Mechanics of Materials

دانشگاه کردستان
University of Kurdistan
زانکۆی کوردستان

Ferdinand P.Beer, E.Russel Johnston, Jr., John T.Dewolf

Other Reference:
J.Wat Oler "Lectures notes on Mechanics od Materials"
Ibrahim A.Assakkaf "Lectures notes on Mechanics od Materials"

Composite Beams

By: Kaveh Karami
Associate Prof. of Structural Engineering
https://prof.uok.ac.ir/Ka.Karami

Bending of Members Made of Several Materials

Composite Beams

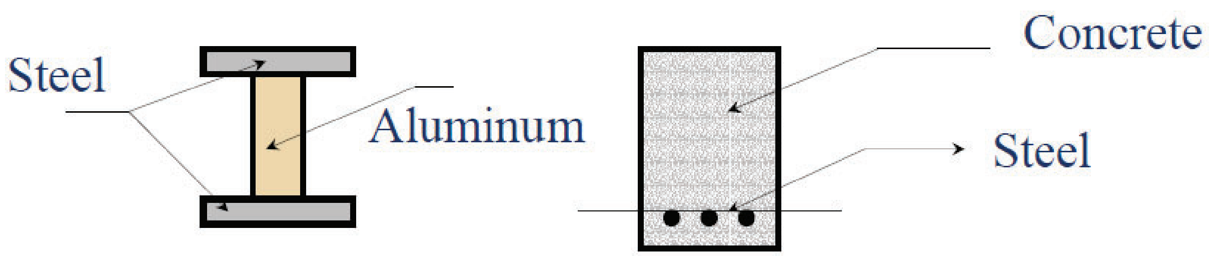
Bending of Composite Beams

- In the previous discussion, we have considered only those beams that are fabricated from a ***single material*** such as steel.
- However, in engineering design there is an increasing trend to use beams fabricated from two or more materials.

Composite Beams

Bending of Composite Beams

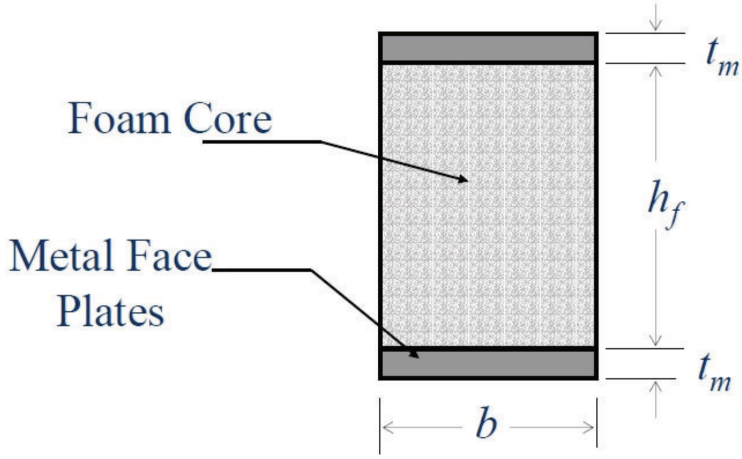
- These are called composite beams.
- They offer the opportunity of using each of the materials employed in their construction advantage.



Composite Beams

Foam Core with Metal Cover Plates

- The design concept of this composite beam is to use light-low strength foam to ***support the load-bearing metal plates*** located at the top and bottom.



Bending of Members Made of Several Materials

Composite Beams

Foam Core with Metal Cover Plates:

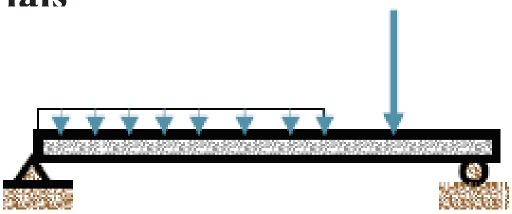
- The ***strain is continuous*** across the interface between the foam and the cover plates.
- The ***stress in the foam is considered zero*** because its modulus of elasticity is small compared to the modulus of elasticity of the metal.

$$E_f << E_m \quad \Rightarrow \quad \sigma_f = E_f \varepsilon_f \approx 0$$

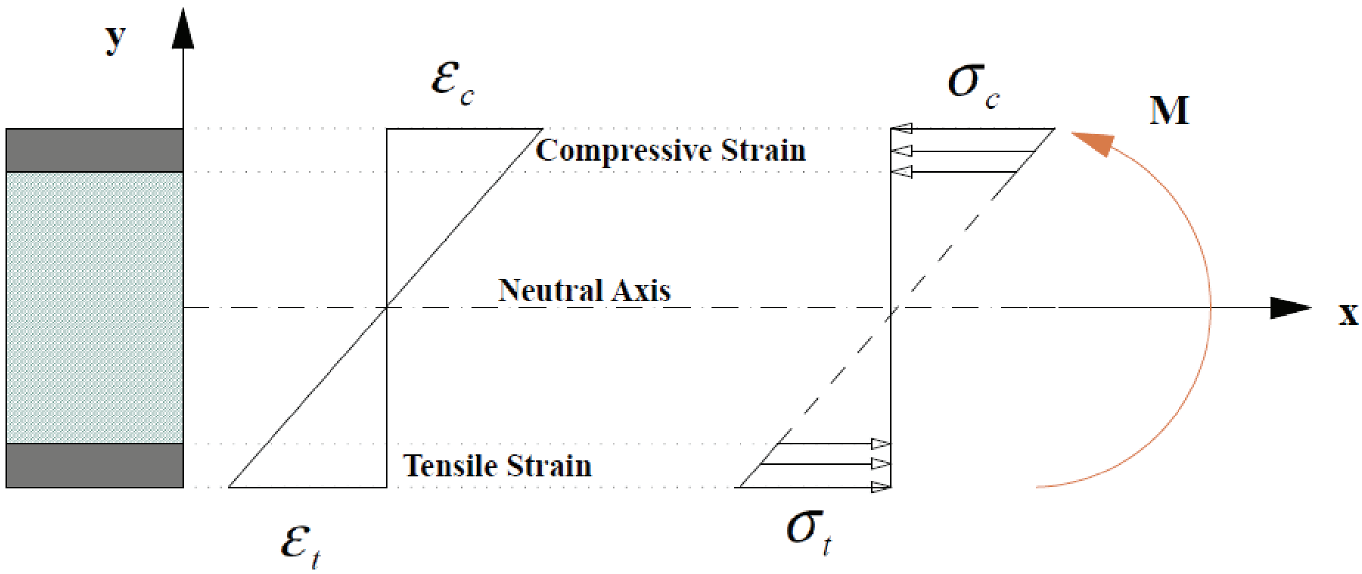
Bending of Members Made of Several Materials

Composite Beams

Foam Core with Metal Cover Plates



–Assumptions:
- Plane sections remain plane before and after loading.
- The strain is linearly distributed.



Composite Beams

Foam Core with Metal Cover Plates

Using Hooke's law, the stress in the metal of the cover plates can be expressed as

$$\varepsilon = \frac{y}{\rho} \quad \& \quad \frac{1}{\rho} = \frac{M}{EI}$$

$$\sigma_m = E\varepsilon_m \quad \Rightarrow \quad \sigma_m = -\frac{E}{\rho} y \quad \Rightarrow \quad \sigma_m = -\frac{M \cdot y}{I}$$

Composite Beams

Foam Core with Metal Cover Plates:

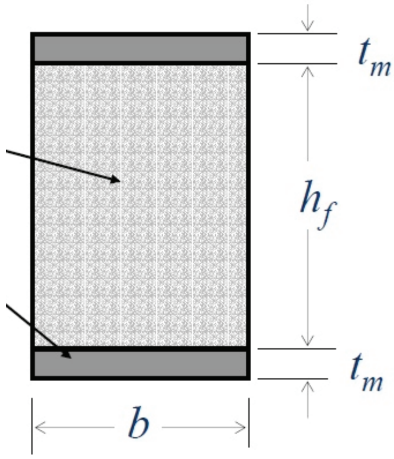
- The relation for the ***stress is the same as that established earlier***, however, the foam does not contribute to the load carrying capacity of the beam because its ***modulus of elasticity is negligible***.
- For this reason, ***the foam is not considered when determining the moment of inertia***.

Bending of Members Made of Several Materials

Composite Beams

Foam Core with Metal Cover Plates

Under these assumptions, the moment of inertia about the neutral axis is given by



$$I_{NA} = 2I_{x'} + 2Ad^2 \Rightarrow$$

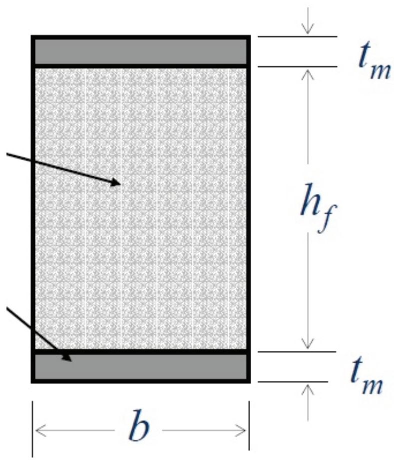
$$I_{NA} = 2\left(\frac{1}{12} bt_m^{\ 3}\right) + 2(bt_m)\left(\frac{h_f}{2} + \frac{t_m}{2}\right)^2 \approx \boxed{\frac{bt_m}{2}(h_f + t_m)^2}$$

Bending of Members Made of Several Materials

Composite Beams

Foam Core with Metal Cover Plates

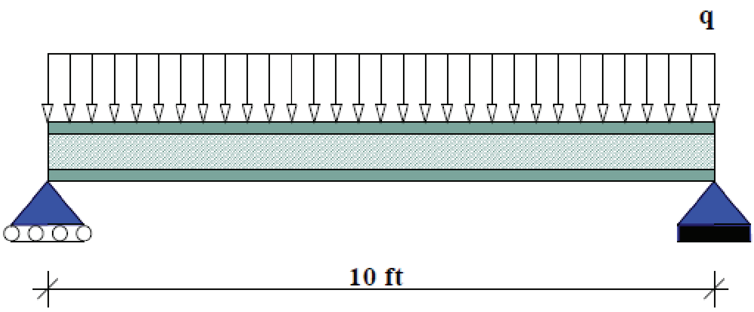
the maximum stress in the metal is computed as

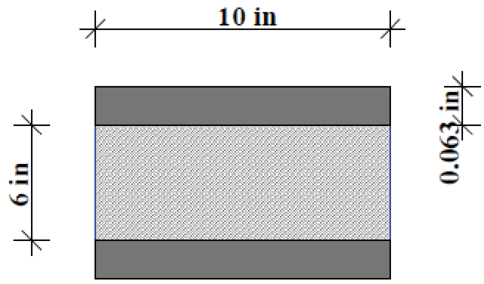


$$\sigma_{\max} = \frac{M.y}{I} = \frac{M.\left(\frac{h_f}{2} + t_m\right)}{\frac{bt_m}{2}(h_f + t_m)^2} \Rightarrow \boxed{\sigma_{\max} = \frac{M.(h_f + 2t_m)}{bt_m(h_f + t_m)^2}}$$

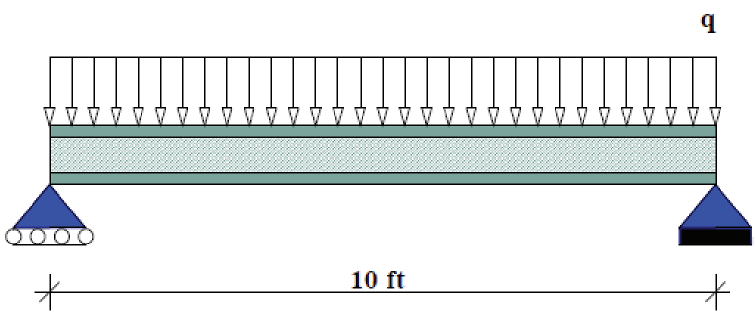
Example 1

A simply-supported, foam core, metal cover plate composite beam is subjected to a uniformly distributed load of magnitude *q*. Aluminum cover plates are adhesively bonded to a polystyrene foam core. **Determine q.** $F_Y = 32\ ksi$





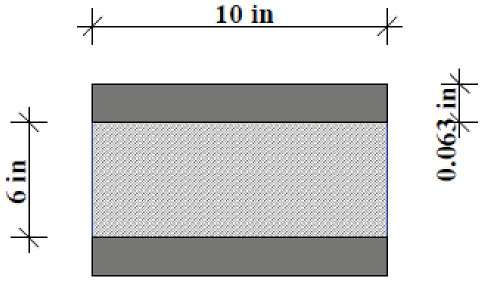
Example 1



The maximum moment for a simply supported beam is given by

When the composite beam yields, the stresses in the cover plates are

Example 1



Composite Beams

Bending of Members Made of Several Materials

- The derivation given for foam core with metal plating was based on the assumption that the modulus of elasticity of the foam is ***so negligible***, that is, ***it does not contribute to the load-carrying capacity*** of the composite beam.

Composite Beams

Bending of Members Made of Several Materials

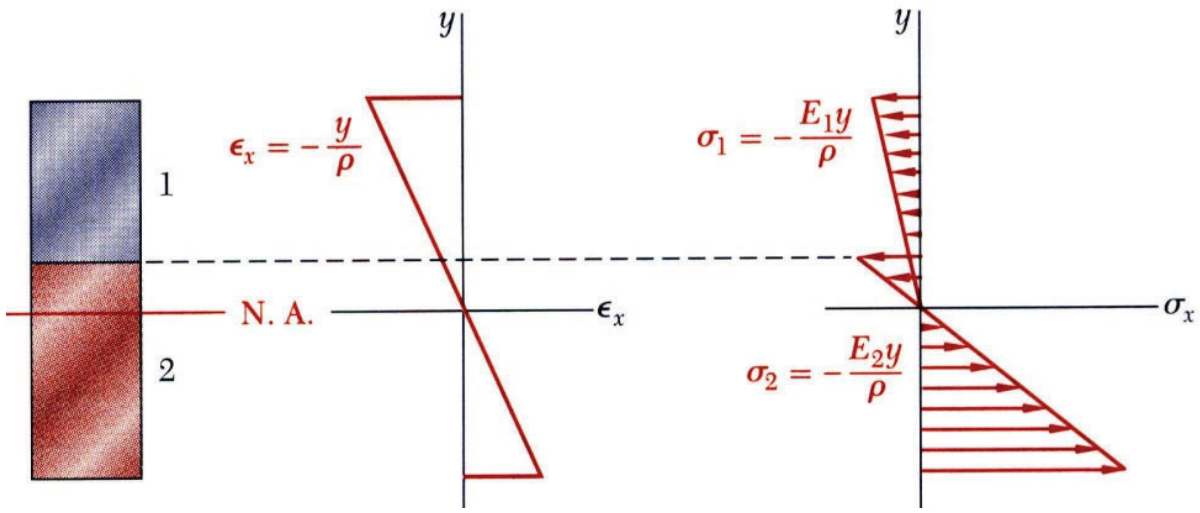
- When the moduli of elasticity of various materials that make up the beam structure are ***not negligible*** and they should be accounted for, then procedure for calculating the normal stresses and shearing stresses on the section will follow different approach, the ***transformed section of the member***.

Composite Beams

Transformed Section

- Consider a composite beam formed from two materials with E_1 and E_2.
- Thus the normal strain still varies linearly with the distance y from the NA.

$$\varepsilon_x = -\frac{y}{\rho}$$



Composite Beams

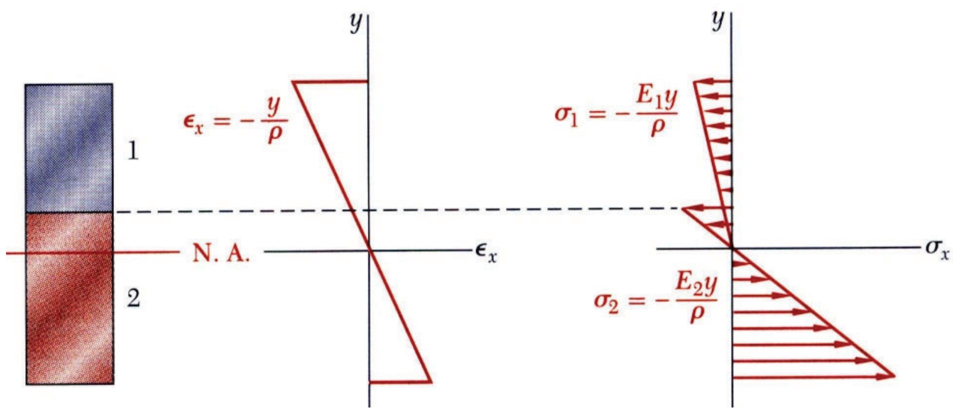
❑ Transformed Section

- Because we have different materials, ***we cannot simply assume that the neutral axis passes through the centroid of the composite section.***

- In fact one of the goal of this discussion will be to ***determine the location of this axis***.

Composite Beams

Transformed Section



$$\begin{cases} \sigma_1 = E_1 \varepsilon_x = -\dfrac{E_1 y}{\rho} \\ \sigma_2 = E_2 \varepsilon_x = -\dfrac{E_2 y}{\rho} \end{cases} \Rightarrow \begin{cases} dF_1 = \sigma_1 dA = -\dfrac{E_1 y}{\rho} dA \\ dF_2 = \sigma_2 dA = -\dfrac{E_2 y}{\rho} dA \end{cases}$$

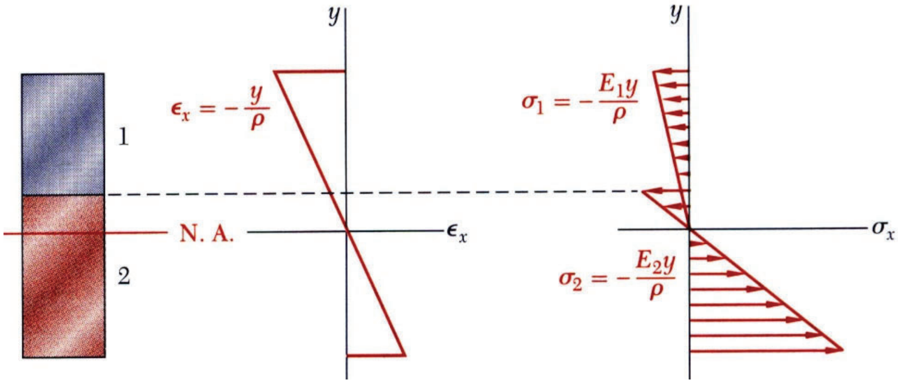
Composite Beams

Transformed Section

$$If \quad \frac{E_2}{E_1} = n \quad \Rightarrow$$

$$dF_2 = -\frac{(nE_1)y}{\rho} dA \quad \Rightarrow$$

$$dF_2 = -\frac{E_1 y}{\rho}(ndA)$$



It is noted that the same force ***dF_2* would be exerted on an element of area ndA of the first material**.

Bending of Members Made of Several Materials

Composite Beams

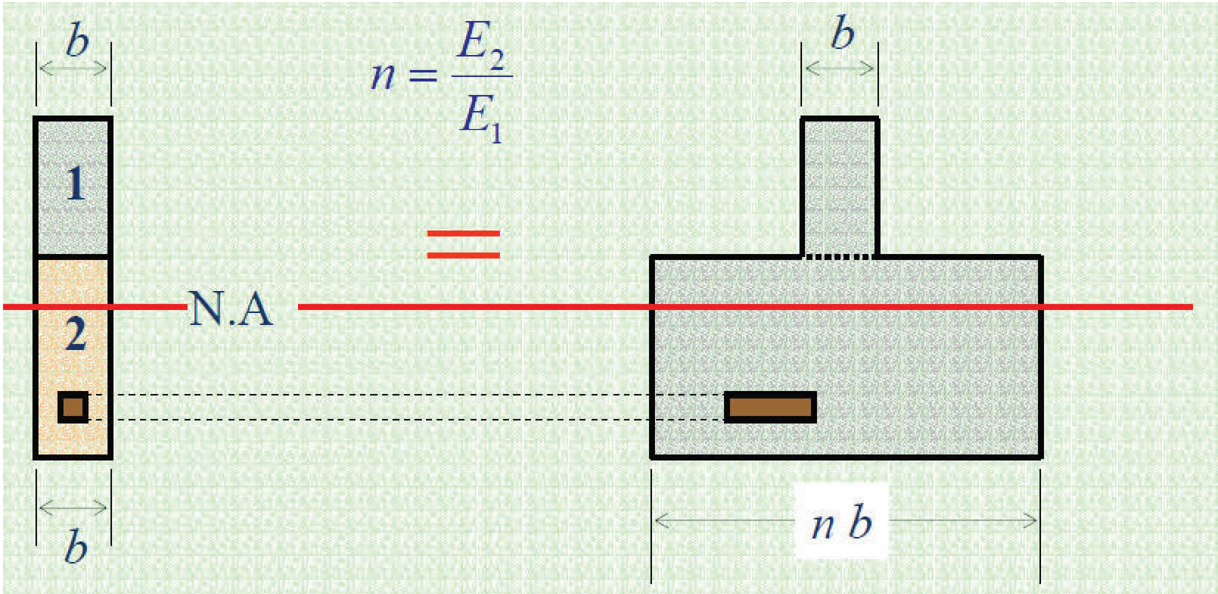
❑ Transformed Section

- This mean that the resistance to bending of the bar would remain the same if both portions were made of the first material, providing that the width of each element of the lower portion were multiplied by the factor n.
- The widening (if n>1) and narrowing (n<1) must be accomplished in ***a direction parallel to the neutral axis of the section.***

Bending of Members Made of Several Materials

Composite Beams

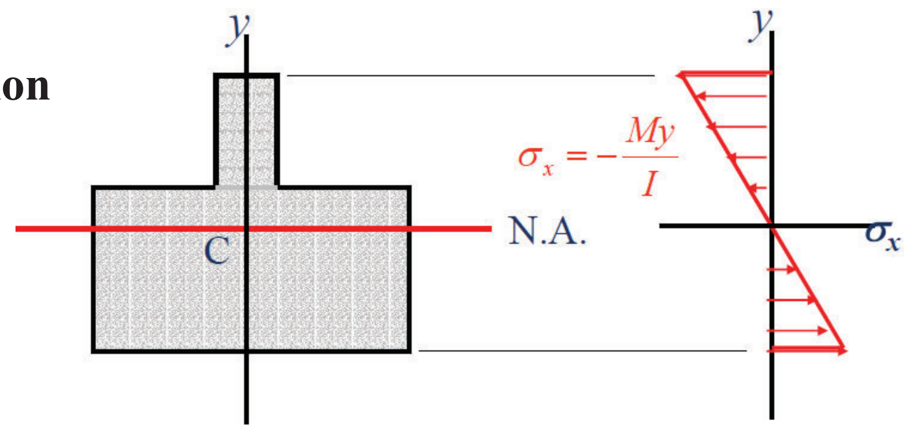
Transformed Section



Bending of Members Made of Several Materials

Composite Beams

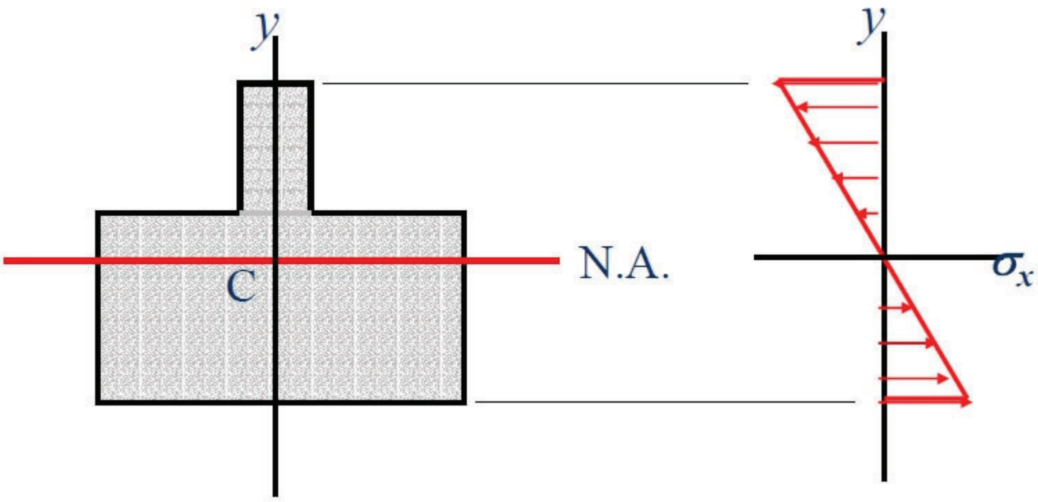
- **Transformed Section**



Since the transformed section represents the cross section of a member made of a homogeneous material with a modulus of elasticity $E1$, the previous method may be used ***to find the neutral axis*** of the section and ***the stresses at various points*** of the section.

Composite Beams

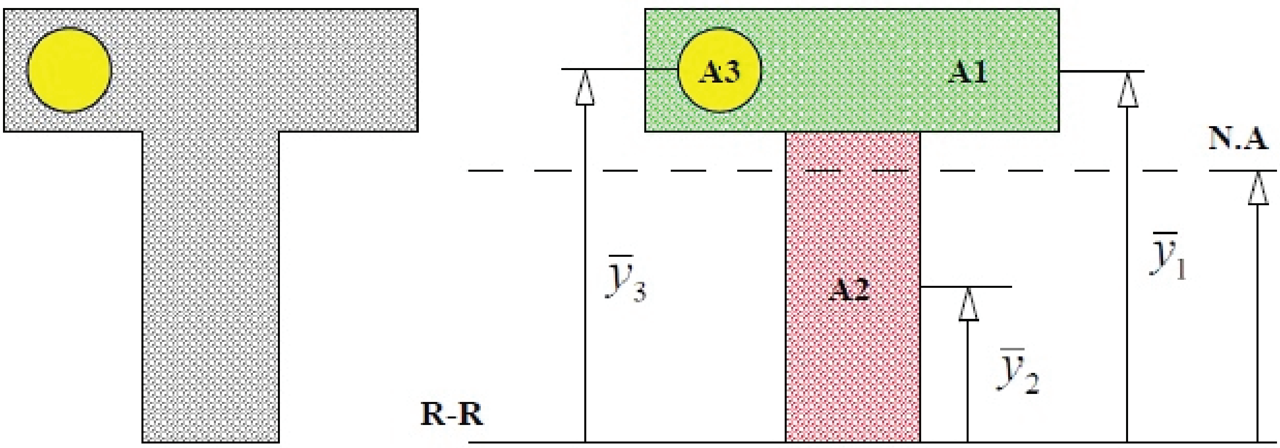
❑ Stresses on Transformed Section



$$\sigma_1 = \frac{My}{I}$$

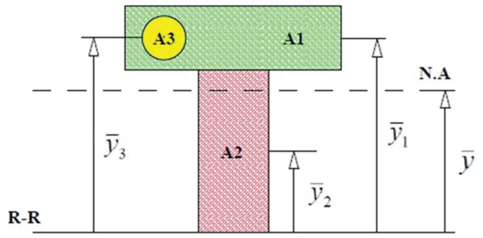
$$\sigma_2 = n \cdot \frac{My}{I}$$

❑ Review: Calculate the Moment of Inertia



Bending of Members Made of Several Materials

- Review: Calculate the Moment of Inertia



Parts	A_i	$\bar{y}_i$	$A_i\bar{y}_i$	$A_i\bar{y}_i^2$	I_{gi}
1	A_1	$\bar{y}_1$	$A_1\bar{y}_1$	$A_1\bar{y}_1^2$	I_{g1}
2	A_2	$\bar{y}_2$	$A_2\bar{y}_2$	$A_2\bar{y}_2^2$	I_{g2}
3	$-A_3$	$\bar{y}_3$	$-A_3\bar{y}_3$	$-A_3\bar{y}_3^2$	$-I_{g3}$
	$\sum A_i$		$\sum A_i\bar{y}_i$	$\sum A_i\bar{y}_i^2$	$\sum I_{gi}$

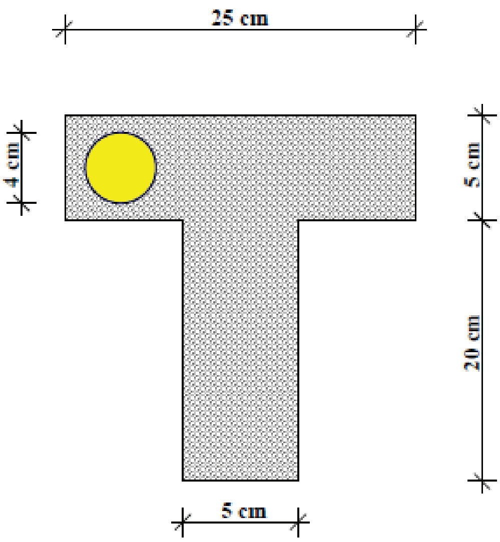
$$A = \sum A_i \qquad I_{R-R} = \sum I_{gi} + \sum A_i\bar{y}_i^2$$

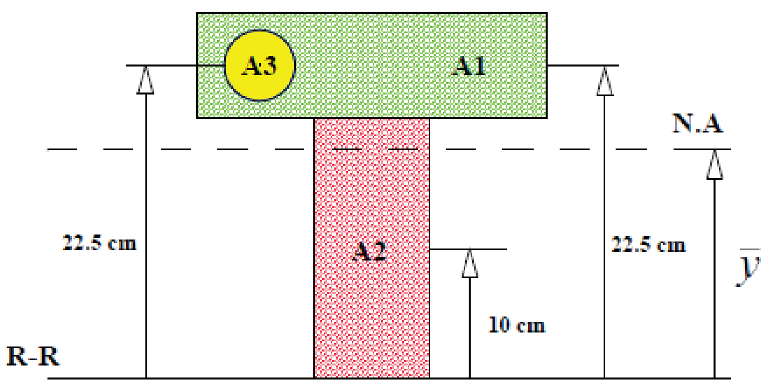
$$\bar{y} = \frac{\sum A_i\bar{y}_i}{\sum A_i} \qquad I_{NA} = \sum I_{gi} + \sum A_i\bar{y}_i^2 - \frac{(\sum A_i\bar{y}_i)^2}{\sum A_i}$$

Bending of Members Made of Several Materials

Example 2

Determine the moment of Inertia.





Bending of Members Made of Several Materials

Example 2

Parts	A_i	$\bar{y}_i$	$A_i\bar{y}_i$	$A_i\bar{y}_i^2$	$I_{g\,i}$
1					
2					
3					

Bending of Members Made of Several Materials

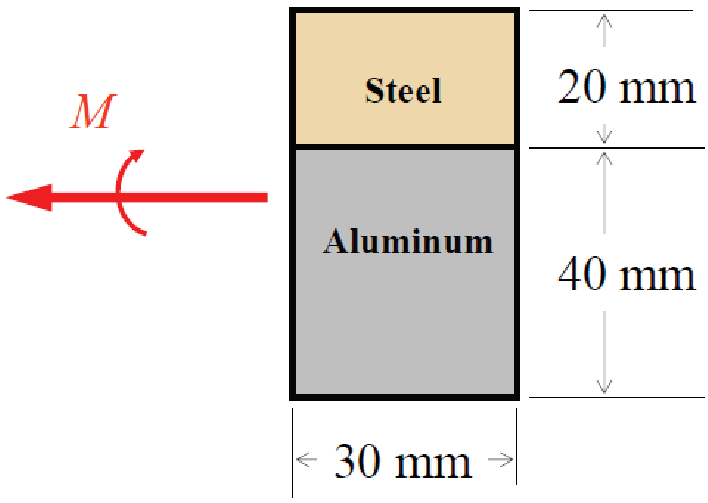
Example 3

A steel bar and aluminum bar are bonded together to form the composite beam shown. Knowing that the beam is bent about a horizontal axis by a moment M, ***determine the maximum stress in (a) the aluminum and (b) the steel.***

$E_a = 70\ Gpa$

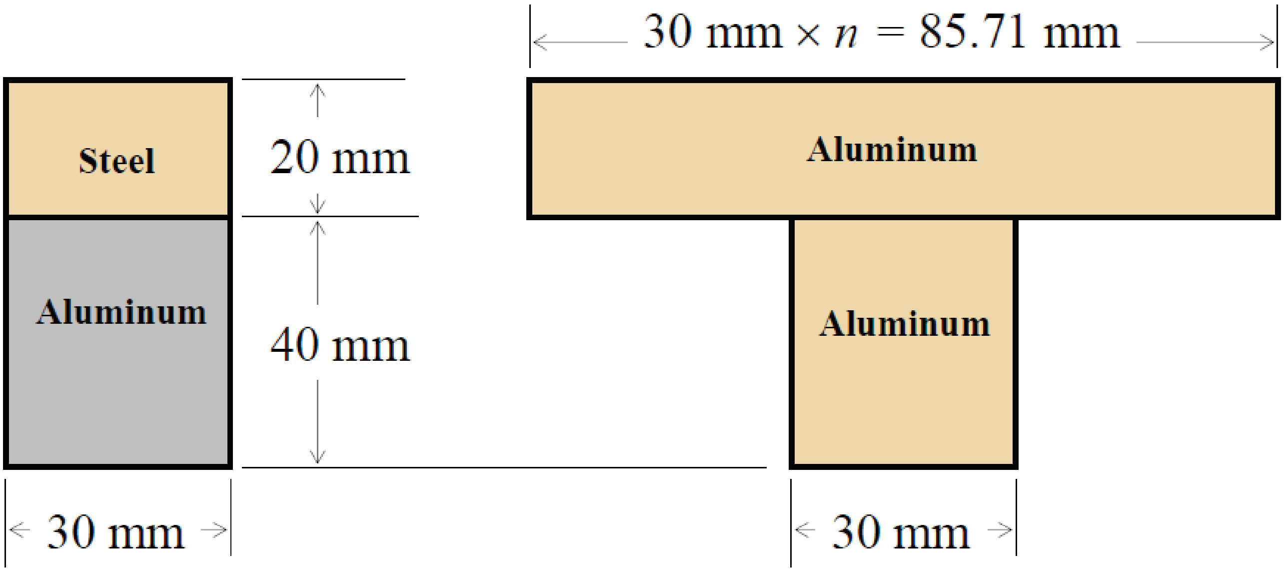
$E_s = 200\ Gpa$

$M = 1500\ N.m$



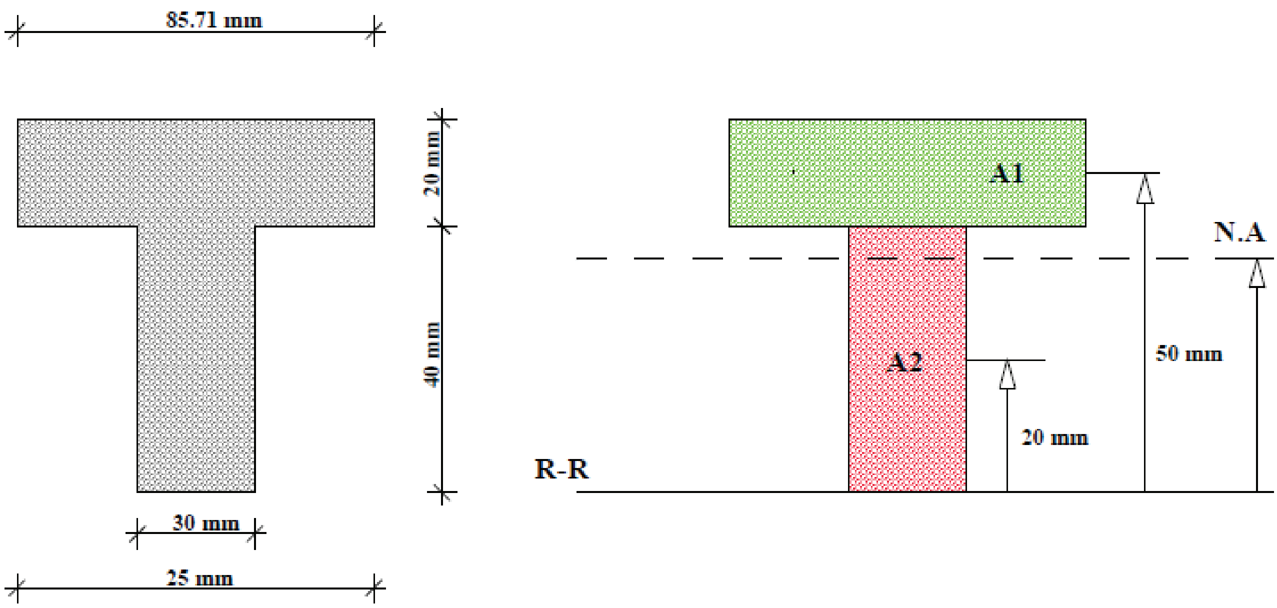
Bending of Members Made of Several Materials

Example 3



Bending of Members Made of Several Materials

Example 3



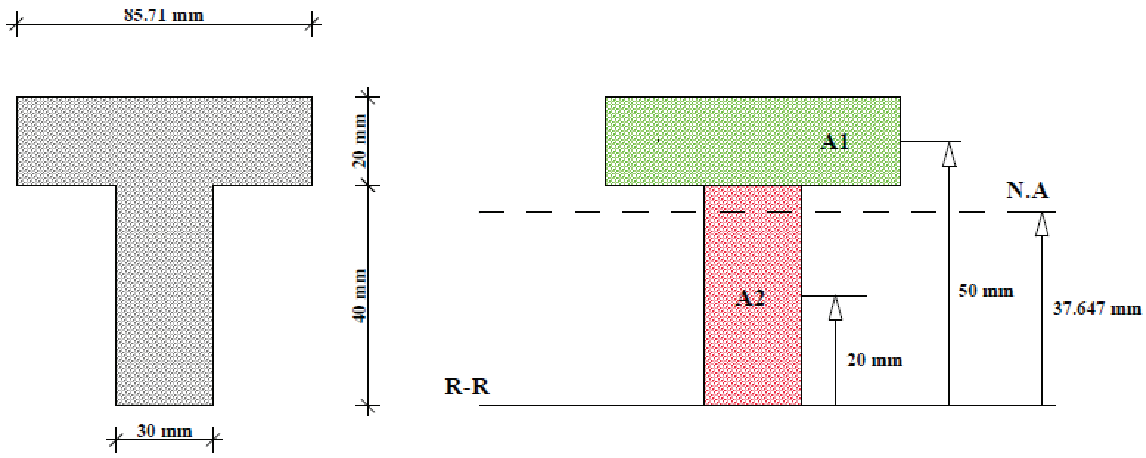
Bending of Members Made of Several Materials

Example 3

Parts	A_i	$\bar{y}_i$	$A_i\bar{y}_i$	$A_i\bar{y}_i^{\,2}$	$I_{g\,i}$
1					
2					

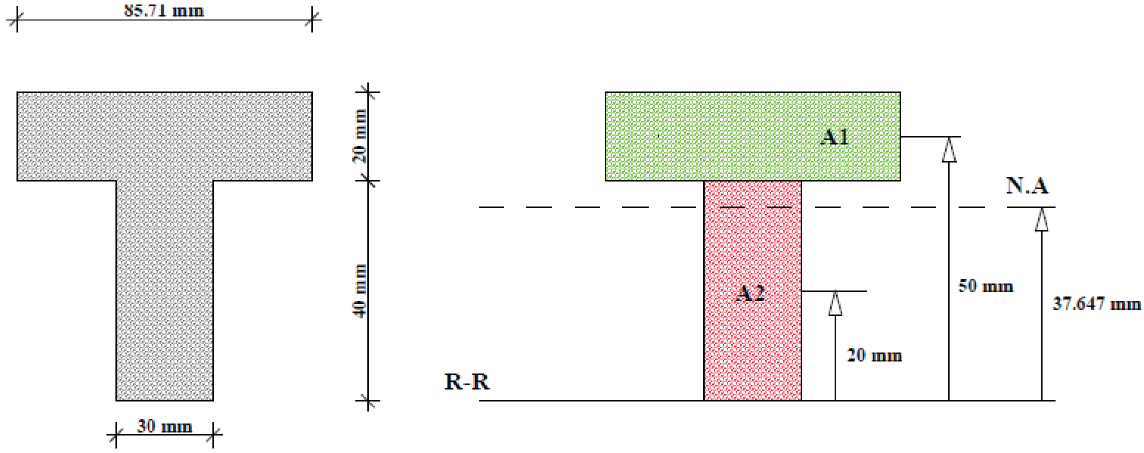
Bending of Members Made of Several Materials

Example 3



Bending of Members Made of Several Materials

Example 3



Bending of Members Made of Several Materials

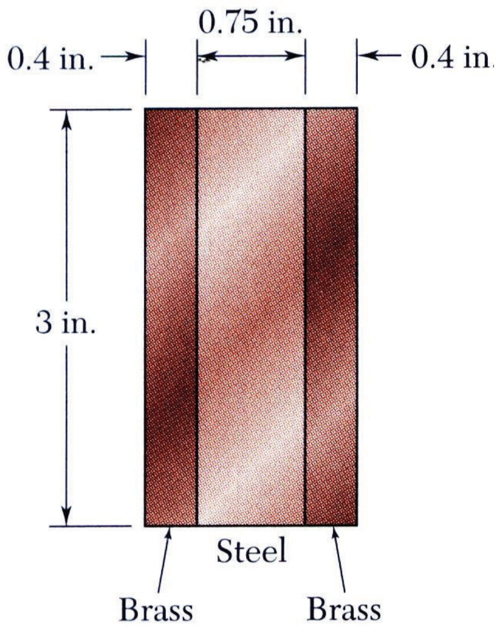
Example 4

Bar is made from bonded pieces of steel and brass. ***Determine the maximum stress in the steel and brass.***

$$E_b = 15\times10^6 \ psi$$

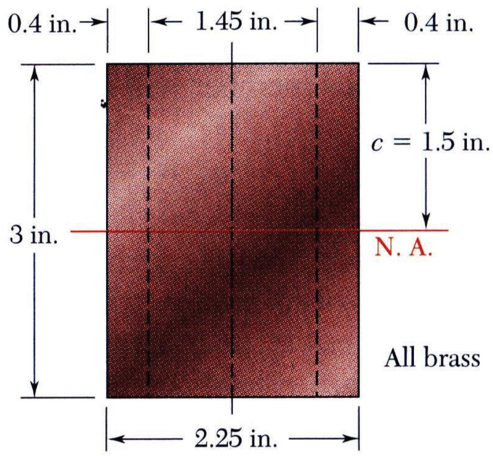
$$E_s = 29\times10^6 \ psi$$

$$M = 1500 \ kip.in$$



Bending of Members Made of Several Materials

Example 4

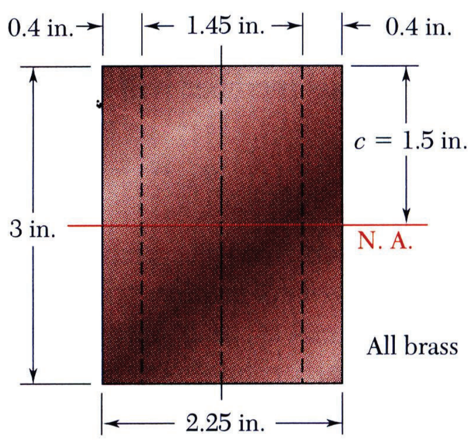


SOLUTION:

- Transform the bar to an equivalent cross section made entirely of brass.

- Evaluate the transformed cross sectional properties

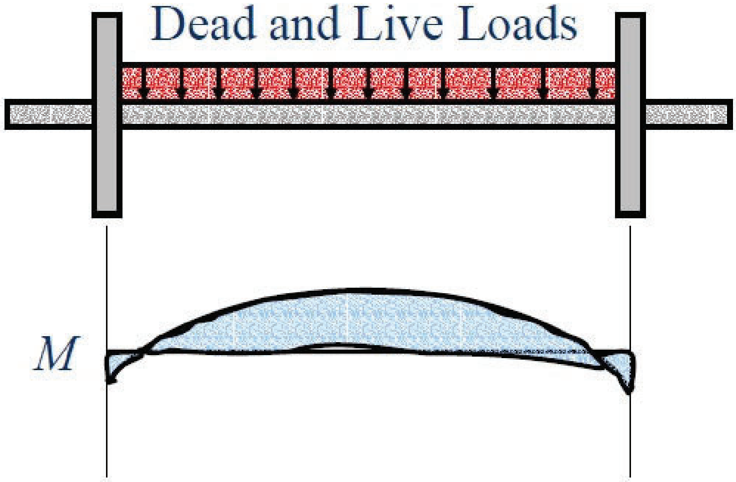
Example 4

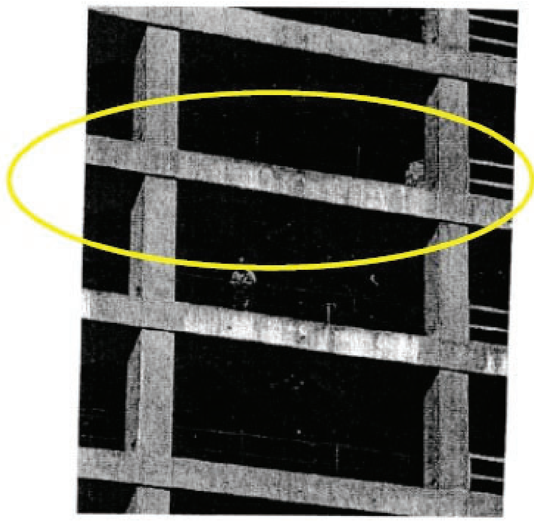


- Calculate the maximum stresses

Bending of Members Made of Several Materials

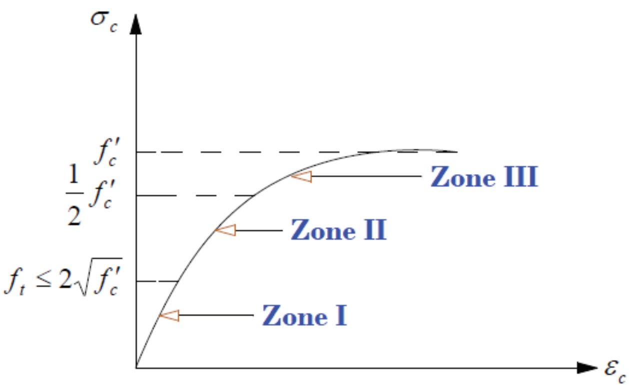
Reinforced Concrete Beams





Bending of Members Made of Several Materials

Reinforced Concrete Beams



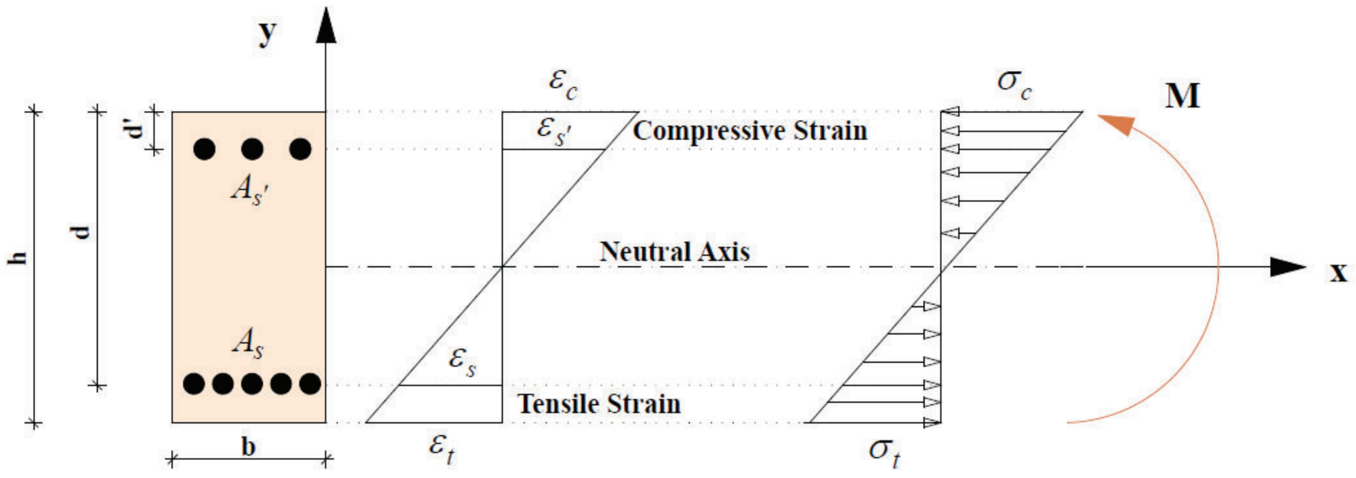
Zone I:

- Stresses are in the low level.
- There are no cracks in concrete. $f_t \le 2\sqrt{f'_c}$
- The section works as homogeneous material
- Strain & Stress has linear behavior.
- Transformed Section

Bending of Members Made of Several Materials

Reinforced Concrete Beams

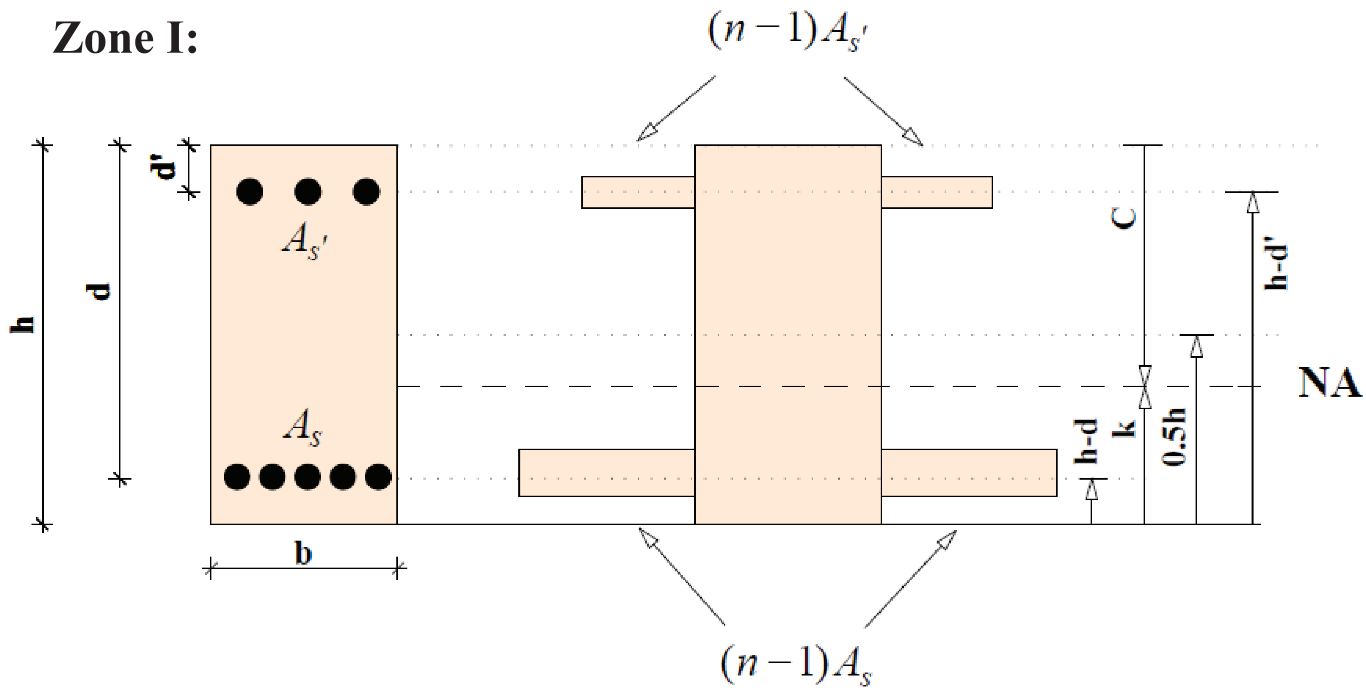
Zone I:



Bending of Members Made of Several Materials

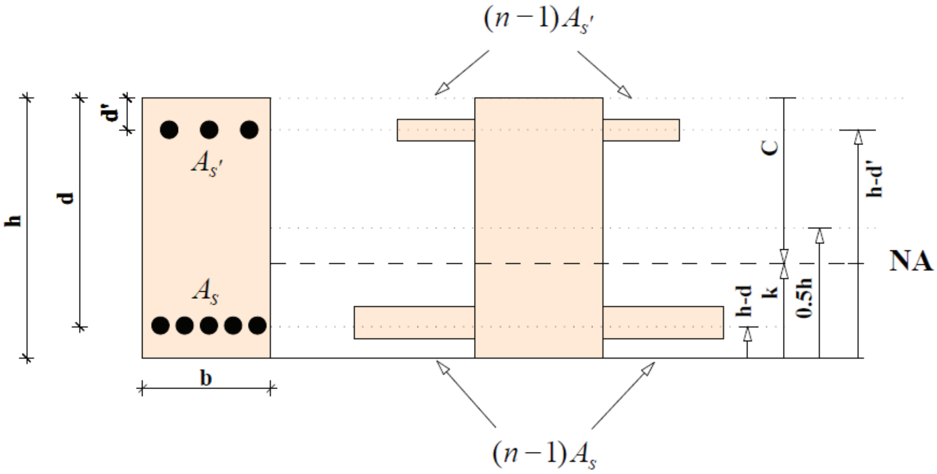
Reinforced Concrete Beams

Zone I:



Reinforced Concrete Beams

Zone I:



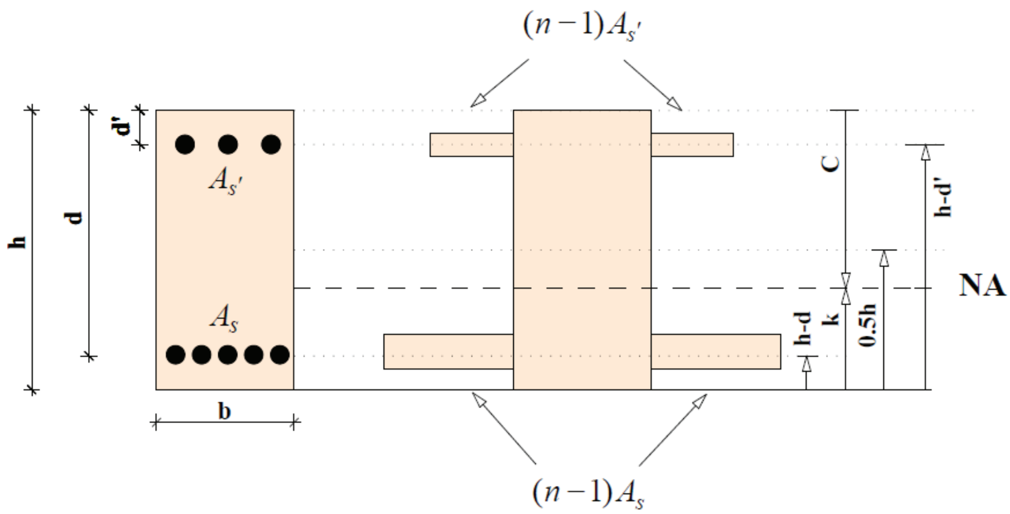
$$k = \frac{\sum A_i y_i}{\sum A_i} = \frac{0.5bh^2 + (n-1)A_s'(h-d') + (n-1)A_s(h-d)}{bh + (n-1)(A_s' + A_s)}$$

$$c = h - k$$

Bending of Members Made of Several Materials

Reinforced Concrete Beams

Zone I:

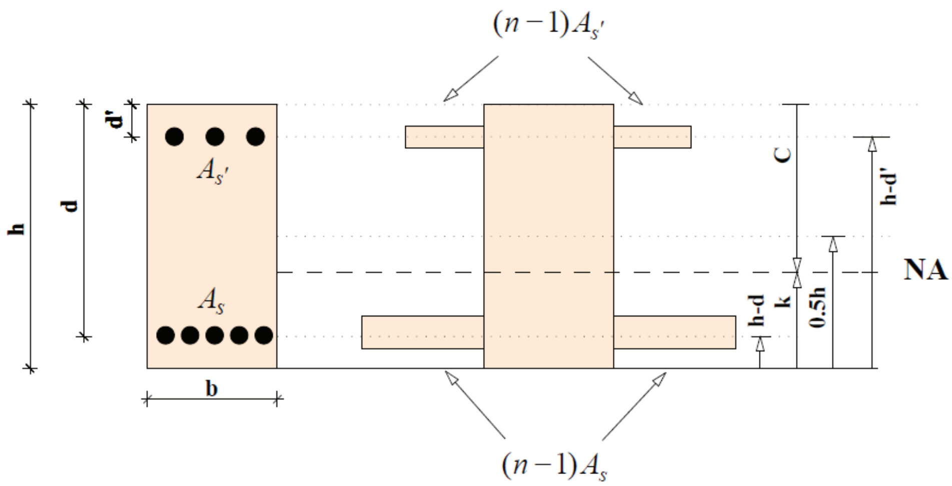


$$I_g = \left[\frac{1}{12}bh^3 + bh\left(c - \frac{h}{2}\right)^2\right] + \left[(n-1)A_s'(c-d')^2\right] + \left[(n-1)A_s(d-c)^2\right]$$

Bending of Members Made of Several Materials

Reinforced Concrete Beams

Zone I:



$$f_{cc\max} = \frac{M \cdot c}{I_g}$$

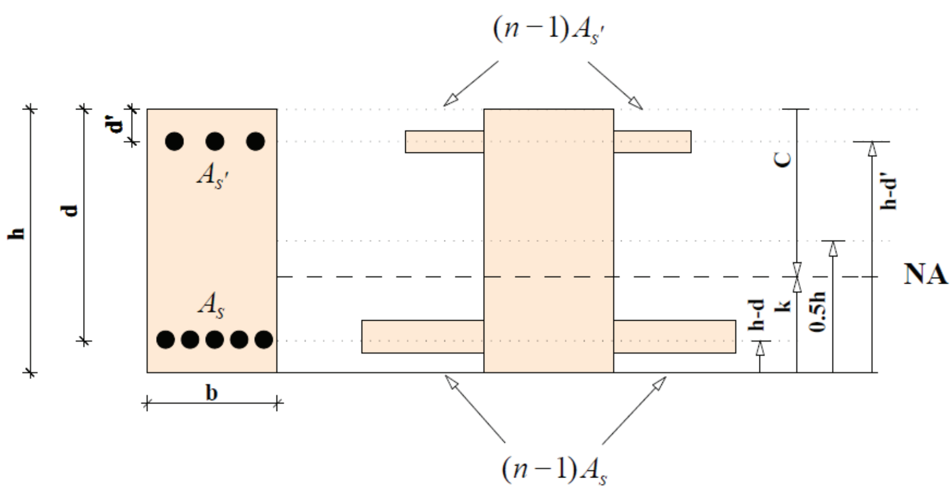
$$f_{ct\max} = \frac{M \cdot (h-c)}{I_g}$$

$$f_t = 2\sqrt{f_c'} \quad \Rightarrow \quad f_{ct\max} \le 2\sqrt{f_c'}$$

Bending of Members Made of Several Materials

Reinforced Concrete Beams

Zone I:



$$f_s = n\frac{M \cdot (d-c)}{I_g}$$

$$f_s' = n\frac{M \cdot (c-d')}{I_g}$$

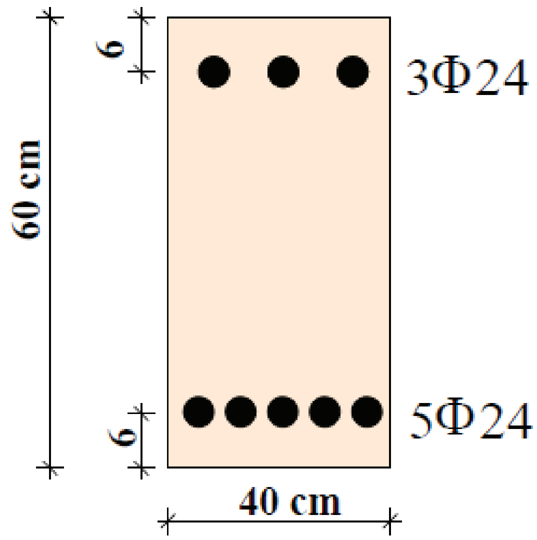
$$M_{cr} = \frac{f_t \cdot I_g}{y_t}$$

Example 5

The beam is under bending. Suppose that the concrete behaves in zone I. Determine:

a) Maximum stresses in concrete and steel.

b) Bending which cause cracks in beam.

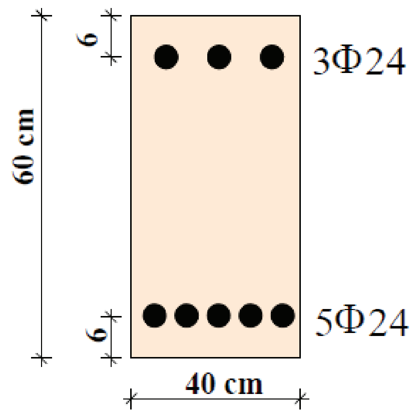


$$f'_c = 200\,\frac{kg}{cm^2} \quad f_y = 4000\,\frac{kg}{cm^2} \quad M = 6\,T.m$$

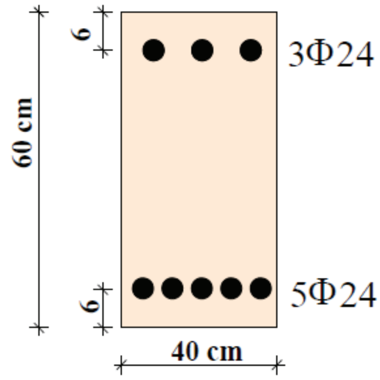
$$E_s = 2\times10^6\,\frac{kg}{cm^2} \quad E_c = 2\times10^5\,\frac{kg}{cm^2}$$

Bending of Members Made of Several Materials

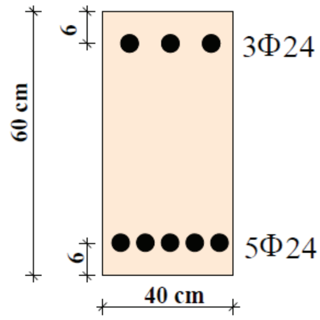
Example 5



Example 5

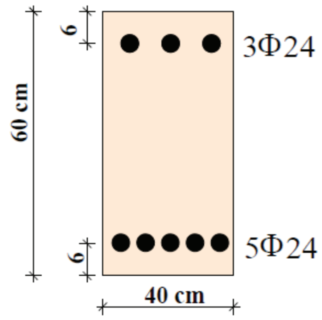


Example 5

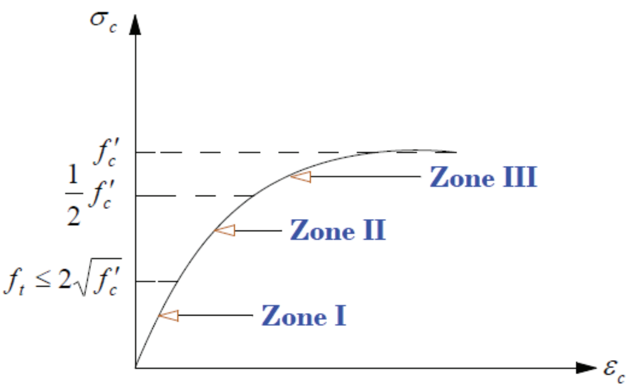


Bending of Members Made of Several Materials

Example 5



Reinforced Concrete Beams

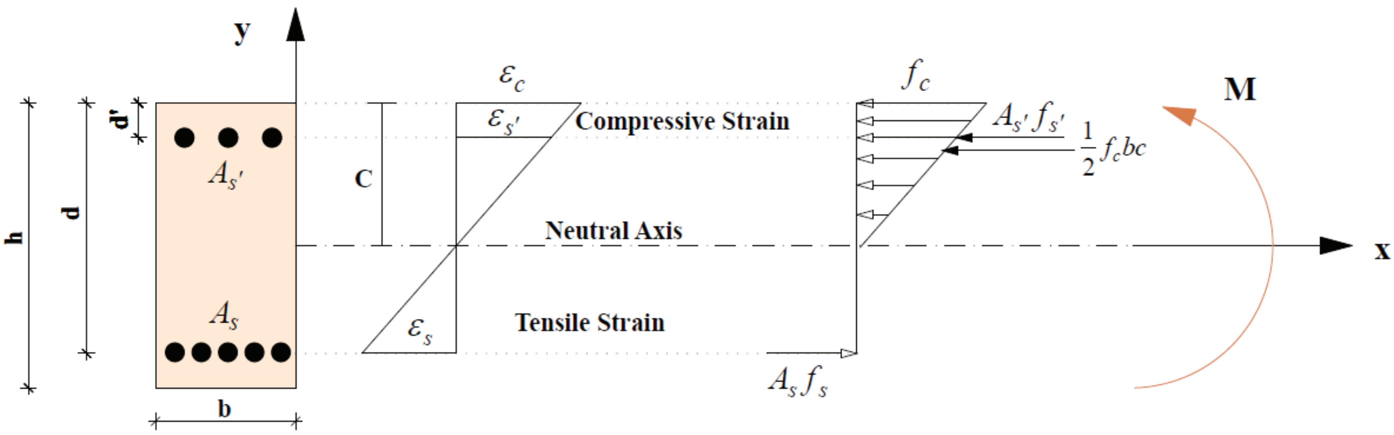


Zone II:

- Stresses are in the low level. $\sigma_c \le \frac{1}{2} f'_c$
- There are cracks in concrete.
- The concrete in tensile zone has no resistance.
- Strain & Stress has linear behavior.
- Transformed Section

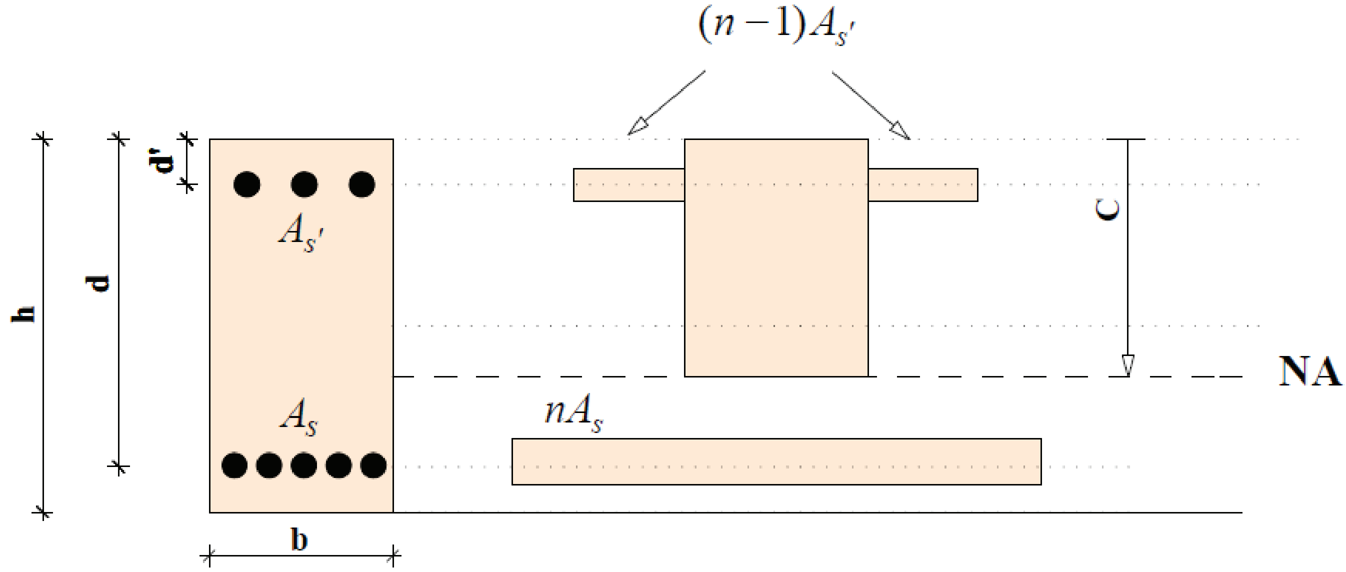
Reinforced Concrete Beams

Zone II:



Reinforced Concrete Beams

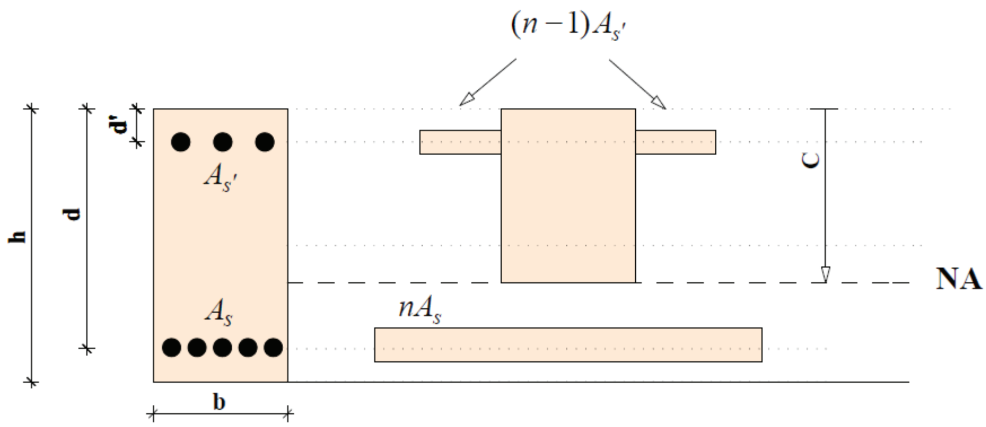
Zone II:



Bending of Members Made of Several Materials

Reinforced Concrete Beams

Zone II:

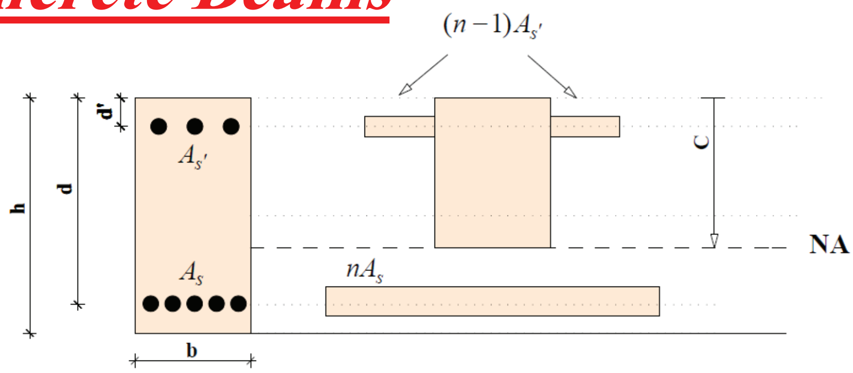


$$\sum A_i \bar{y}_i = 0 \quad \Rightarrow \quad \frac{1}{2}bc^2 + (n-1)A_s'(c-d') - nA_s(d-c) = 0$$

Bending of Members Made of Several Materials

Reinforced Concrete Beams

Zone II:



$$\frac{1}{2}bc^2 + (n-1)A_s'(c-d') - nA_s(d-c) = 0 \quad \Rightarrow$$

$$A = \frac{b}{2}$$

$$B = [(n-1)A_s' + nA_s]$$

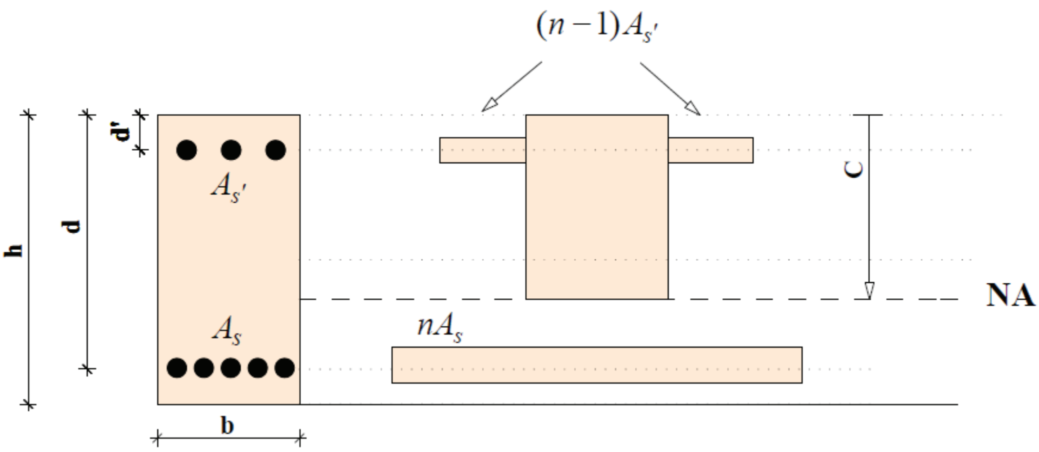
$$D = -[(n-1)A_s'd' + nA_s d]$$

$$c = \frac{-B + \sqrt{B^2 - 4AD}}{2A}$$

Bending of Members Made of Several Materials

Reinforced Concrete Beams

Zone II:

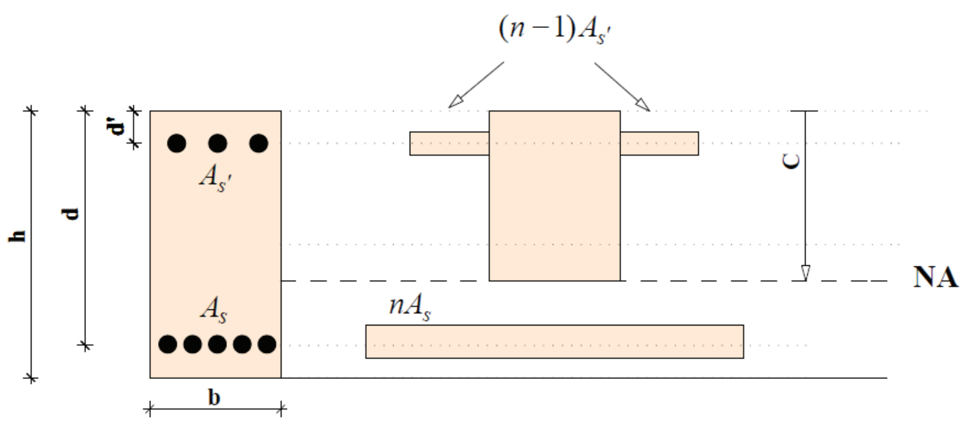


$$I_{crack} = \frac{1}{3}bc^3 + (n-1)A_s'(c-d')^2 + nA_s(d-c)^2$$

Bending of Members Made of Several Materials

Reinforced Concrete Beams

Zone II:



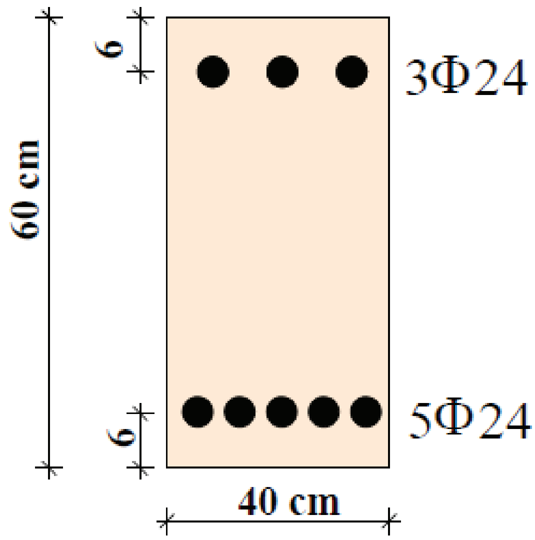
$$f_{cc\,max} = \frac{M \cdot c}{I_{crack}}$$

$$f_s = n\frac{M \cdot (d-c)}{I_{crack}}$$

$$f_s' = n\frac{M \cdot (c-d')}{I_{crack}}$$

Bending of Members Made of Several Materials

Example 6



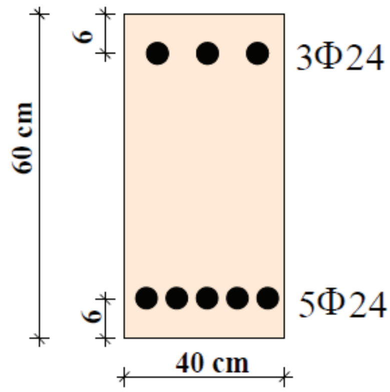
The beam is under bending. Suppose that the concrete behaves in zone II. Determine Maximum stresses in concrete and steel.

$$f'_c = 200\ \frac{kg}{cm^2} \qquad f_y = 4000\ \frac{kg}{cm^2} \qquad M = 20\ T.m$$

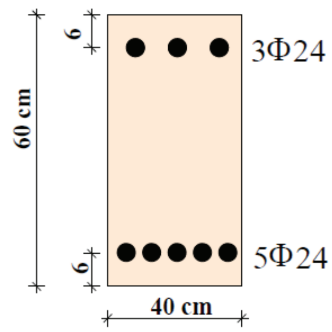
$$E_s = 2\times10^6\ \frac{kg}{cm^2} \qquad E_c = 2\times10^5\ \frac{kg}{cm^2}$$

Bending of Members Made of Several Materials

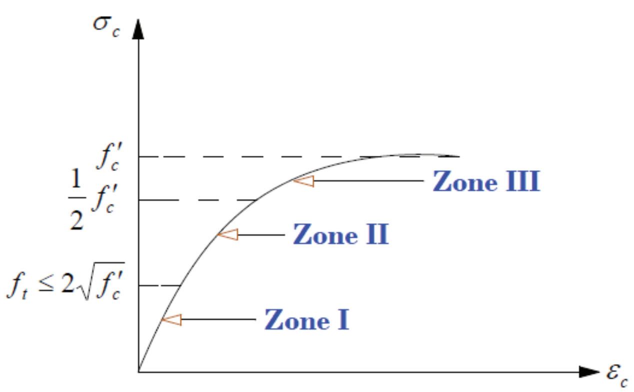
Example 6



Example 6



Reinforced Concrete Beams

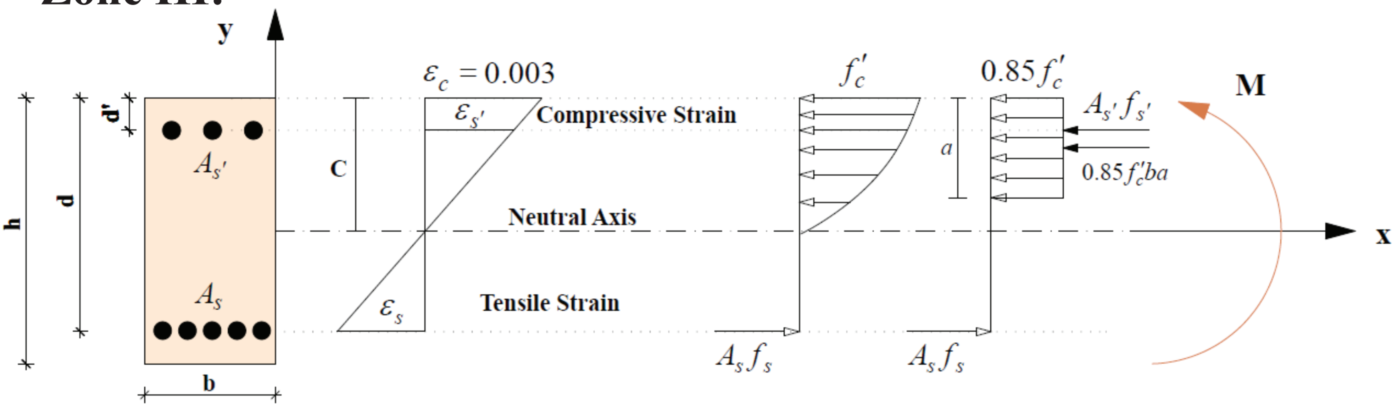


Zone III:

- Stresses are in the high level. $\sigma_c > \frac{1}{2} f'_c$
- There are cracks in concrete.
- The concrete in tensile zone has no resistance.
- Strain has linear behavior.
- Stress has nonlinear behavior.
- Transformed Section

Reinforced Concrete Beams

Zone III:



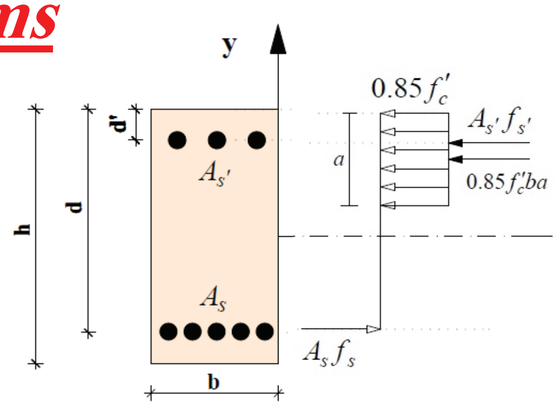
$$a = \beta_1 \cdot c$$

$$\beta_1 = 0.85 - 0.0008(f_c - 300) \quad if \quad f'_c \succ 300 \ \frac{Kg}{Cm^2}$$

$$\beta_1 = 0.85 \quad if \quad f'_c \le 300 \ \frac{Kg}{Cm^2}$$

Reinforced Concrete Beams

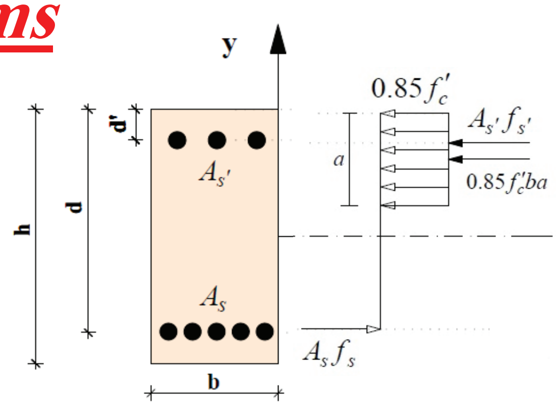
Zone III:



Bending of Members Made of Several Materials

Reinforced Concrete Beams

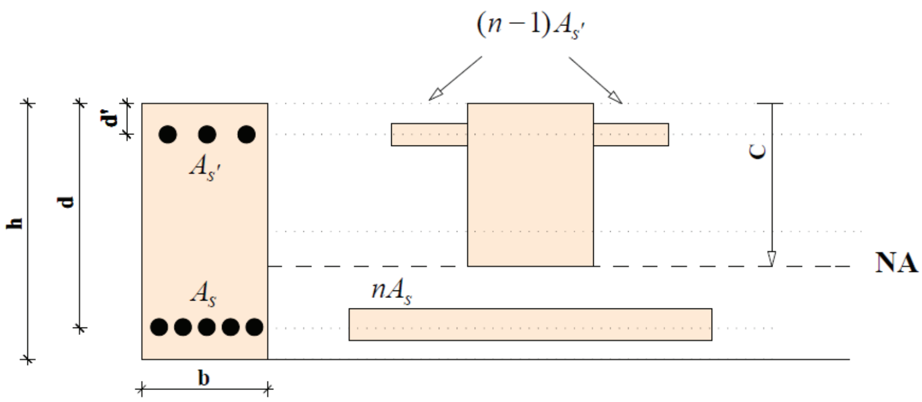
Zone III:



Bending of Members Made of Several Materials

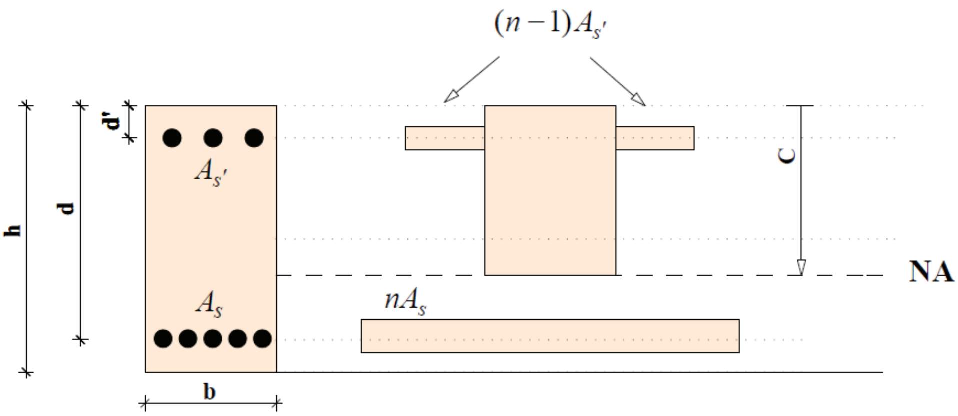
Reinforced Concrete Beams

Zone III:



Reinforced Concrete Beams

Zone III:



Example 7

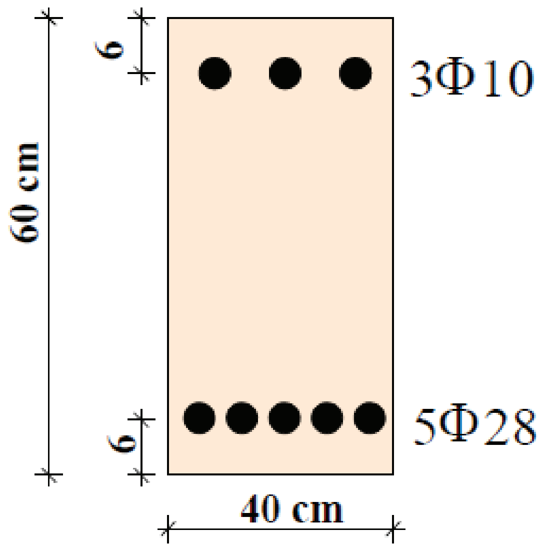
The beam is under bending. Suppose that the concrete behaves in zone III. Determine:

a) Maximum stresses in concrete and steel.

b) Ultimate resistance bending.

$$f'_c = 200\,\frac{kg}{cm^2} \quad f_y = 4000\,\frac{kg}{cm^2} \quad M = 50\,T.m$$

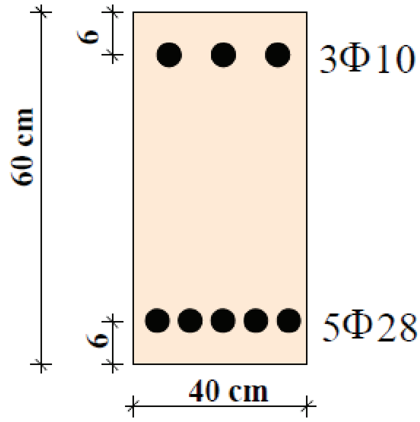
$$E_s = 2\times10^6\,\frac{kg}{cm^2} \quad E_c = 2\times10^5\,\frac{kg}{cm^2}$$



$$f_s = f_y$$
$$f_{s'} = f_y$$

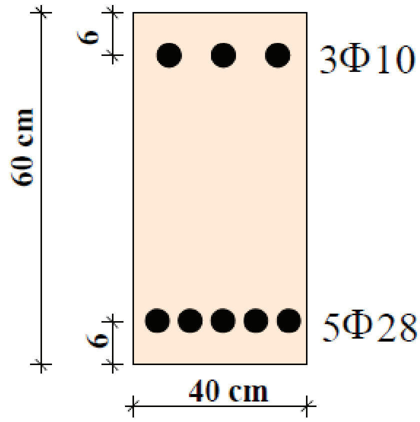
Bending of Members Made of Several Materials

Example 7



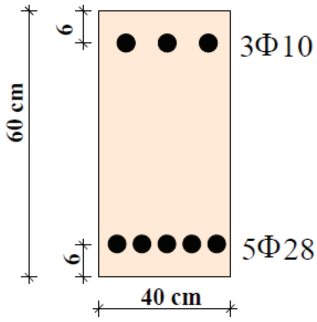
Bending of Members Made of Several Materials

Example 7



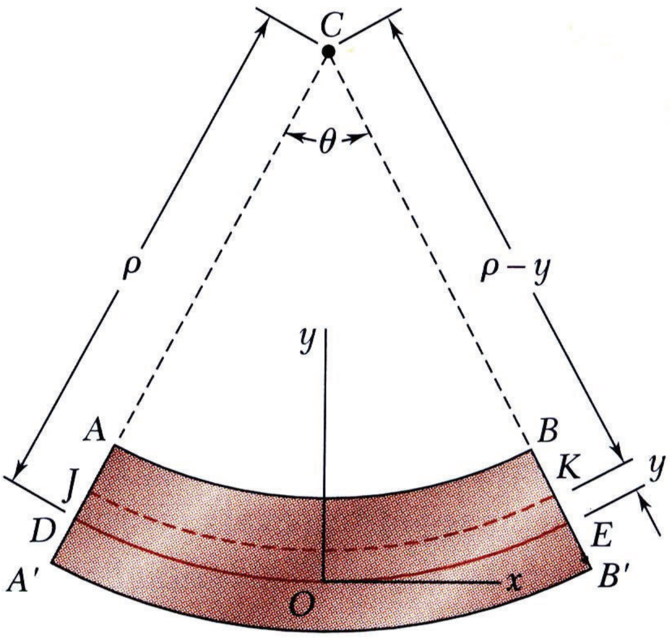
Bending of Members Made of Several Materials

Example 7



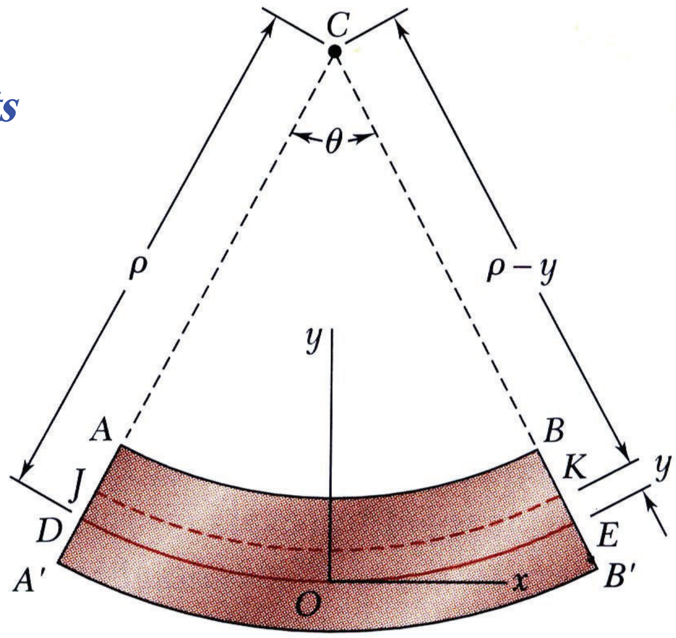
Bending of Curved Members

In this section we will consider the stresse caused by the application of equal and opposite couples to members that are ***initially curved***. Our discussion will b limited to curved members of ***uniform cross section*** possessing a plane of symmetry in which the bending couples are applied, and it will be assumed that ***al stresses remain below the proportional limit.***

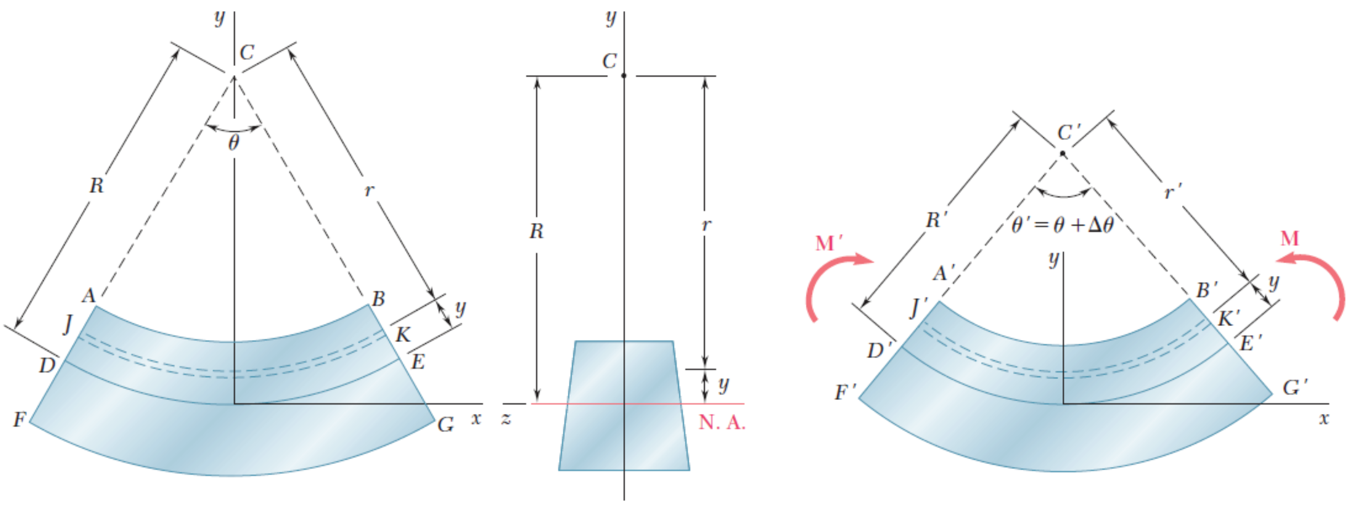


Bending of Curved Members

If the initial curvature of the member is small, ***if its radius of curvature is large compared to the depth of its cross section***, a good approximation can be obtained for the distribution of stresses by assuming
the member to be straight and using the formulas derived.



Bending of Curved Members

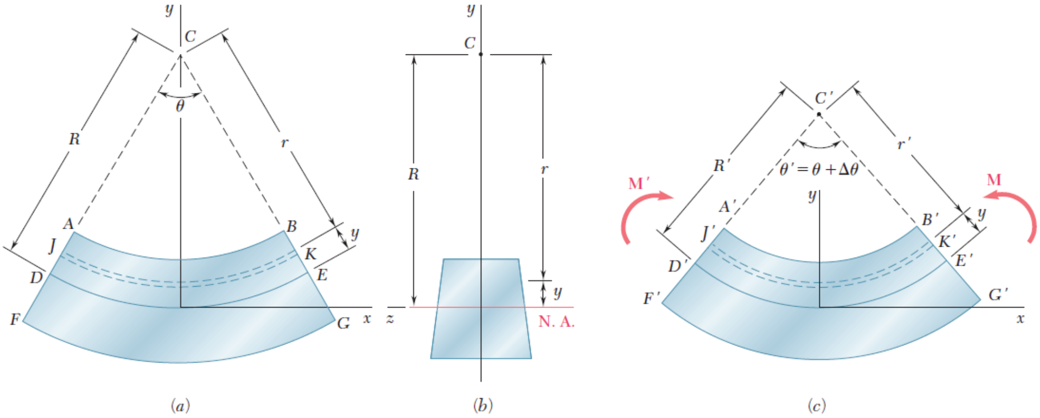


(a) (b) (c)

We express the fact that the length of the neutral surface remains constant

$$R\theta = R'\theta'$$

Bending of Curved Members



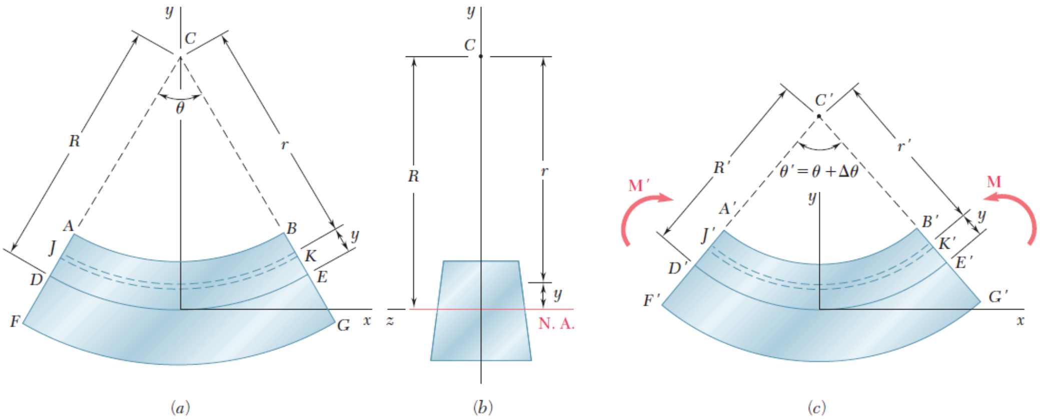
Consider Curve JK. The deformation of JK is

$$\delta_{JK} = r'\theta' - r\,\theta$$

$$r = R - y \qquad r' = R' - y$$

$$\Rightarrow \quad \delta_{JK} = (R' - y)\theta' - (R - y)\theta \quad \Rightarrow \quad \delta = -y\Delta\theta$$

Bending of Curved Members



$$\varepsilon_x = \frac{\delta_{JK}}{L_{JK}} = \frac{\delta}{r\theta} = -\frac{y\Delta\theta}{r\theta} \quad \Rightarrow \quad \varepsilon_x = -\frac{\Delta\theta}{\theta}\frac{y}{R-y}$$

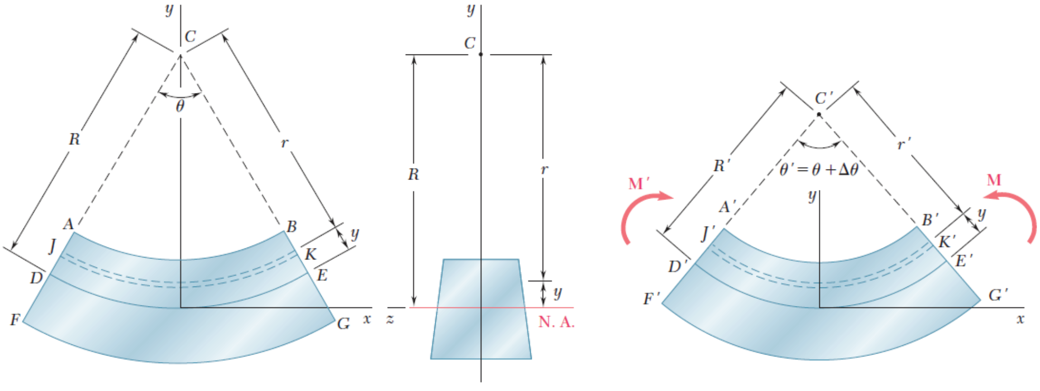
$$r = R - y$$

Strain varies nonlinearly with the distance y from the neutral surface

$$\Rightarrow \quad \sigma_x = E\varepsilon_x \quad \Rightarrow \quad \sigma_x = -\frac{E\Delta\theta}{\theta}\frac{y}{R-y} \quad or \quad \sigma_x = -\frac{E\Delta\theta}{\theta}\frac{R-r}{r}$$

like strain the normal stress does not vary linearly with the distance y from the neutral surface

Bending of Curved Members



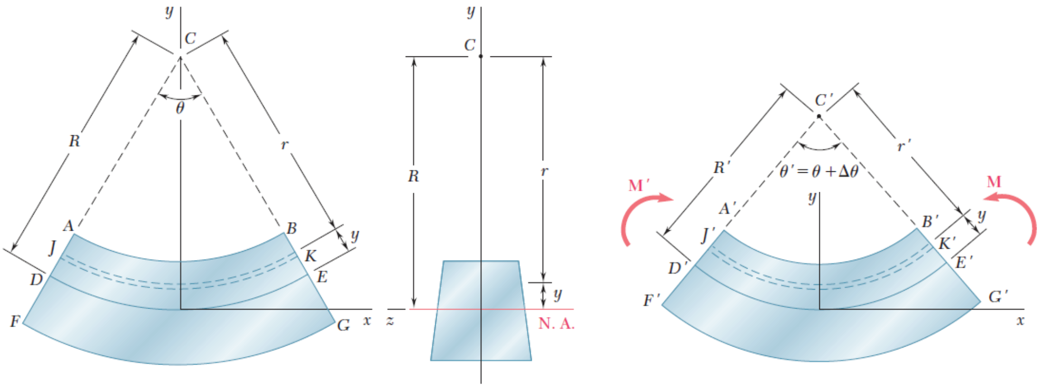
(a) (b) (c)

Equations of Equilibrium

$$\int dF = 0 \quad \Rightarrow \quad \boxed{\int \sigma_x dA = 0}$$

$$\int -y \cdot dF = M \quad \Rightarrow \quad \boxed{\int (-y \cdot \sigma_x) dA = M}$$

Bending of Curved Members



(a) (b) (c)

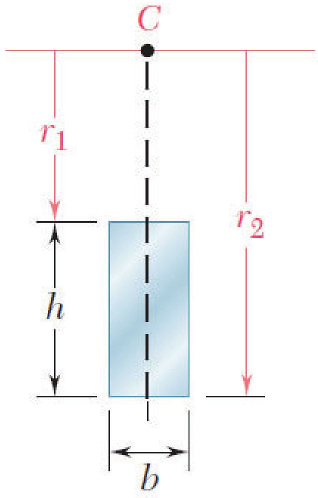
The distance R from the center of curvature C to the neutral surface is defined by the relation

$$\int \sigma_x dA = 0$$

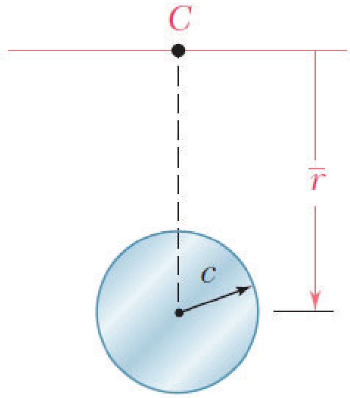
$$\sigma_x = -\frac{E\Delta\theta}{\theta}\frac{R-r}{r}$$

$$\Rightarrow \quad \boxed{R = \frac{A}{\int \frac{dA}{r}}}$$

Bending of Curved Members



Rectangle

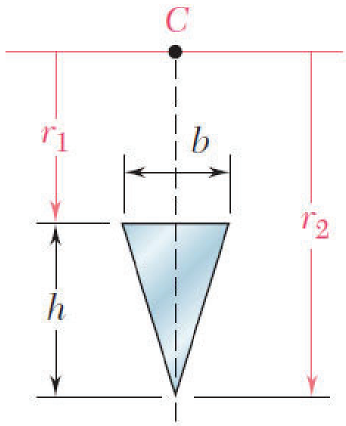


Circle

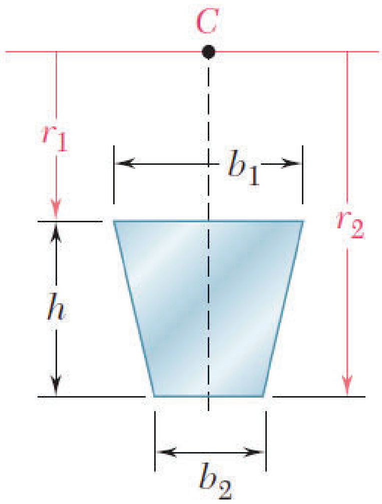
$$R = \frac{h}{\ln \frac{r_2}{r_1}}$$

$$R = \frac{1}{2}\left(\bar{r} + \sqrt{\bar{r}^2 - c^2}\right)$$

Bending of Curved Members



Triangle



Trapezoid

$$R = \frac{\frac{1}{2}h}{\frac{r_2}{h}\ln \frac{r_2}{r_1} - 1}$$

$$R = \frac{\frac{1}{2}h^2(b_1 + b_2)}{(b_1 r_2 - b_2 r_1)\ln \frac{r_2}{r_1} - h(b_1 - b_2)}$$

Bending of Curved Members

r: The distance from C to the centroid of the cross section.

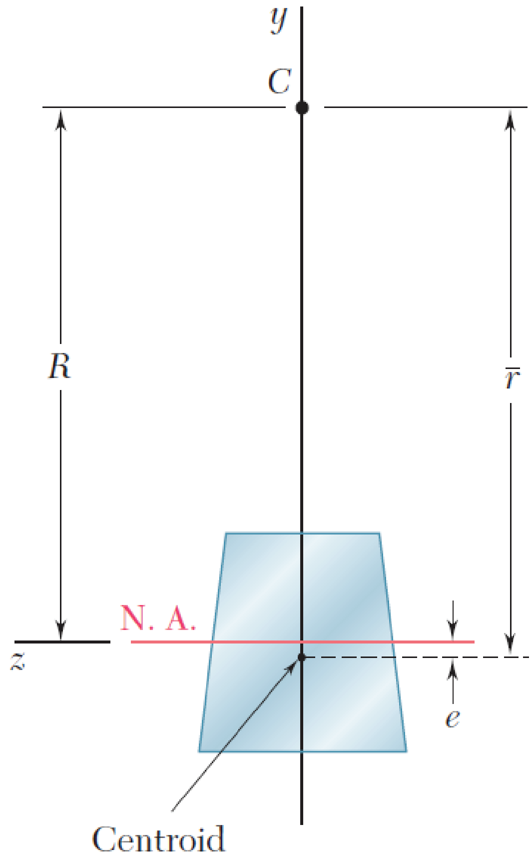
$R \neq \bar{r}$

$\bar{r} - R = e$

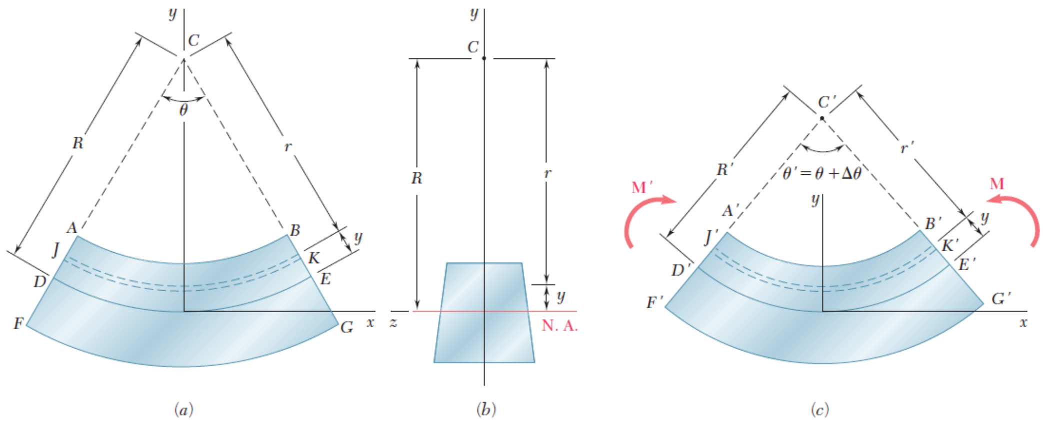
$$R = \frac{A}{\int \frac{dA}{r}}$$

$$\bar{r} = \frac{1}{A}\int r\,dA$$

We thus conclude that, in a curved member, the neutral axis of a transverse section does not pass through the centroid of that section



Bending of Curved Members

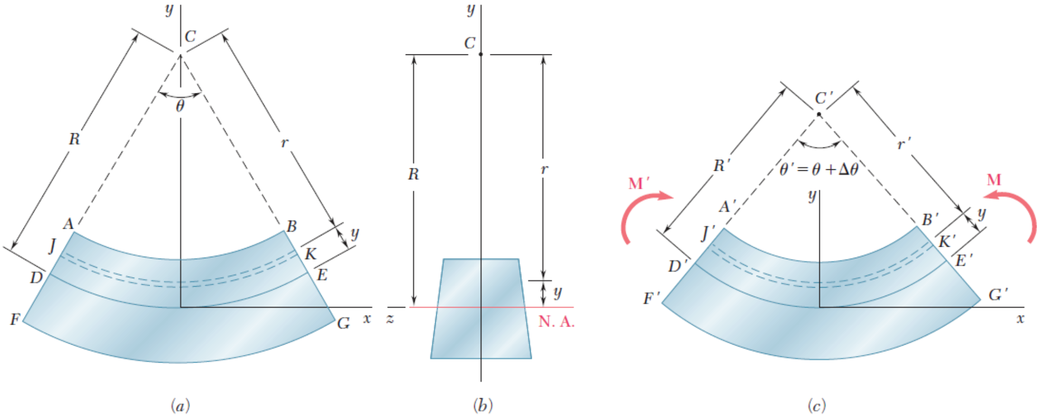


$$\int(-y \cdot \sigma_x)\,dA = M$$

$$\sigma_x = -\frac{E\Delta\theta}{\theta}\frac{R-r}{r}$$

$$\Rightarrow \quad \frac{E\Delta\theta}{\theta} = \frac{M}{A(\bar{r} - R)}$$

Bending of Curved Members



$$\frac{E\Delta\theta}{\theta}=\frac{M}{A(\bar{r}-R)} \qquad \bar{r}-R=e$$

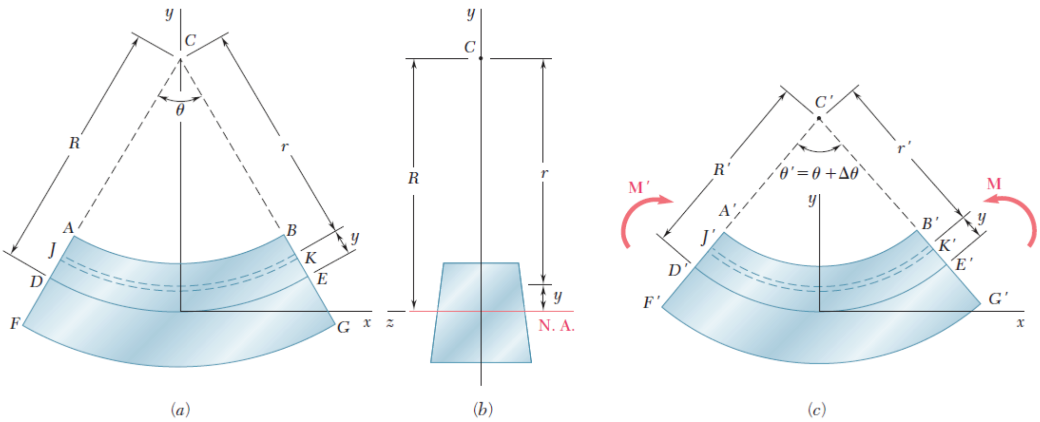
$$\sigma_x=-\frac{E\Delta\theta}{\theta}\frac{y}{R-y}$$

$$\sigma_x=-\frac{E\Delta\theta}{\theta}\frac{R-r}{r}$$

$$\Rightarrow \quad \sigma_x=-\frac{My}{Ae(R-y)}$$

$$\sigma_x=\frac{M(r-R)}{Aer}$$

Bending of Curved Members



$$R\theta=R'\theta' \quad \Rightarrow \quad \frac{1}{R'}=\frac{1}{R}\frac{\theta'}{\theta}$$

$$\theta'=\theta+\Delta\theta$$

$$\frac{E\Delta\theta}{\theta}=\frac{M}{Ae}$$

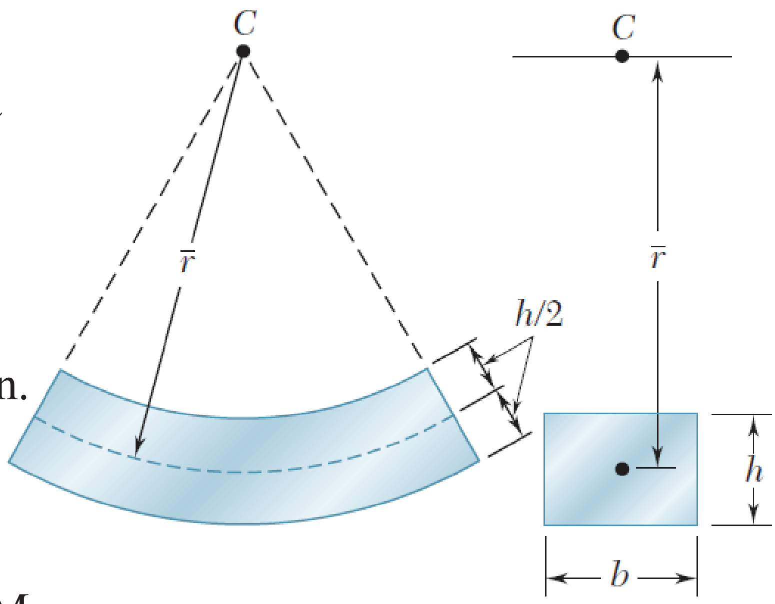
The change in curvature of the neutral surface

$$\Rightarrow \quad \frac{1}{R'}-\frac{1}{R}=\frac{M}{EAeR}$$

Bending of Curved Members

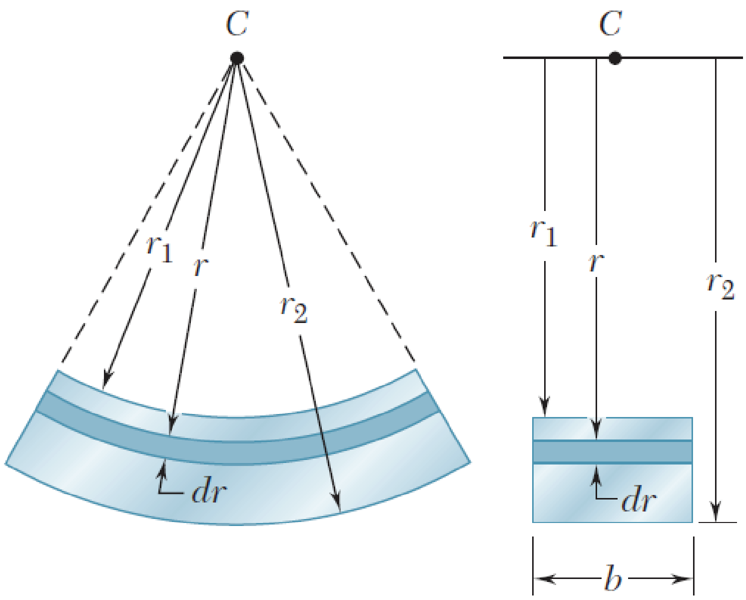
Example 8

A curved rectangular bar has a mean radius $\bar{r} = 150mm$ and a cross section of width b = 60 mm and depth h =36 mm. ***Determine the distance e*** between the centroid and the neutral axis of the cross section. Also, ***determine the largest tensile and compressive stresses***, knowing that the bending moment in the bar is M = 900 N. m



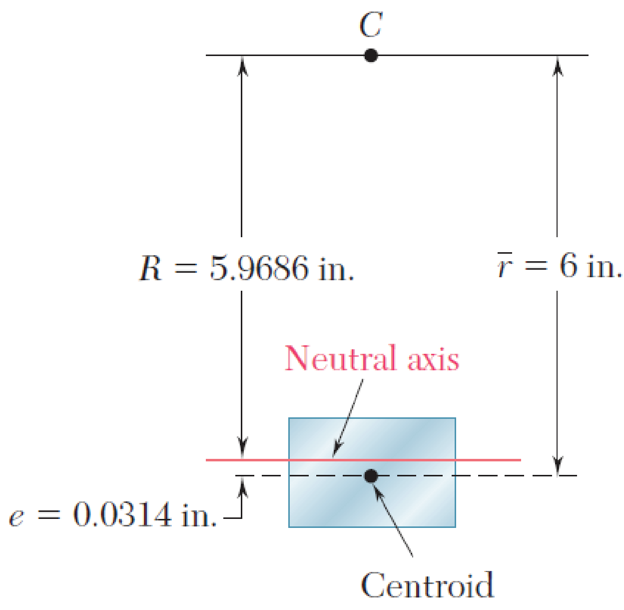
Bending of Curved Members

Example 8



Bending of Curved Members

Example 8



Bending of Curved Members

Example 8

Bending of Curved Members

Example 8

Bending of Curved Members

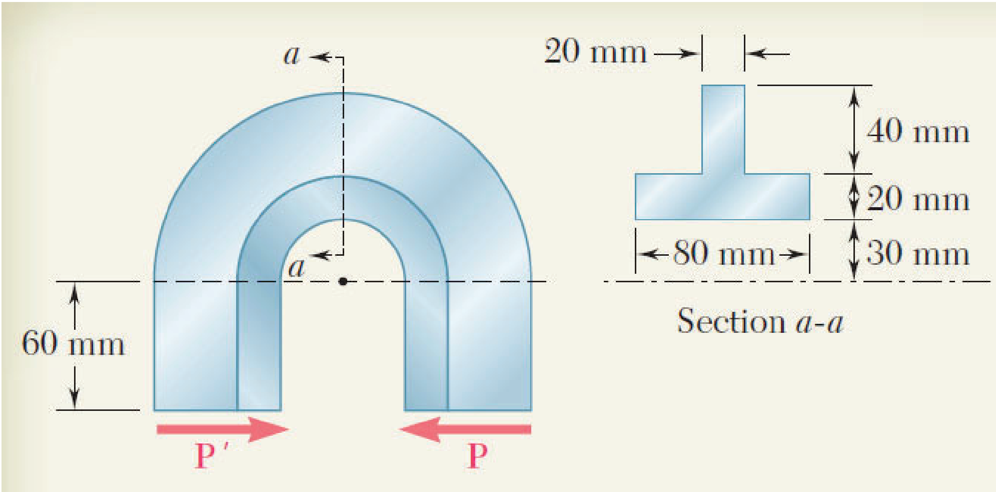
Example 8

Let us compare the obtained values with the result we would get for a straight bar

Bending of Curved Members

Example 9

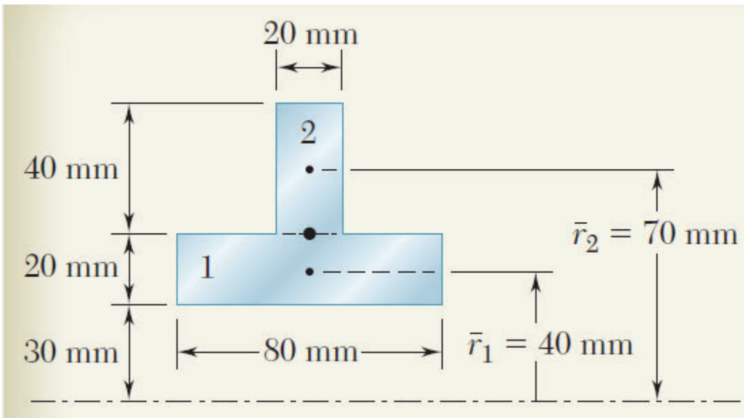
A machine component has a T-shaped cross section and is loaded as shown. Knowing that the allowable compressive stress is 50 MPa, determine the largest force P that can be applied to the component.



Bending of Curved Members

Example 9

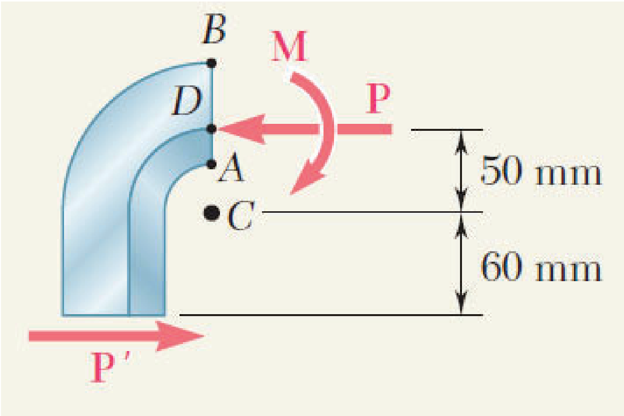
Centroid of the Cross Section

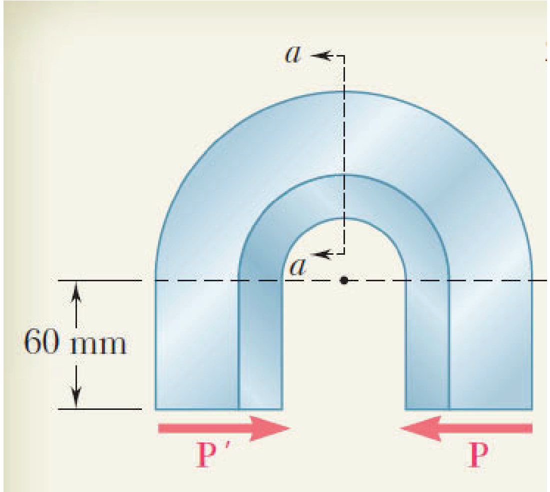


Part	$A_i\ (mm^2)$	$\bar{r}_i\ (mm)$	$\bar{r}_i\ A_i (mm^3)$
1			
2			

Bending of Curved Members

Example 9



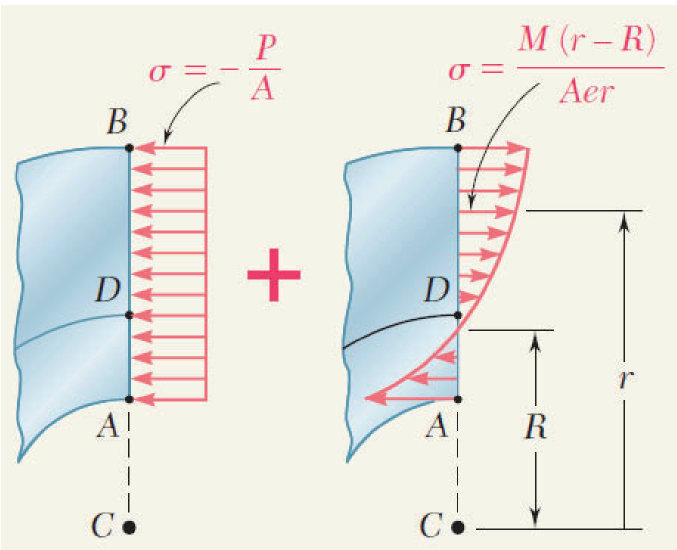


Force and Couple at *D.* The internal forces in section *a-a* are equivalent to a force P acting at D and a couple M of moment

Bending of Curved Members

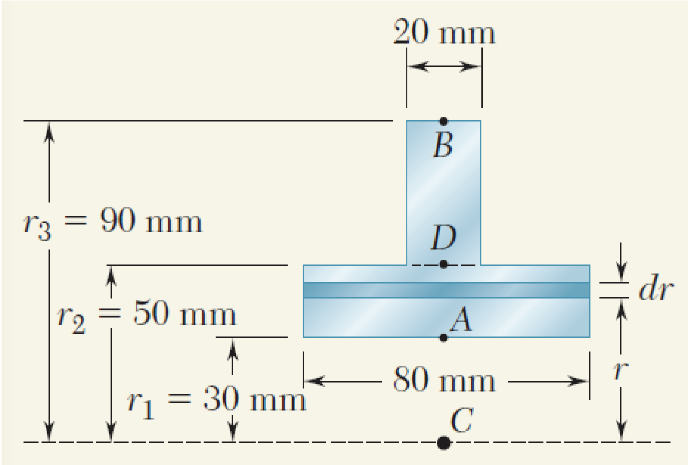
Example 9

Superposition. The centric force P causes a uniform compressive stress on section a-a. The bending couple M causes a varying stress distribution



Bending of Curved Members

Example 9



Bending of Curved Members

Example 9

Allowable Load.