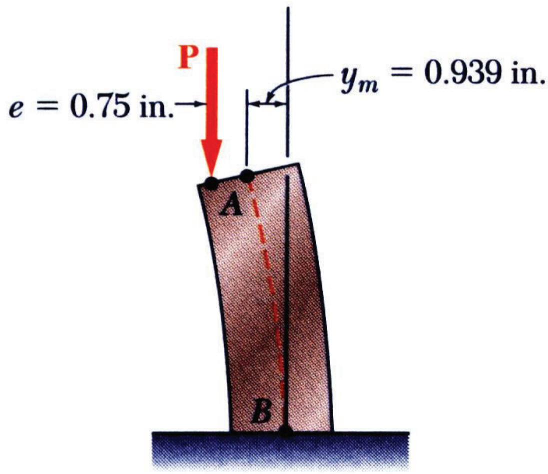
❑ Critical Buckling Stress

Example 10

- Eccentric load:

  - End deflection,

  - Maximum normal stress,

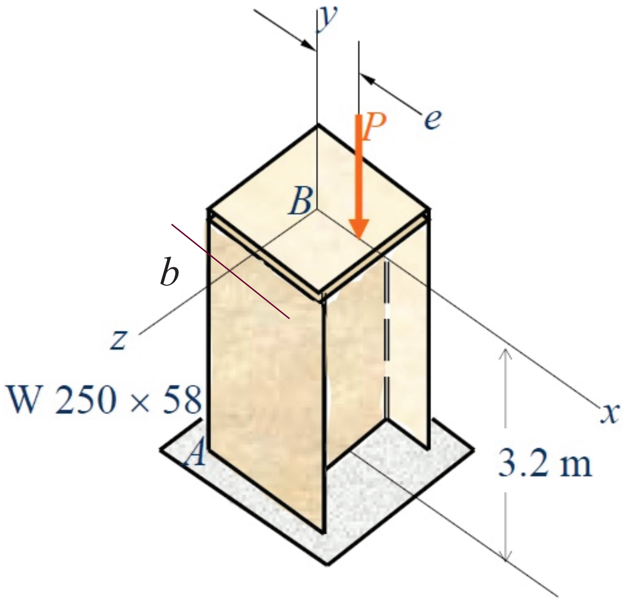


Buckling of Columns

❑ Critical Buckling Stress

Example 11

The axial load P is applied at a point located on the x axis at a distance e from the geometric axis of the W 250 × 58 rolled-steel column AB. When P = 350 kN, it is observed that the horizontal deflection of the top of the column is 5 mm. Using E = 200 GPa, determine (a) the eccentricity e of the load, (b) the maximum stress in the column.



$$I_x = 87\times10^{-6}\, m^4 \quad r_x = 0.1085\, m$$

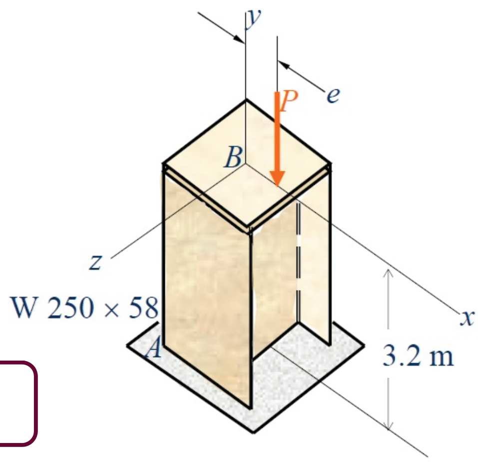
$$I_y = 18.73\times10^{-6}\, m^4 \quad r_y = 0.0502\, m$$

$$S_y = 184.5\times10^{-6}\, m^3 \quad A = 7.42\times10^{-3}\ m^2 \quad b = 203mm$$

Buckling of Columns

❑ Critical Buckling Stress

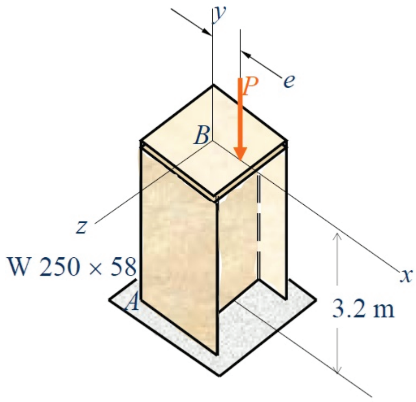
Example 11



Buckling of Columns

❑ Critical Buckling Stress

Example 11

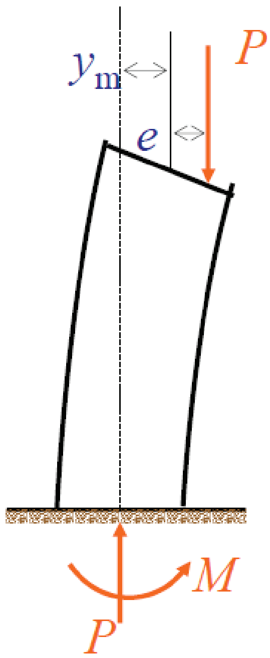


Buckling of Columns

❑ Critical Buckling Stress

Example 11

An alternate solution for Part (b):

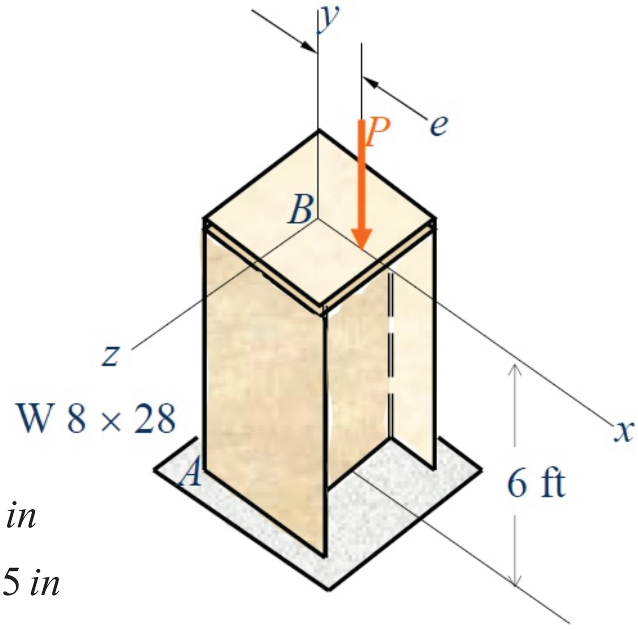


Buckling of Columns

❑ Critical Buckling Stress

Example 12

An axial load P is applied at a point located on the x axis at a distance e = 0.60 in. from the geometric axis of the W8 × 28 rolledsteel column AC. Knowing that the column is free at its top B and fixed at its base A. Determine the allowable load P if a factor of safety of 2.5 with respect to yield is required.



$E = 29\times10^6\ psi$ $\quad I_z = 21.7\ in^4$ $\quad r_z = 1.62\ in$

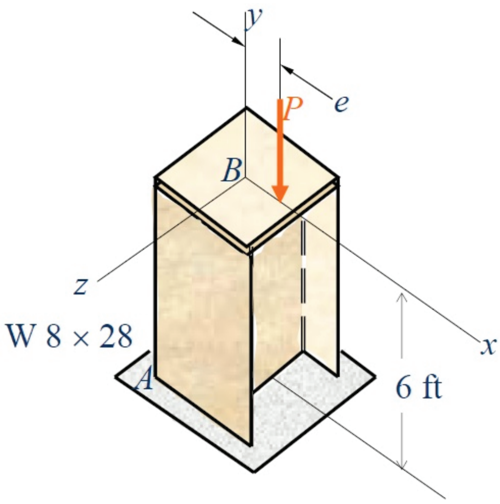
$\sigma_y = 36\ ksi$ $\quad A = 8.25\ in^2$ $\quad c = 3.2675\ in$

Buckling of Columns

❑ Critical Buckling Stress

Example 12

One-end fixed, one-end free column



Buckling of Columns

❑ Critical Buckling Stress

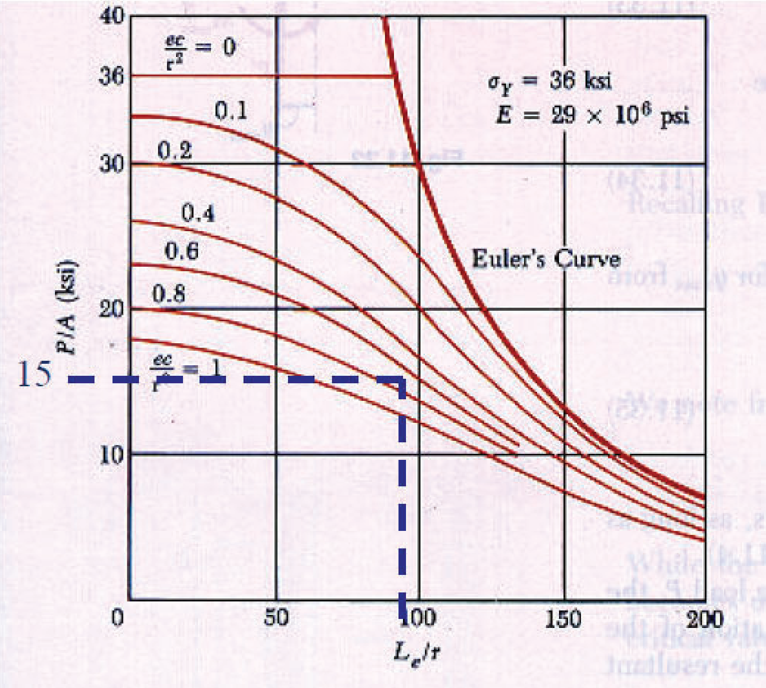
Example 12

Since $E = 29 \times 10^6\ psi$

$\sigma_y = 36\ ksi$

can be used:

We read



Buckling of Columns

❑ Critical Buckling Stress

Example 12

Suppose that we do not have the curves provided in design curves, or we do have the curves but our problem consists of a column that has different material (e.g., σy = 50 ksi), how can we evaluate the eccentric load P for Example 12?

❑ Critical Buckling Stress

Example 12

A general trial and error (iterative) procedure can be used as follows:

we assume an initial (guess) value for P in the right-hand side of the equation; **let it be 20 kips**, hence

❑ Critical Buckling Stress

Example 12

The revised value P= 163.80 kips can now be substituted in the right-hand side of the same equation to produce yet another revised value as follows:

Buckling of Columns

❑ Critical Buckling Stress

Example 12

A third iteration using a revised value for P= 103.01 kips, gives

Buckling of Columns

❑ Critical Buckling Stress

Example 12

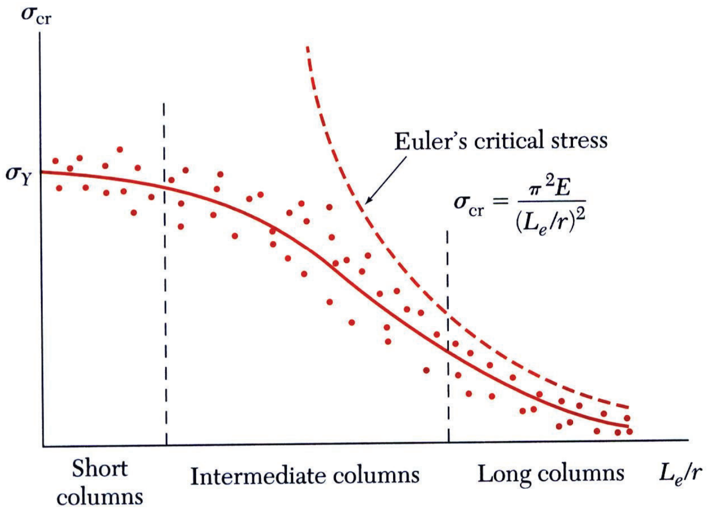
Initial Value of P →

P (kip)	
20.00	123.63
163.79	123.48
103.01	123.55
132.79	123.52
119.10	123.53
125.59	123.53
122.56	123.53
123.98	123.53
123.31	123.53

The iterative procedure is continued until the value of the eccentric load P converges to the exact solution of 123.53 kips, as shown in the spreadsheet result of Table

Buckling of Columns

❑ Design of Columns Under Centric Load



- Previous analyses assumed stresses below the proportional limit and initially straight, homogeneous columns

- Experimental data demonstrate
  - for large L_e/r, σ_{cr} follows Euler's formula and depends upon E but not σ_Y.
  - for small L_e/r, σ_{cr} is determined by the yield strength σ_Y and not E.
  - for intermediate L_e/r, σ_{cr} depends on both σ_Y and E.

Buckling of Columns

❑ Design of Columns Under Centric Load

Structural Steel

American Inst. of Steel Construction

- For $L_e/r > C_c$

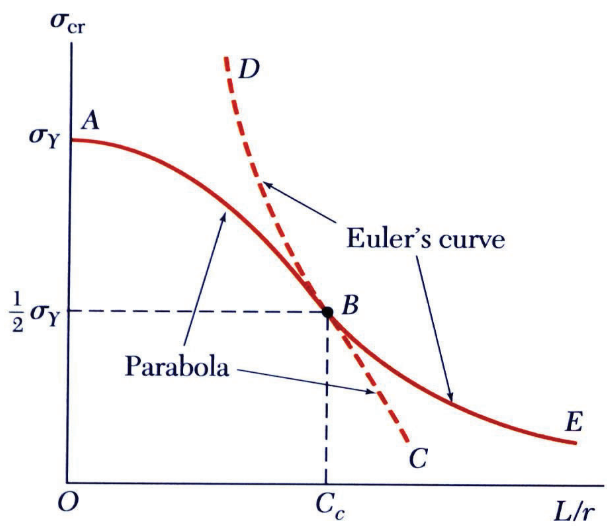
$$\sigma_{cr} = \frac{\pi^2 E}{(L_e/r)^2} \qquad \sigma_{all} = \frac{\sigma_{cr}}{FS}$$

$$FS = 1.92$$

- For $L_e/r < C_c$

$$\sigma_{cr} = \sigma_Y\left[1 - \frac{(L_e/r)^2}{2C_c^2}\right] \qquad \sigma_{all} = \frac{\sigma_{cr}}{FS}$$

$$FS = \frac{5}{3} + \frac{3}{8}\frac{L_e/r}{C_c} - \frac{1}{8}\left(\frac{L_e/r}{C_c}\right)^3$$



- At $L_e/r = C_c$

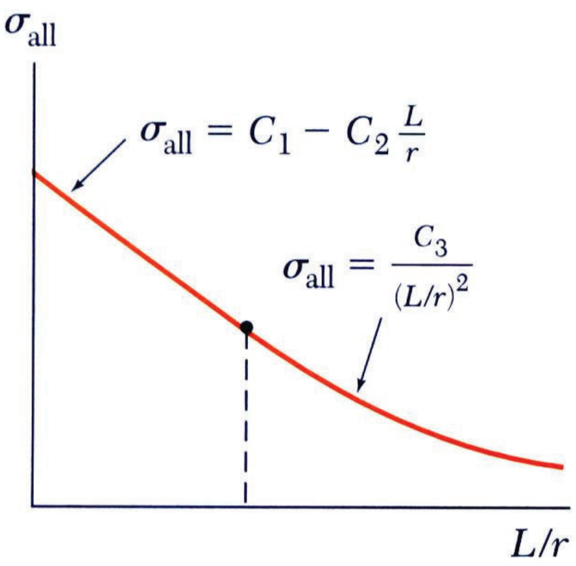
$$\sigma_{cr} = \tfrac{1}{2}\sigma_Y \qquad C_c^2 = \frac{2\pi^2 E}{\sigma_Y}$$

Buckling of Columns

❑ Design of Columns Under Centric Load

Aluminum

Aluminum Association, Inc.



- Alloy 6061-T6

$L_e/r < 66$:

$$\sigma_{all} = [20.2 - 0.126(KL/r)]\,\text{ksi}$$
$$\sigma_{all} = [139 - 0.868(KL/r)]\,\text{MPa}$$

$L_e/r \geq 66$:

$$\sigma_{all} = \frac{51000\,\text{ksi}}{(KL/r)^2} = \frac{351\times10^3\,\text{MPa}}{(KL/r)^2}$$

- Alloy 2014-T6

$L_e/r < 55$:

$$\sigma_{all} = [30.7 - 0.23(KL/r)]\,\text{ksi}$$
$$\sigma_{all} = [212 - 1.585(KL/r)]\,\text{MPa}$$

$L_e/r \geq 55$:

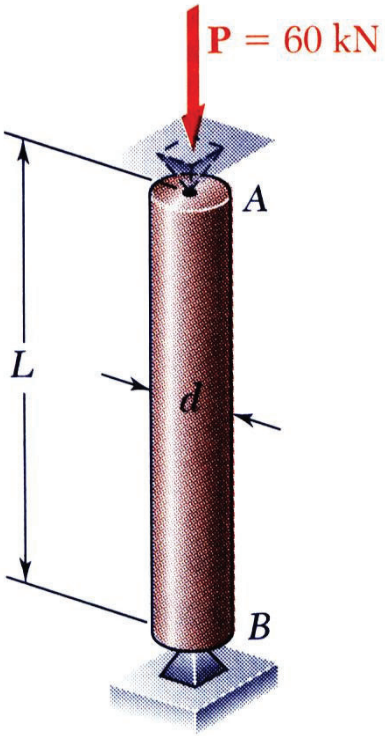
$$\sigma_{all} = \frac{54000\,\text{ksi}}{(KL/r)^2} = \frac{372\times10^3\,\text{MPa}}{(KL/r)^2}$$

Buckling of Columns

❑ Design of Columns Under Centric Load

Example 13

Using the aluminum alloy2014-T6, determine the smallest diameter rod which can be used to support the centric load P = 60 kN if a) L = 750 mm, b) L = 300 mm

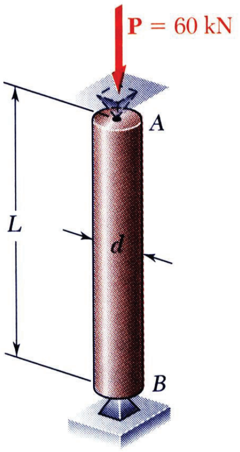


Buckling of Columns

❑ Design of Columns Under Centric Load

Example 13

$$r = \sqrt{\frac{I}{A}} = \sqrt{\frac{\pi c^4/4}{\pi c^2}} = \frac{c}{2}$$



- For $L = 750$ mm, assume $L/r > 55$
- Determine cylinder radius:
- Check slenderness ratio assumption:

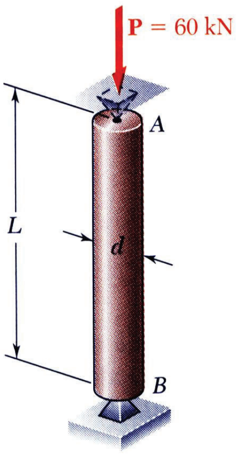
assumption was correct

Buckling of Columns

❑ Design of Columns Under Centric Load

Example 13

$$r = \sqrt{\frac{I}{A}} = \sqrt{\frac{\pi c^4/4}{\pi c^2}} = \frac{c}{2}$$

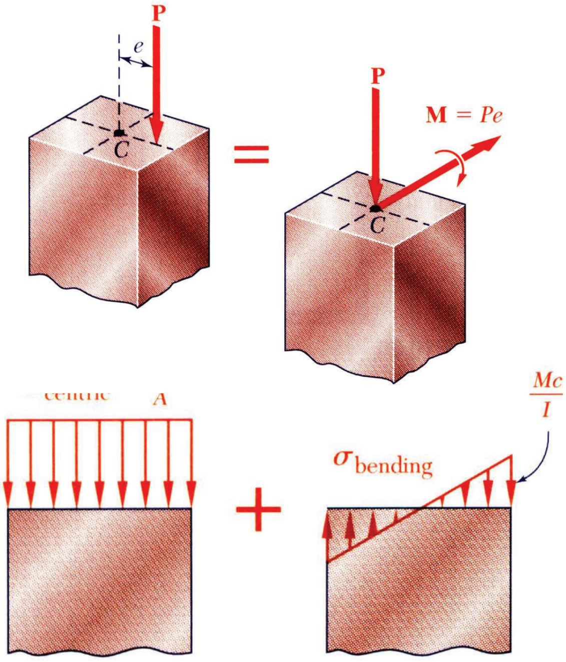


- For $L = 300$ mm, assume $L/r < 55$
- Determine cylinder radius:
- Check slenderness ratio assumption:

assumption was correct

Buckling of Columns

❑ Design of Columns Under Centric Load



- An eccentric load P can be replaced by a centric load P and a couple $M = Pe$.

- Normal stresses can be found from superposing the stresses due to the centric load and couple,

$$\sigma = \sigma_{centric} + \sigma_{bending}$$

$$\sigma_{\max} = \frac{P}{A} + \frac{Mc}{I}$$

- **Allowable stress method:**

$$\frac{P}{A} + \frac{Mc}{I} \le \sigma_{all}$$

- **Interaction method:**

$$\frac{P/A}{(\sigma_{all})_{centric}} + \frac{Mc/I}{(\sigma_{all})_{bending}} \le 1$$