Mechanics of Materials

Ferdinand P.Beer, E.Russel Johnston, Jr., John T.Dewolf

Other Reference:
J.Wat Oler "Lectures notes on Mechanics od Materials"
Ibrahim A.Assakkaf "Lectures notes on Mechanics od Materials"

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Torsion

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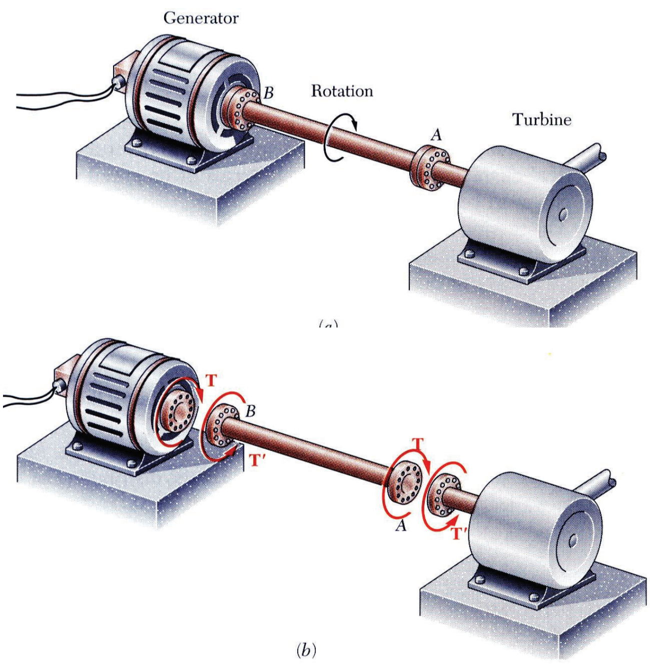
Torsion

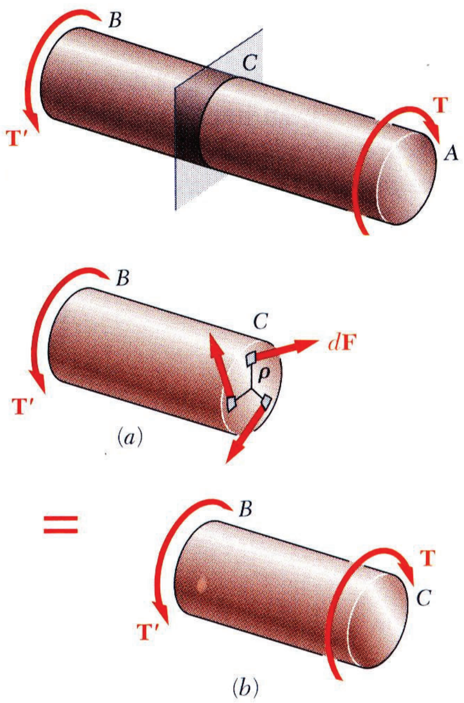
❑ Torsional Loads on Circular Shafts



- Interested in stresses and strains of circular shafts subjected to twisting couples or *torques*
- Turbine exerts torque *T* on the shaft
- Shaft transmits the torque to the generator
- Generator creates an equal and opposite torque *T'*

Torsion

❑ Net Torque Due to Internal Stresses



- Net of the internal shearing stresses is an internal torque, equal and opposite to the applied torque,

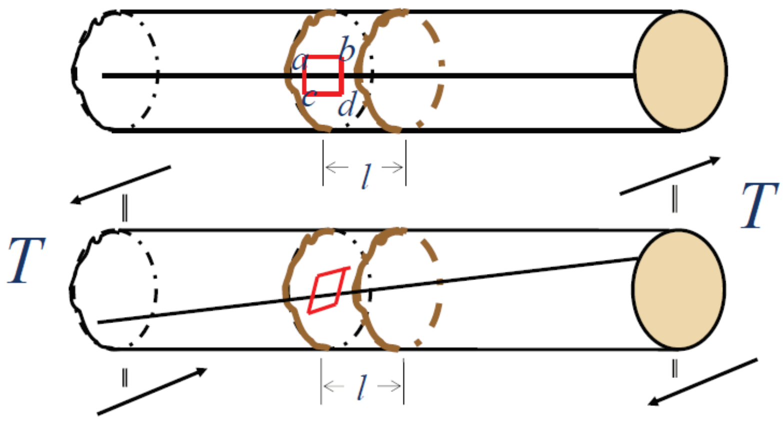
$$T = \int \rho \, dF = \int \rho(\tau \, dA)$$

- Although the net torque due to the shearing stresses is known, the distribution of the stresses is not
- Distribution of shearing stresses is statically indeterminate – must consider shaft deformations
- Unlike the normal stress due to axial loads, the distribution of shearing stresses due to torsional loads can not be assumed uniform.

Torsion

❑ Deformation of Circular Shaft

The distance l between the outside circumferential lines does not change Significantly.



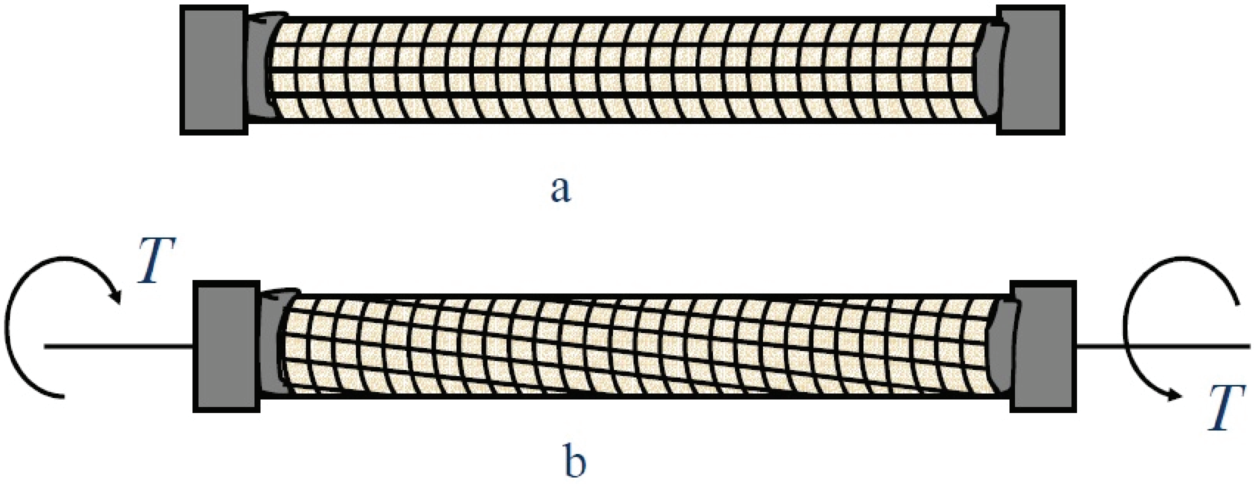
The rectangles become parallelograms whose sides have the same length as those of the original rectangles.

The circumferential lines do not become zigzag; that is ; they remain in parallel planes.

The original straight parallel longitudinal lines, such as ***ab*** and ***cd***, remain parallel to each other but do not remain parallel to the longitudinal axis of the member. These lines become helices.

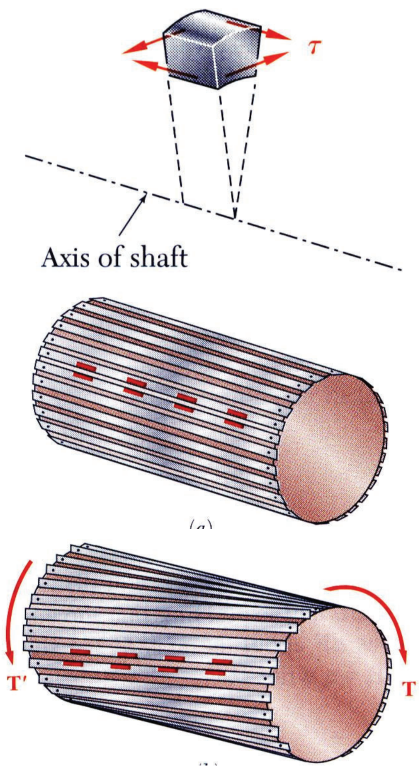
Torsion

❑ Deformation of Circular Shaft



Torsion

❑ Deformation of Circular Shaft



- Torque applied to shaft produces shearing stresses on the faces perpendicular to the axis.
- Conditions of equilibrium require the existence of equal stresses on the faces of the two planes containing the axis of the shaft
- The existence of the axial shear components is demonstrated by considering a shaft made up of axial slats.

  The slats slide with respect to each other when equal and opposite torques are applied to the ends of the shaft.

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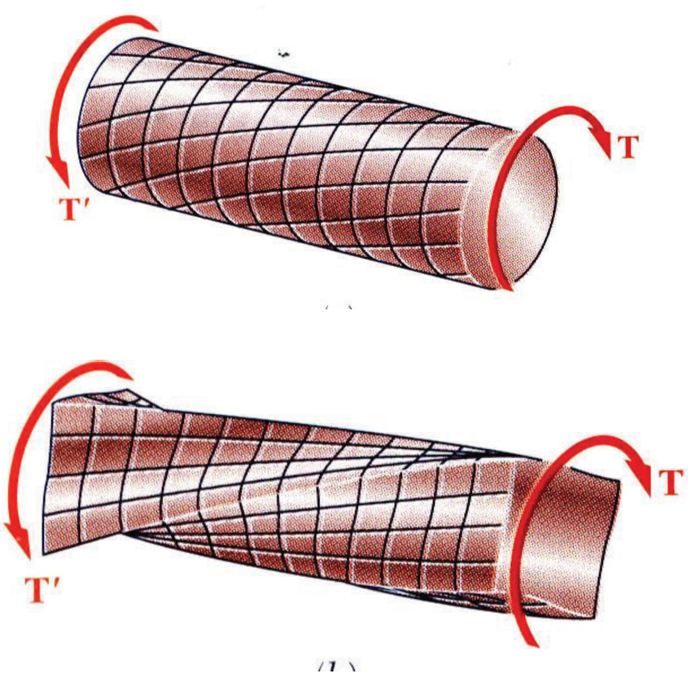
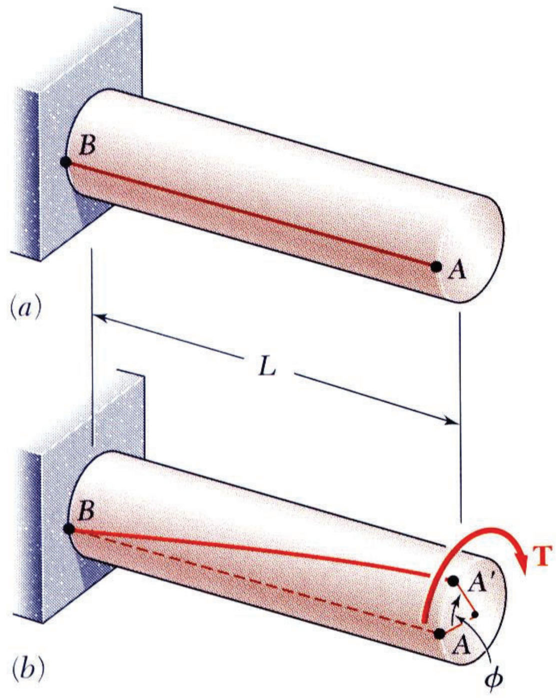
❑ Deformation of Circular Shaft



- When subjected to torsion, every cross-section of a circular shaft remains plane and undistorted.
- Cross-sections for hollow and solid circular shafts remain plain and undistorted because a circular shaft is axisymmetric.
- Cross-sections of noncircular (non-axisymmetric) shafts are distorted when subjected to torsion.

Torsion

❑ Deformation of Circular Shaft



- From observation, the angle of twist of the shaft is proportional to the applied torque and to the shaft length.

$$\phi \propto T$$
$$\phi \propto L$$

Torsion

❑ Shearing Strain

- Consider an interior section of the shaft. As a torsional load is applied, an element on the interior cylinder deforms into a rhombus.

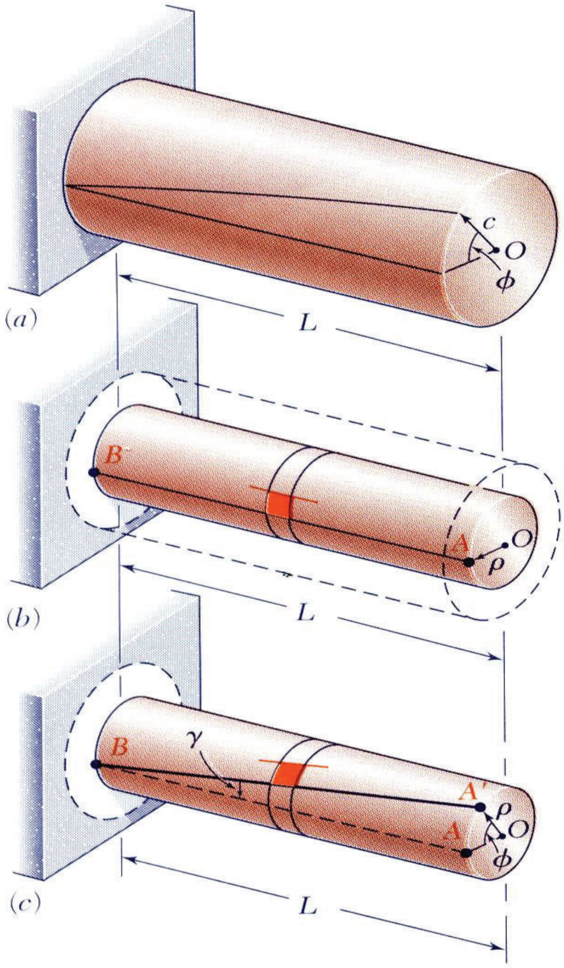
- Since the ends of the element remain planar, the shear strain is equal to angle of twist.

- It follows that

$$L\gamma = \rho\phi \quad \text{or} \quad \boxed{\gamma = \frac{\rho\phi}{L}}$$

- Shear strain is proportional to twist and radius

$$\boxed{\gamma_{\max} = \frac{c\phi}{L}} \quad \text{and} \quad \boxed{\gamma = \frac{\rho}{c}\gamma_{\max}}$$



Torsion

❑ Torsional Shearing Stress

The Elastic Torsion Formula

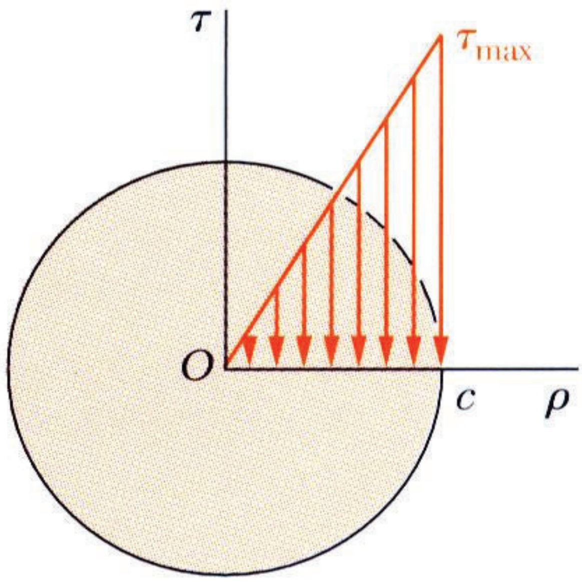
If Hooke's law applies, the shearing stress τ is related to the shearing strain γ by the equation

$$\boxed{\tau = G\,\gamma}$$

$$\gamma = \frac{\rho}{c}\gamma_{\max} \quad \Rightarrow \quad G\gamma = \frac{\rho}{c}G\gamma_{\max}$$

$$\Rightarrow \quad \boxed{\tau = \frac{\rho}{c}\tau_{\max}}$$

The shearing stress varies linearly with the radial position in the section.



Torsion

❑ Torsional Shearing Stress

- Recall that the sum of the moments from the internal stress distribution is equal to the torque on the shaft at the section,

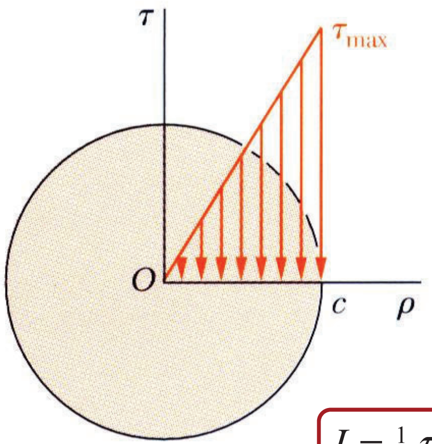
$$T = \int \rho\tau \, dA = \int \rho\left(\frac{\rho}{c}\tau_{\max}\right) dA = \frac{\tau_{\max}}{c}\int \rho^2 \, dA \quad \Rightarrow \quad T = \frac{\tau_{\max}}{c} J$$

- The results are known as the elastic torsion formulas,

$$\tau_{\max} = \frac{Tc}{J} \quad \text{and} \quad \tau = \frac{T\rho}{J}$$

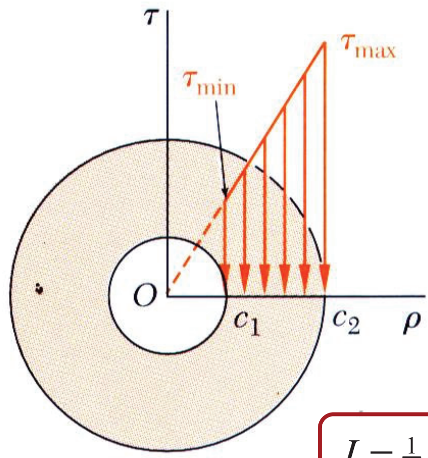
Torsion

❑ Torsional Shearing Stress



$$J = \tfrac{1}{2}\pi c^4$$

$$\tau_{\max} = \frac{Tc}{J}$$



$$J = \tfrac{1}{2}\pi\left(c_2^4 - c_1^4\right)$$

$$\tau_{\min} = \frac{Tc_1}{J}$$

$$\tau_{\max} = \frac{Tc_2}{J}$$

Torsion

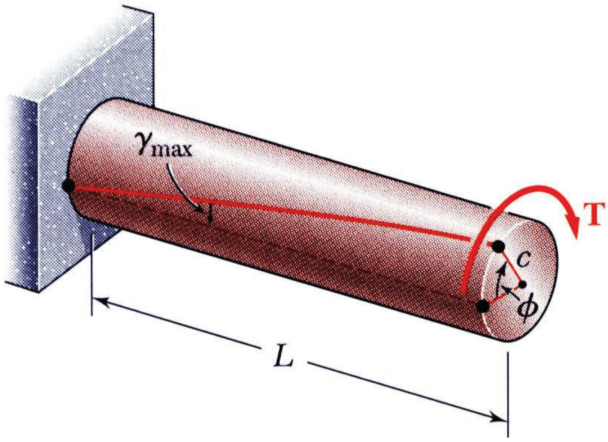
❑ Angle of Twist in Elastic Range

- Recall that the angle of twist and maximum shearing strain are related,

$$\gamma_{\max} = \frac{c\phi}{L}$$

- In the elastic range, the shearing strain and shear are related by Hooke's Law,

$$\gamma_{\max} = \frac{\tau_{\max}}{G} = \frac{Tc}{JG}$$



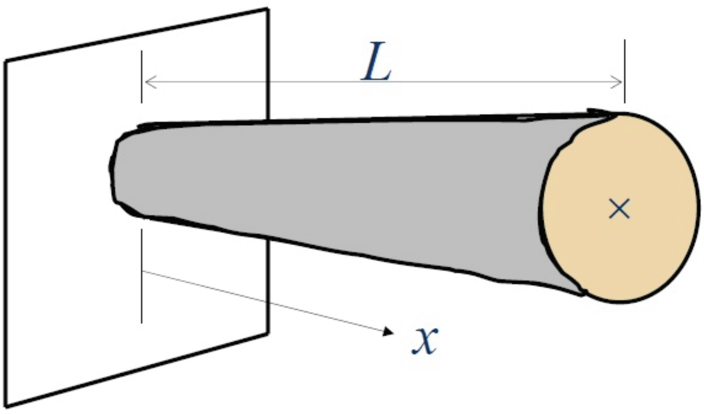
- Equating the expressions for shearing strain and solving for the angle of twist,

$$\phi = \frac{TL}{JG}$$

Torsion

❑ Angle of Twist in Elastic Range

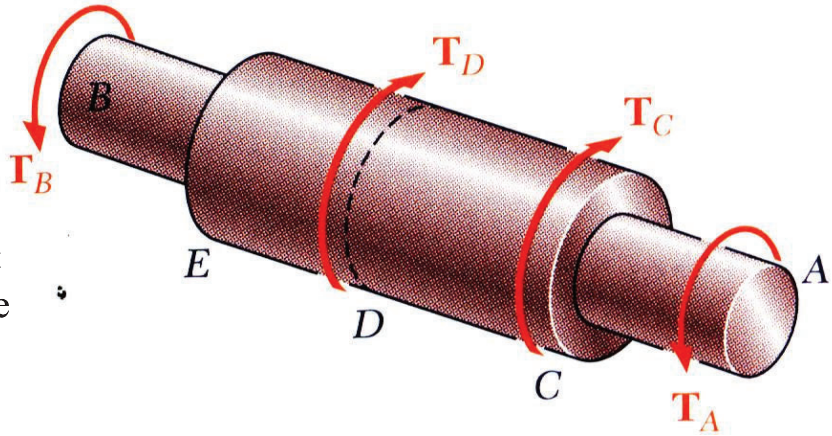
If the properties (T, G, or J) of the shaft are functions of the length of the shaft, then



$$\phi = \int_0^l \frac{T}{JG}\,dx$$

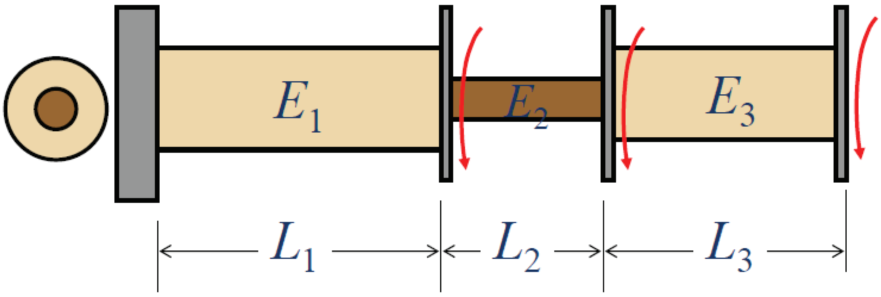
Torsion

❑ Angle of Twist in Elastic Range



- If the torsional loading or shaft cross-section changes along the length, the angle of rotation is found as the sum of segment rotations

$$\phi = \sum_i \frac{T_i L_i}{J_i G_i}$$

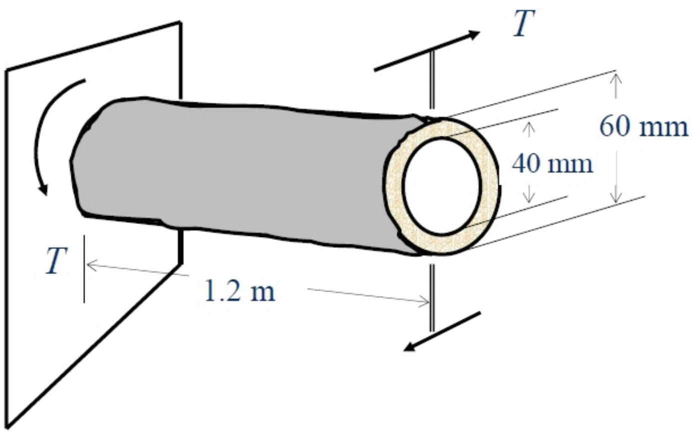


Torsion

❑ Torsional Shearing Stress

Example 1

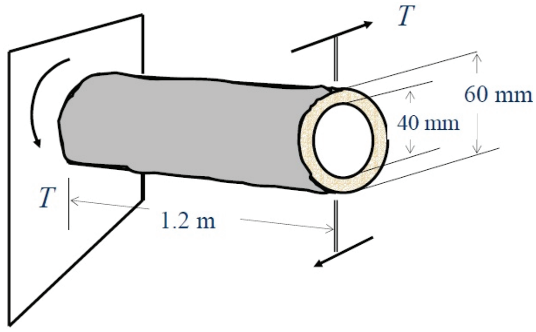
A hollow cylindrical steel shaft is 1.2 m long and has inner and outer diameters equal to 40 mm and 60 mm. (a) What is the largest torque which may be applied to the shaft if the shearing stress is not to exceed 120 MPa? (b) What is the corresponding minimum value of the shearing stress in the shaft?



Torsion

- **Torsional Shearing Stress**

Example 1



(a) Largest Permissible Torque

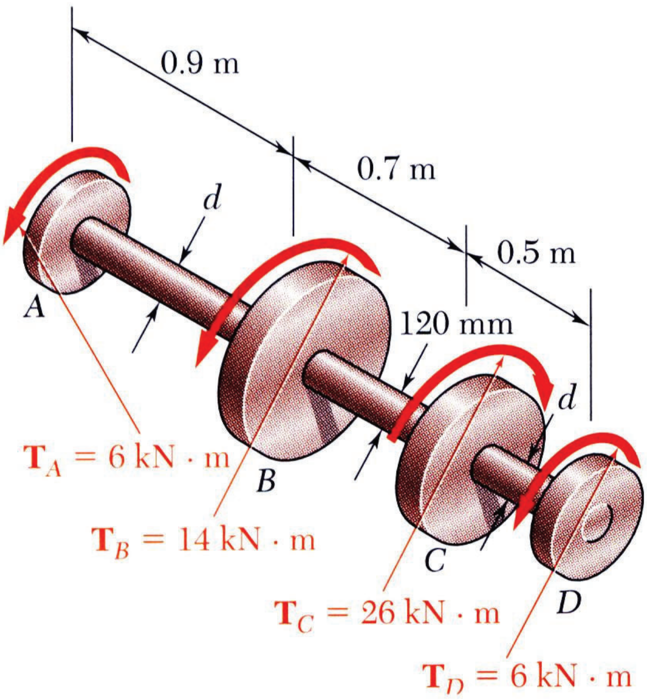
(b) Minimum Shearing Stress

Torsion

- **Torsional Shearing Stress**

Example 2

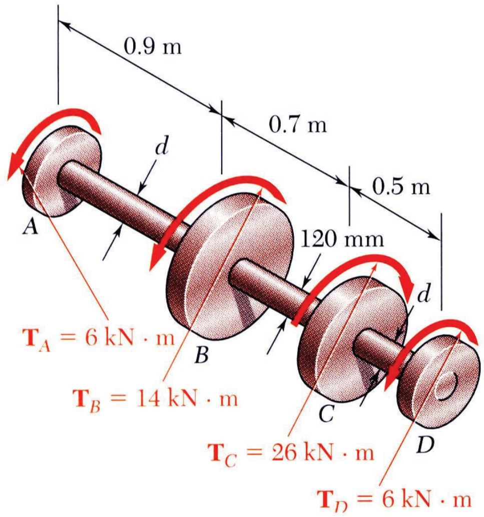
Shaft *BC* is hollow with inner and outer diameters of 90 mm and 120 mm, respectively. Shafts *AB* and *CD* are solid of diameter *d*. For the loading shown, determine (*a*) the minimum and maximum shearing stress in shaft *BC*, (*b*) the required diameter *d* of shafts *AB and CD* if the allowable shearing stress in these shafts is 65 MPa.



Torsion

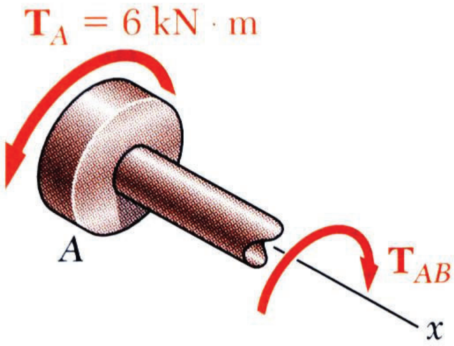
❑ Torsional Shearing Stress

Example 2



SOLUTION:

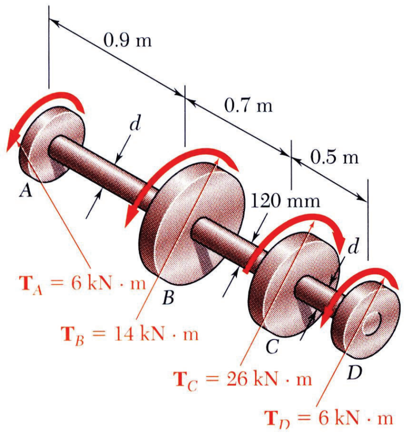
- Cut sections through shafts *AB* and *BC* and perform static equilibrium analysis to find torque loadings



Torsion

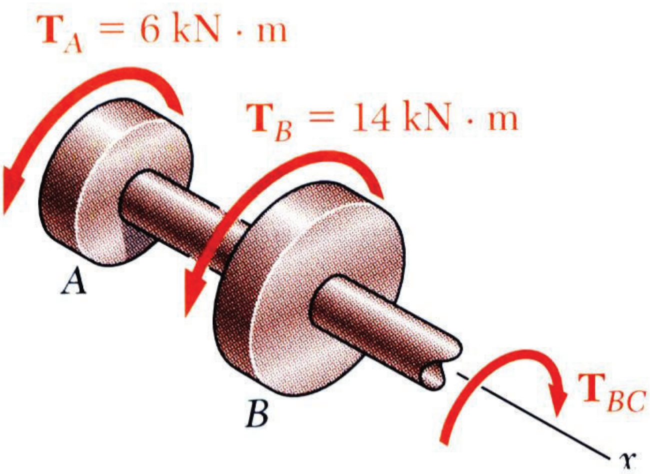
❑ Torsional Shearing Stress

Example 2



SOLUTION:

- Cut sections through shafts *AB* and *BC* and perform static equilibrium analysis to find torque loadings

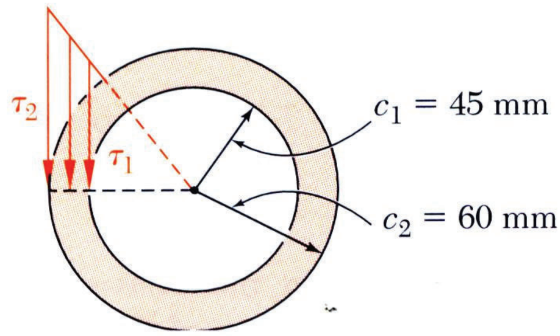


Torsion

❑ Torsional Shearing Stress

Example 2

- Apply elastic torsion formulas to find minimum and maximum stress on shaft *BC*

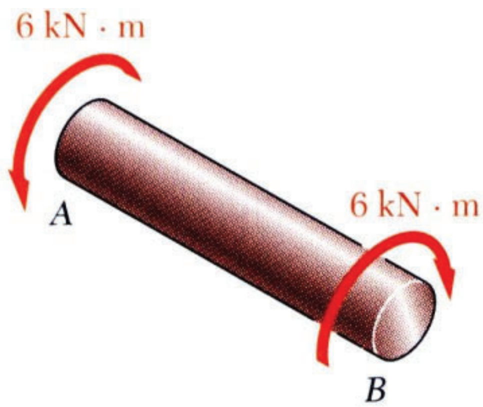


Torsion

❑ Torsional Shearing Stress

Example 2

- Given allowable shearing stress and applied torque, invert the elastic torsion formula to find the required diameter

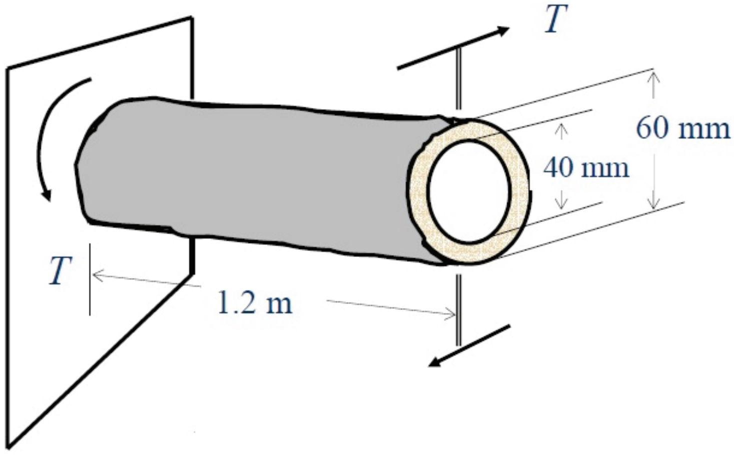


Torsion

❑ Torsional Shearing Stress

Example 3

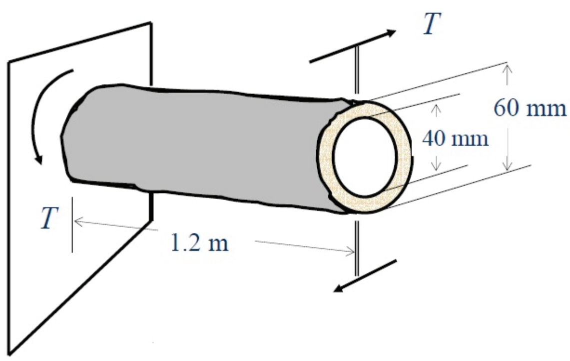
What torque should be applied to the end of the shaft to produce a twist of 2 degree? Use the value $G = 80$ GPa for the modulus of rigidity of steel.



Torsion

❑ Torsional Shearing Stress

Example 3

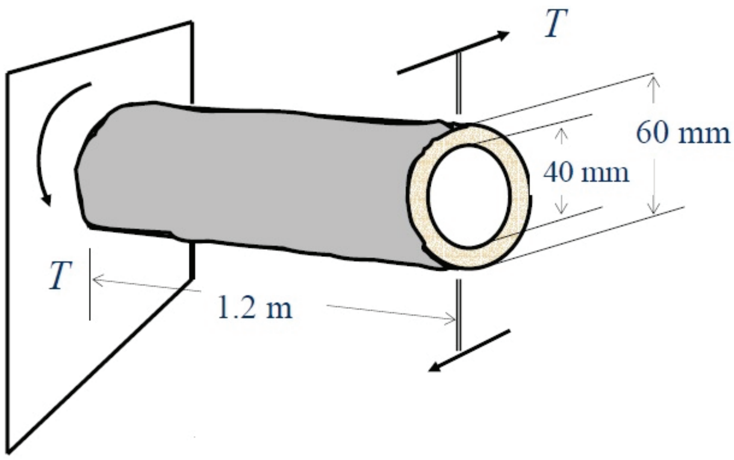


Torsion

❑ Torsional Shearing Stress

Example 4

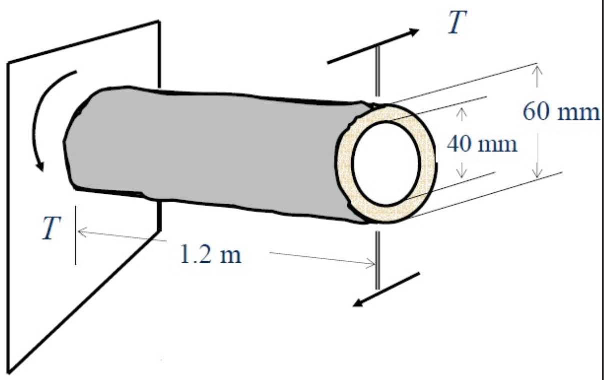
What angle of twist will create a shearing stress of 70 MPa on the inner surface of the hollow steel shaft?



Torsion

❑ Torsional Shearing Stress

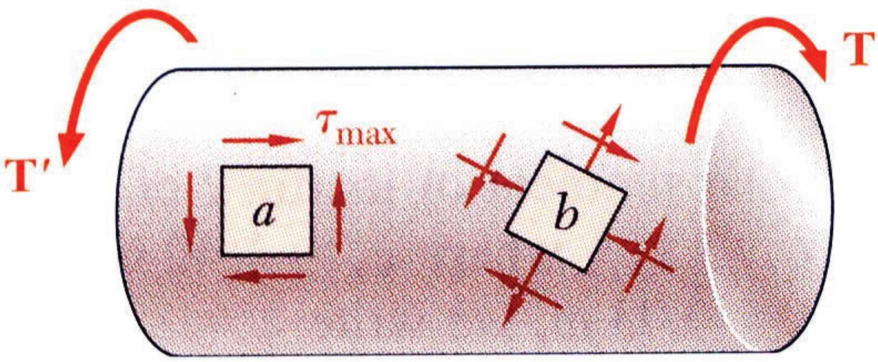
Example 4



Torsion

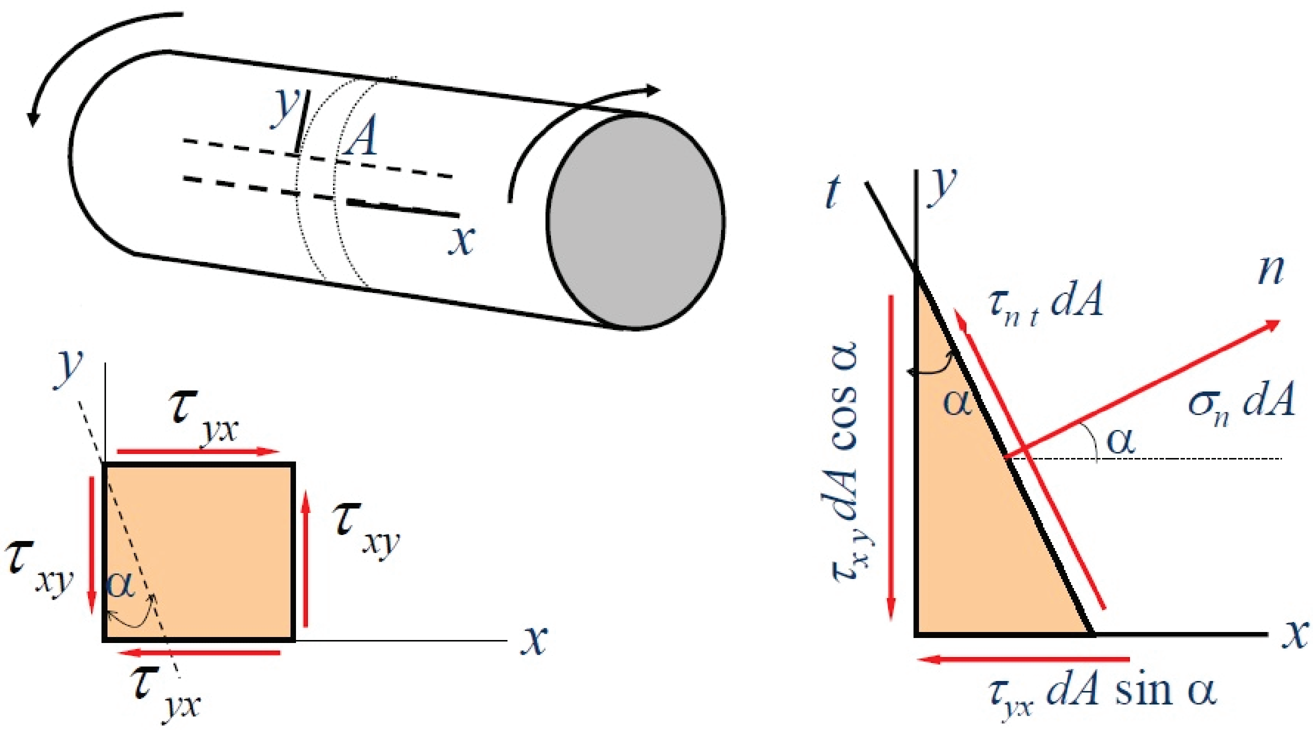
❑ Normal Stresses

- Elements with faces parallel and perpendicular to the shaft axis are subjected to shear stresses only. Normal stresses, shearing stresses or a combination of both may be found for other orientations.



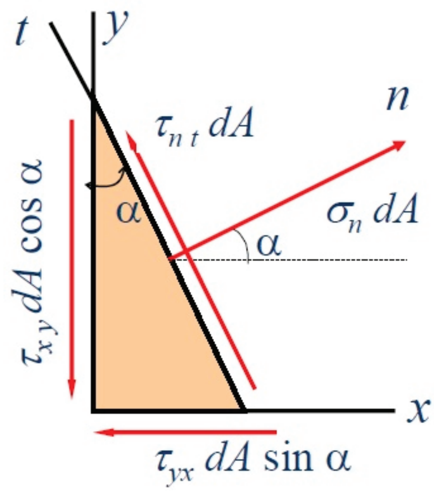
Torsion

❑ Normal Stresses



Torsion

- **Normal Stresses**



$$\sum F_t = 0 \quad \Rightarrow$$

$$\tau_{nt} dA - \tau_{xy}(dA\cos\alpha)\cos\alpha + \tau_{yx}(dA\sin\alpha)\sin\alpha = 0$$

$$\Rightarrow \boxed{\tau_{nt} = \tau_{xy}(\cos^2\alpha - \sin^2\alpha) = \tau_{xy}\cos 2\alpha}$$

$$\sum F_n = 0 \quad \Rightarrow$$

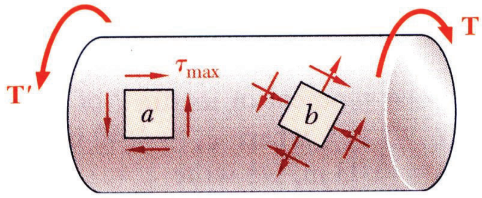
$$\sigma_n dA - \tau_{xy}(dA\cos\alpha)\sin\alpha + \tau_{yx}(dA\sin\alpha)\cos\alpha = 0$$

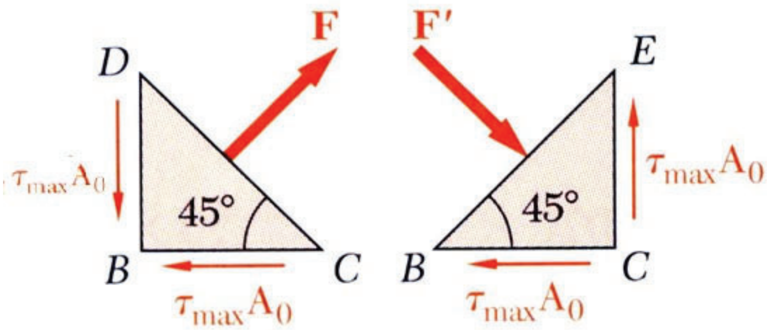
$$\Rightarrow \boxed{\sigma_n = 2\tau_{xy}\sin\alpha\cos\alpha = \tau_{xy}\sin 2\alpha}$$

Torsion

- **Normal Stresses**

- Consider an element at 45° to the shaft axis,





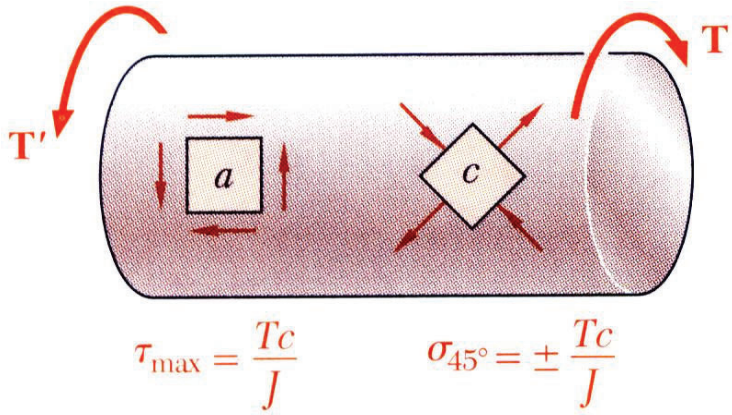
$$\boxed{\begin{aligned}\sigma_n &= \tau_{xy}\sin 2\alpha \\ \tau_{nt} &= \tau_{xy}\cos 2\alpha\end{aligned}}$$

$$\theta = 45^o \quad \Rightarrow \quad \boxed{\sigma_{n\,max} = \tau_{nt\,max} = \tau_{max}}$$

Torsion

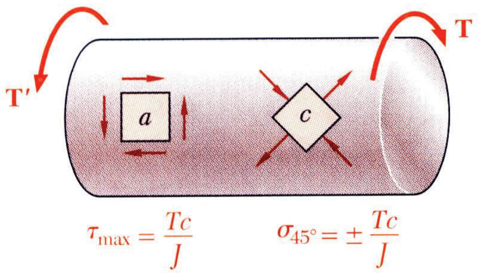
❑ Normal Stresses

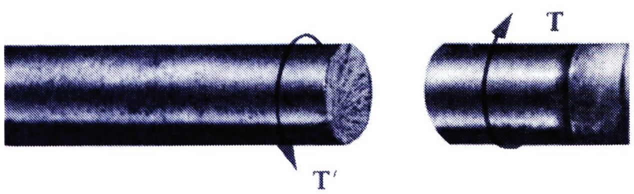
- Element *a* is in pure shear.
- Element *c* is subjected to a tensile stress on two faces and compressive stress on the other two.
- Note that all stresses for elements *a* and *c* have the same magnitude

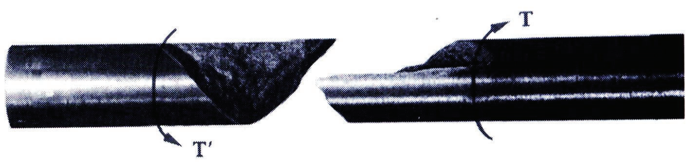


Torsion

❑ Torsional Failure Modes







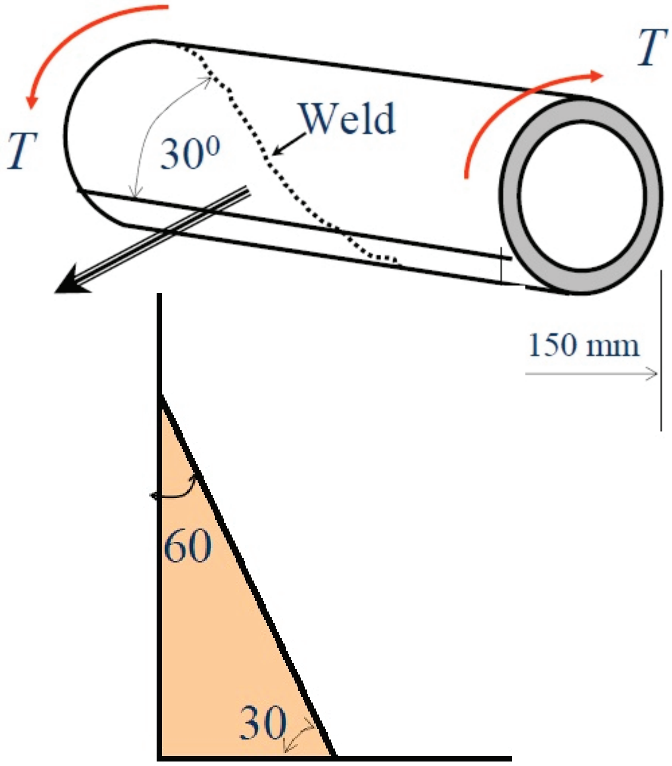
- Ductile materials generally fail in shear. Brittle materials are weaker in tension than shear.
- When subjected to torsion, a ductile specimen breaks along a plane of maximum shear, i.e., a plane perpendicular to the shaft axis.
- When subjected to torsion, a brittle specimen breaks along planes perpendicular to the direction in which tension is a maximum, i.e., along surfaces at 45° to the shaft axis.

Torsion

❑ Torsional Shearing Stress

Example 5

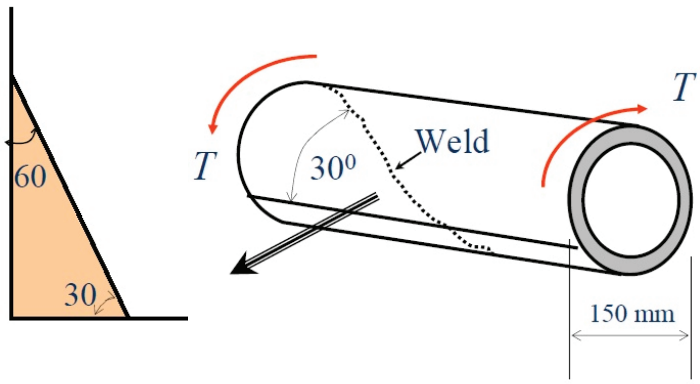
A cylindrical tube is fabricated by butt-welding a 6 mm-thick steel plate along a spiral seam as shown. If the maximum compressive stress in the tube must be limited to 80 MPa, determine (a) the maximum torque T that can be applied and (b) the factor of safety with respect to the failure by fracture for the weld, when a torque of 12 kN.m is applied, if the ultimate strengths of the weld metal are 205 MPa in shear and 345 MPa in tension.



Torsion

❑ Torsional Shearing Stress

Example 5



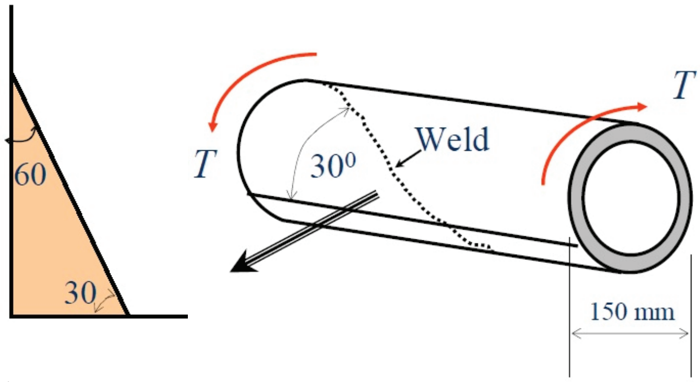
The polar moment of area for the cylindrical tube can be determined

The maximum torque can be computed from

Torsion

❑ Torsional Shearing Stress

Example 5

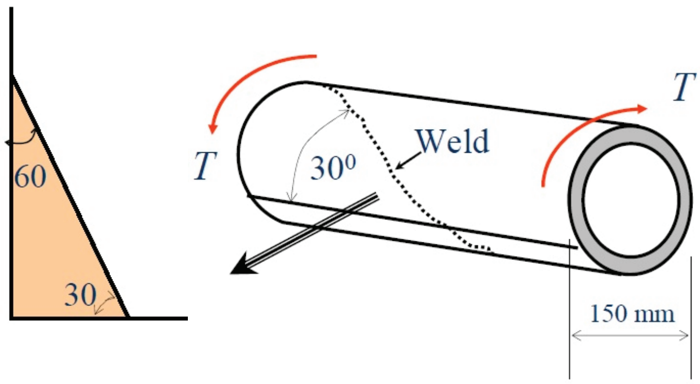


The normal stress and shear stress on the weld surface are given by

Torsion

❑ Torsional Shearing Stress

Example 5



The factors of safety with respect to failure by fracture for the weld are

Torsion

❑ Statically Indeterminate Shafts

- It is often for torsionally loaded members to be statically indeterminate in real engineering applications.

- When this occurs, distortion equations involving angle of twist θ must written until the total number of equations agrees with the number of unknowns to be determined.

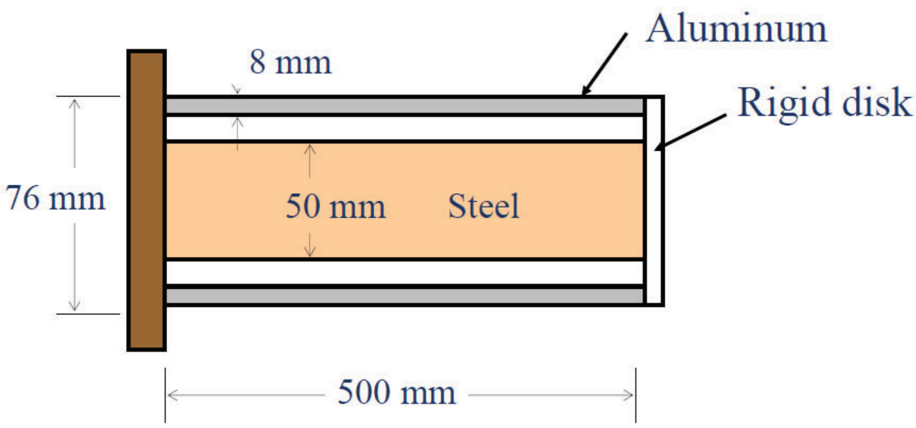
- A simplified angle of twist diagram will often be of great assistance in obtaining the correct equations.

Torsion

❑ Statically Indeterminate Shafts

Example 6

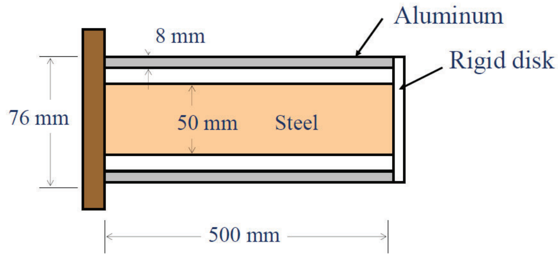
A steel shaft and aluminum tube are connected to a fixed support and to a rigid disk as shown in the figure. Knowing that the initial stresses are zero, determine the maximum torque T_0 that may be applied to the disk if the allowable stresses are 120 MPa in the steel shaft and 70 MPa in the aluminum tube. Use G = 80 GPa for steel and G = 27 GPa for aluminum.



Torsion

❑ Statically Indeterminate Shafts

Example 6



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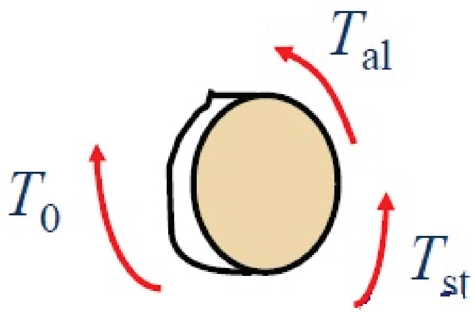
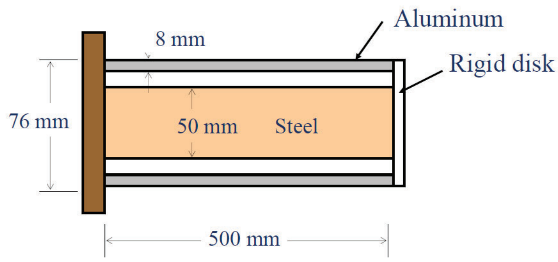


Deformation

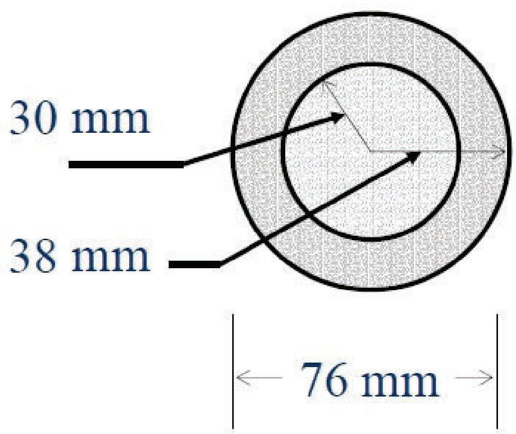
Torsion

❑ Statically Indeterminate Shafts

Example 6



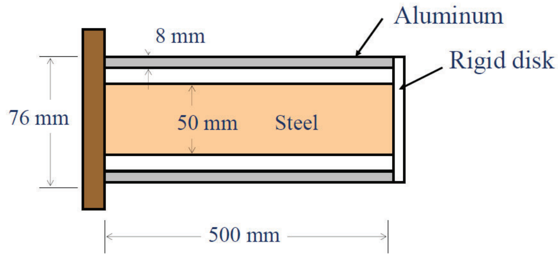
Properties of the aluminum tube



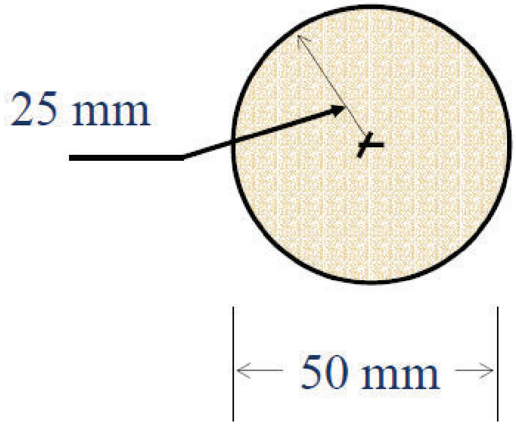
Torsion

❑ Statically Indeterminate Shafts

Example 6



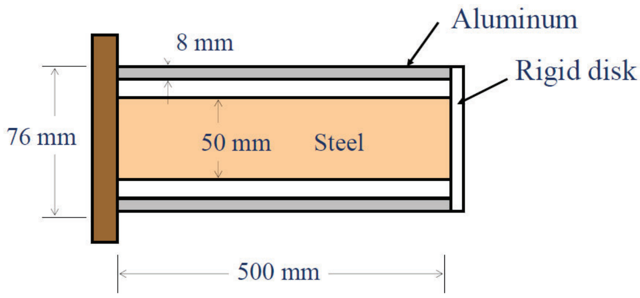
Properties of the steel tube



Torsion

❑ Statically Indeterminate Shafts

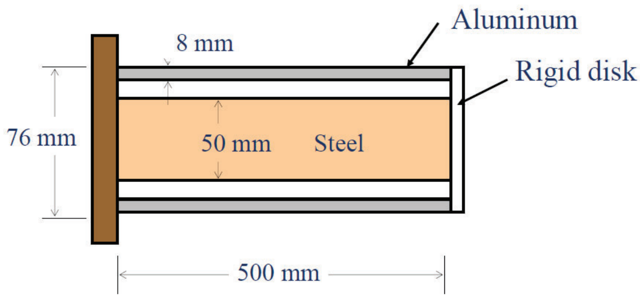
Example 6



Torsion

❑ Statically Indeterminate Shafts

Example 6

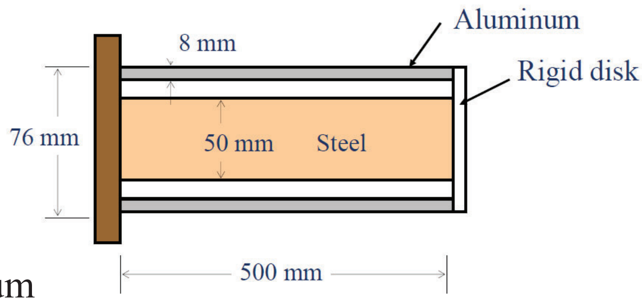


Let's assume that the requirement τ_{st} is less or to equal to 120 MPa, therefore

Torsion

❑ Statically Indeterminate Shafts

Example 6



Let's check the maximum stress τ_{al} in aluminum tube corresponding to T_{al} = 3244N·m:

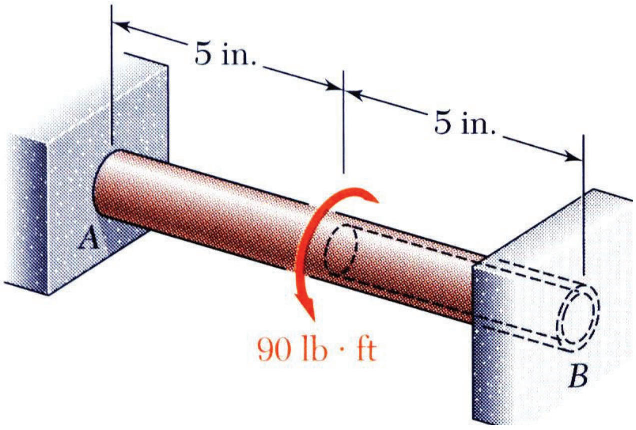
Hence, the max permissible torque T_0 is computed from

Torsion

❑ Statically Indeterminate Shafts

Example 7

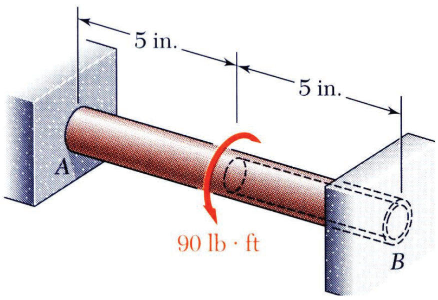
A circular shaft *AB* consists of a 10 in long, 7/8 in diameter steel cylinder, in which a 5 in. long, 5/8 in. diameter cavity has been drilled from end *B*. The shaft is attached to fixed supports at both ends, and a 90 lb – ft torque is applied at its mid-section. Determine the torque exerted on the shaft by each of the supports.

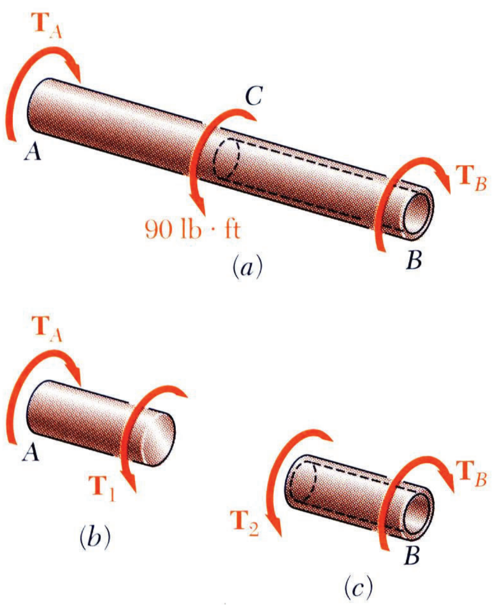


Torsion

❑ Statically Indeterminate Shafts

Example 7





- From a free-body analysis of the shaft,

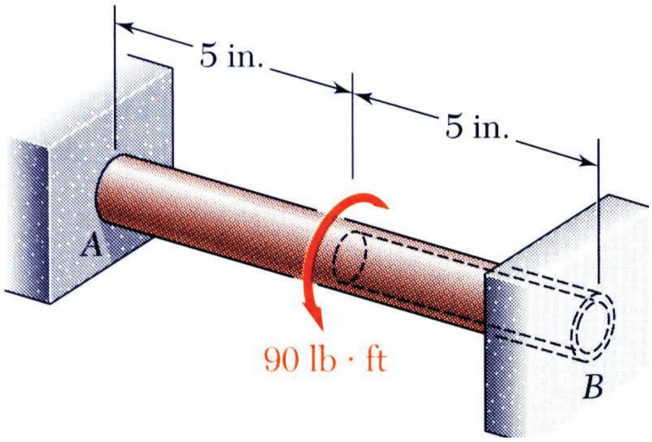
- which is not sufficient to find the end torques. The problem is statically indeterminate.

- Divide the shaft into two components which must have compatible deformations,

Torsion

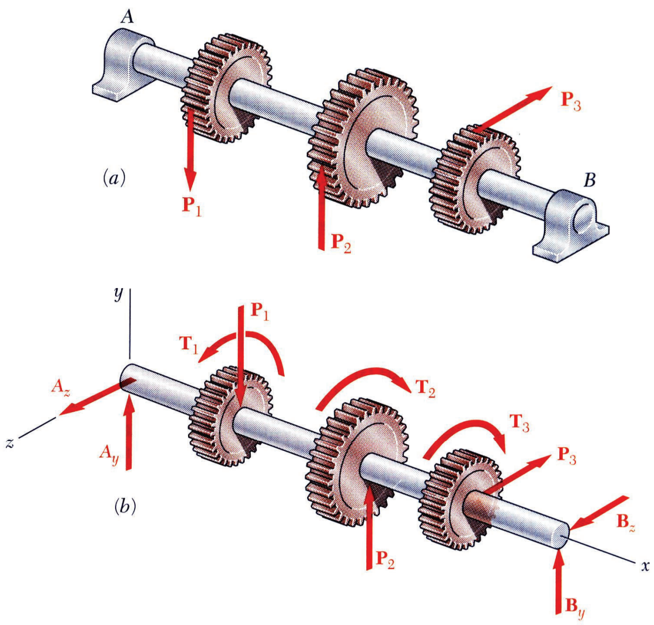
❑ Statically Indeterminate Shafts

Example 7



Torsion

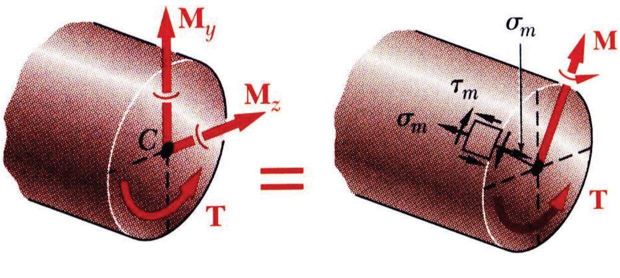
❑ Design of a Transmission Shaft



- If power is transferred to and from the shaft by gears or sprocket wheels, the shaft is subjected to transverse loading as well as shear loading.
- Normal stresses due to transverse loads may be large and should be included in determination of maximum shearing stress.
- Shearing stresses due to transverse loads are usually small and contribution to maximum shear stress may be neglected.

Torsion

❑ Design of a Transmission Shaft



- At any section,

$$\sigma_m = \frac{Mc}{I} \quad \text{where} \quad M^2 = M_y^2 + M_z^2$$
$$\tau_m = \frac{Tc}{J}$$

- Maximum shearing stress,

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_m}{2}\right)^2 + (\tau_m)^2} = \sqrt{\left(\frac{Mc}{2I}\right)^2 + \left(\frac{Tc}{J}\right)^2}$$

for a circular or annular cross - section, $2I = J$

$$\tau_{\max} = \frac{c}{J}\sqrt{M^2 + T^2}$$

- Shaft section requirement,

$$\left(\frac{J}{c}\right)_{\min} = \frac{\left(\sqrt{M^2 + T^2}\right)_{\max}}{\tau_{all}}$$

Torsion

❑ Design of Transmission Shafts

- Designer must select shaft material and cross-section to meet performance specifications without exceeding allowable shearing stress.

- Find shaft cross-section which will not exceed the maximum allowable shearing stress,

$$\tau_{\max} = \frac{Tc}{J}$$

$$\frac{J}{c} = \frac{\pi}{2}c^3 = \frac{T}{\tau_{\max}} \quad \text{(solid shafts)}$$

$$\frac{J}{c_2} = \frac{\pi}{2c_2}(c_2^4 - c_1^4) = \frac{T}{\tau_{\max}} \quad \text{(hollow shafts)}$$

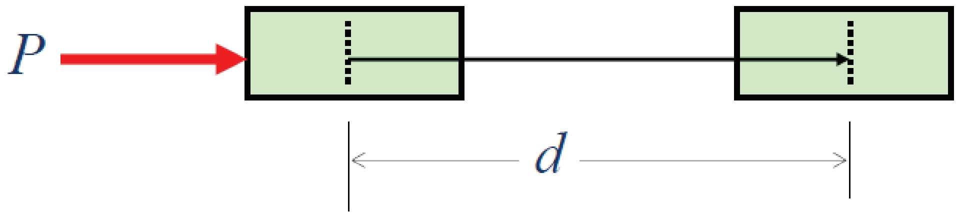
Torsion

❑ Power Transmission

Work of a Force

– A force does work only when the particle to which the force is applied moves.

$$\text{Work} = U = Pd$$



Torsion

❑ Power Transmission

Work of a Couple

– The work of a couple is defined as the magnitude of the couple **C** times the angular movement of the body.

$$U_{1\to 2} = C\,\Delta\theta$$

$$dU = \vec{C}\cdot d\vec{\theta}$$

Torsion

❑ Power Transmission by Torsional Shaft

The power is defined as the time rate of doing work, that is

$$P = \frac{dU}{dt} = C \cdot \frac{d\theta}{t} = T \cdot \frac{d\theta}{t} = T\omega$$

ω = angular velocity of the shaft in radians per minute

Torsion

❑ Power Transmission by Torsional Shaft

– But ω = 2π *f*, where *f* = frequency. The unit of frequency is 1/s and is called *hertz* (Hz).
– If this is the case, then the power is given by

$$P = 2\pi f \cdot T$$

or

$$T = \frac{P}{2\pi f}$$

Units of Power

SI	US Customary
watt (1 N·m/s)	hp (33,000 ft·lb/min)

Some useful relations

$$1\ rpm = \frac{1}{60} s^{-1} = \frac{1}{60} Hz$$

$$1 hp = 550\ ft.lb/s = 6600\ \frac{in \cdot lb}{s}$$

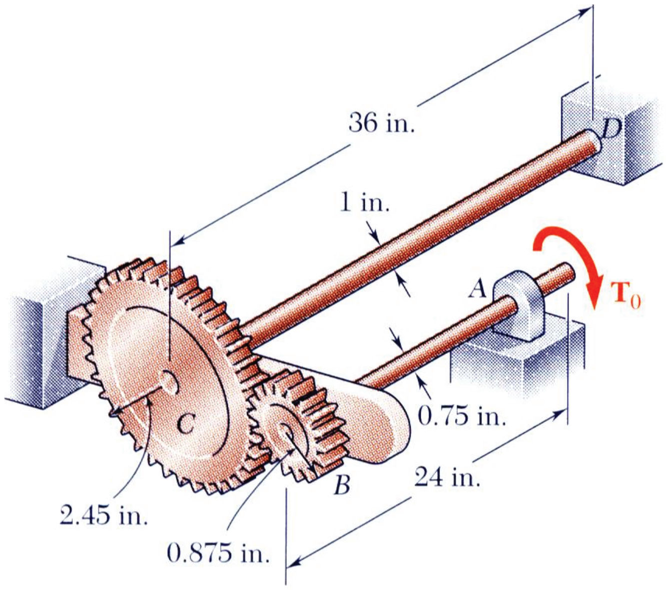
rpm = revolution per minute

Torsion

❑ Statically Indeterminate Shafts

Example 8

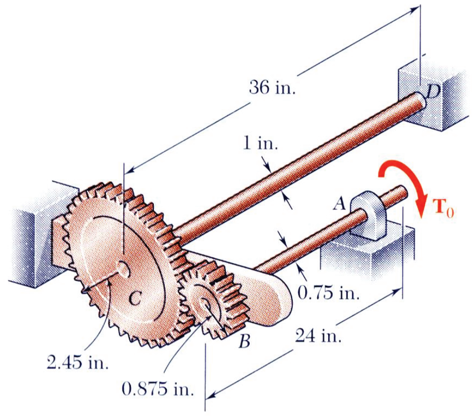
Two solid steel shafts are connected by gears. Knowing that for each shaft $G = 11.2 \times 10^6$ psi and that the allowable shearing stress is 8 ksi, determine (*a*) the largest torque T_0 that may be applied to the end of shaft *AB*, (*b*) the corresponding angle through which end *A* of shaft *AB* rotates.



Torsion

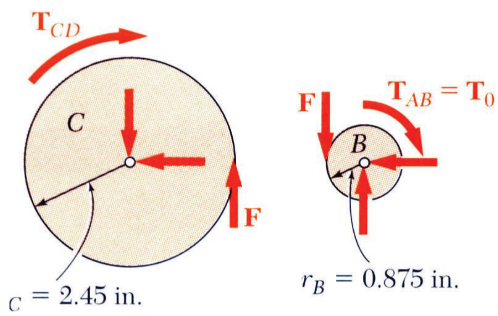
❑ Statically Indeterminate Shafts

Example 8



SOLUTION:

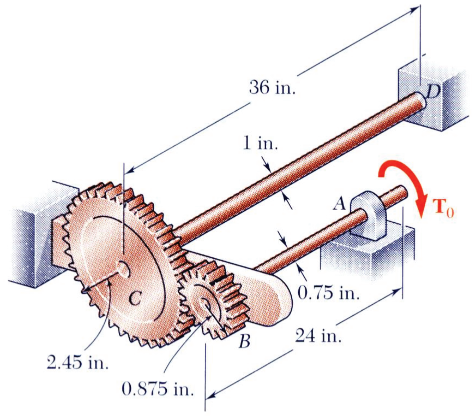
- Apply a static equilibrium analysis on the two shafts to find a relationship between T_{CD} and T_0



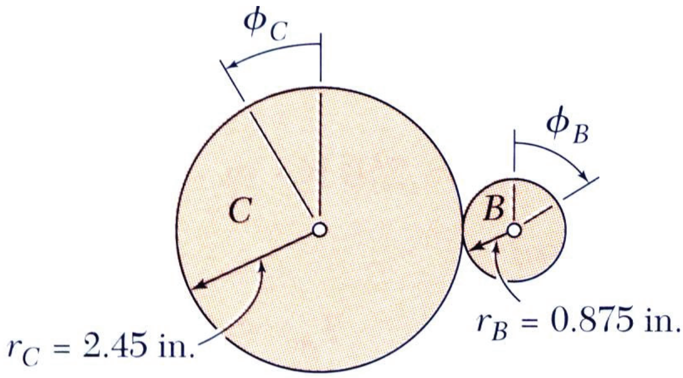
Torsion

❑ Statically Indeterminate Shafts

Example 8



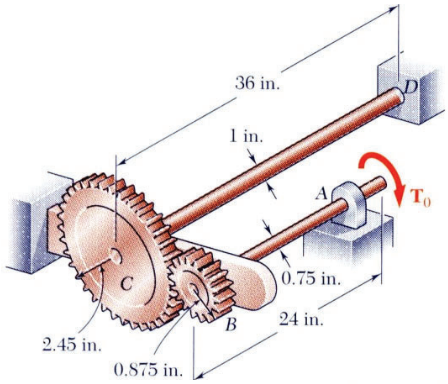
- Apply a kinematic analysis to relate the angular rotations of the gears



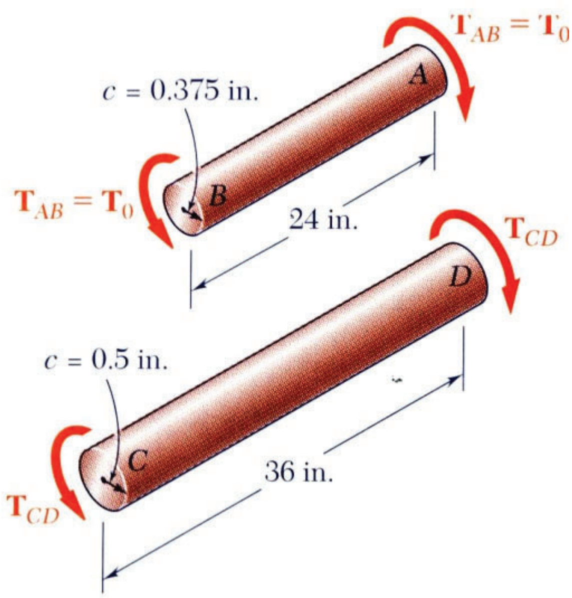
Torsion

❑ Statically Indeterminate Shafts

Example 8



- Find the T_0 for the maximum allowable torque on each shaft – choose the smallest

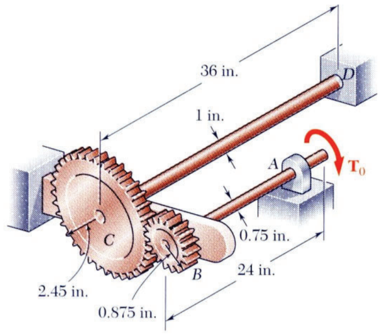


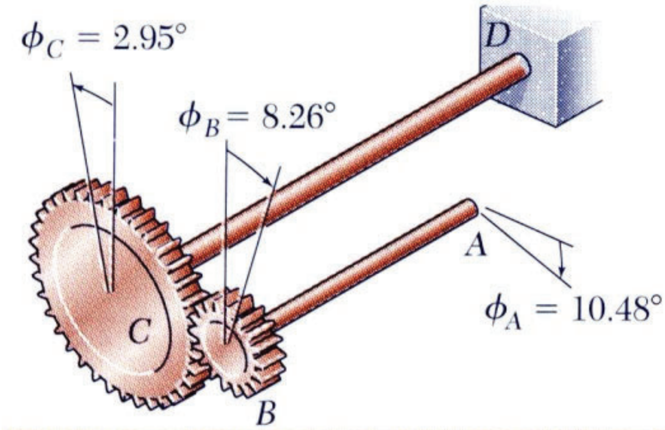
Torsion

❑ Statically Indeterminate Shafts

Example 8

- Find the corresponding angle of twist for each shaft and the net angular rotation of end *A*

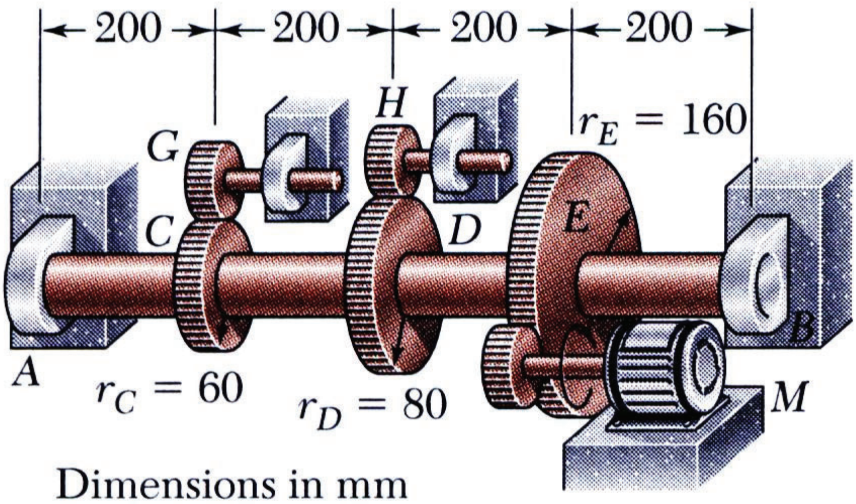




Torsion

❑ Design of a Transmission Shaft

Example 9



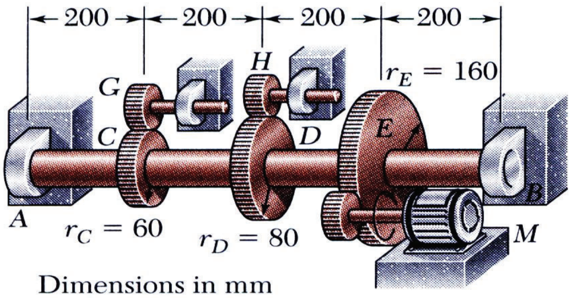
Solid shaft rotates at 480 rpm and transmits 30 kW from the motor to gears *G* and *H*; 20 kW is taken off at gear *G* and 10 kW at gear *H*. Knowing that s_{all} = 50 MPa, determine the smallest permissible diameter for the shaft.

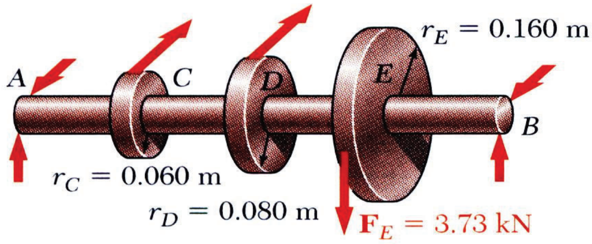
Loading combination

❑ Design of a Transmission Shaft

Example 9

- Determine the gear torques and corresponding tangential forces.

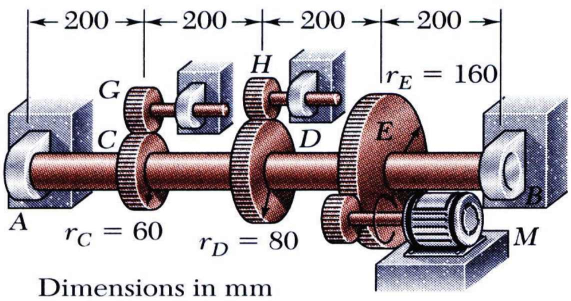


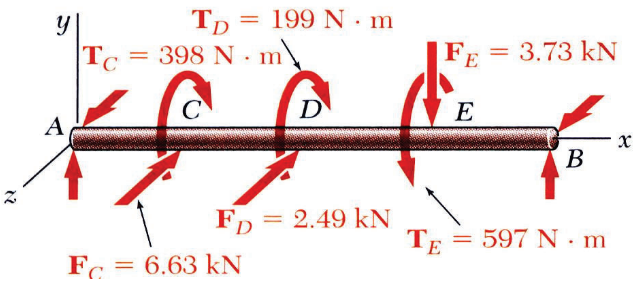


Loading combination

❑ Design of a Transmission Shaft

Example 9



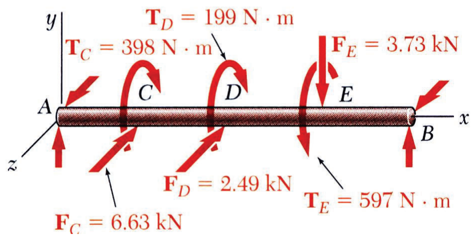


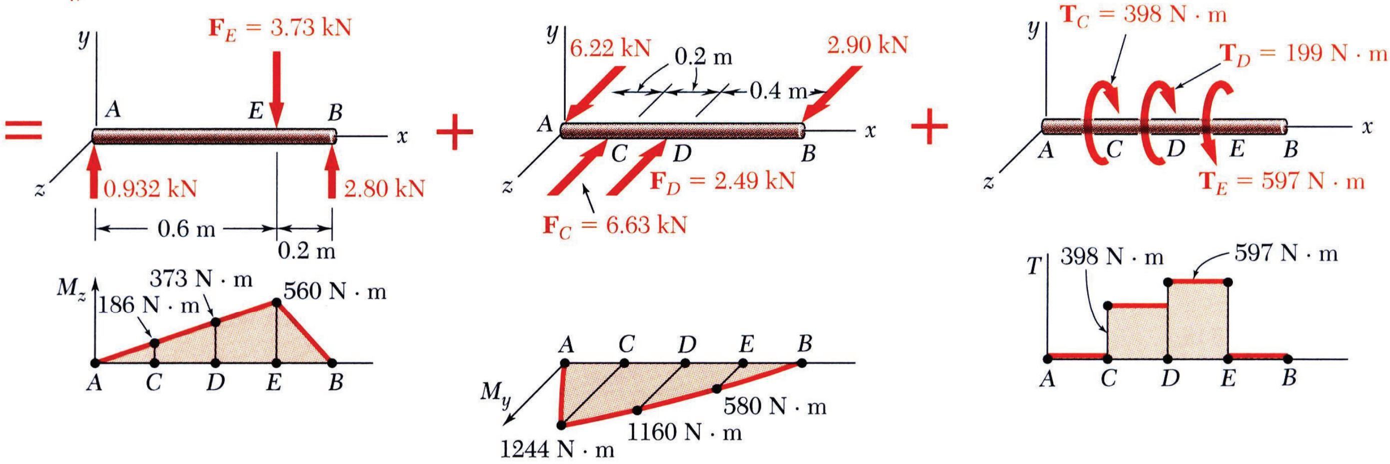
Loading combination

❑ Design of a Transmission Shaft

Example 9

- Identify critical shaft section from torque and bending moment diagrams.





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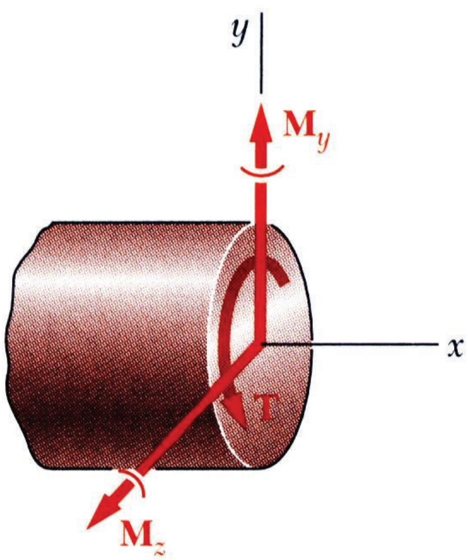
❑ Design of a Transmission Shaft

Example 9

- Calculate minimum allowable shaft diameter.

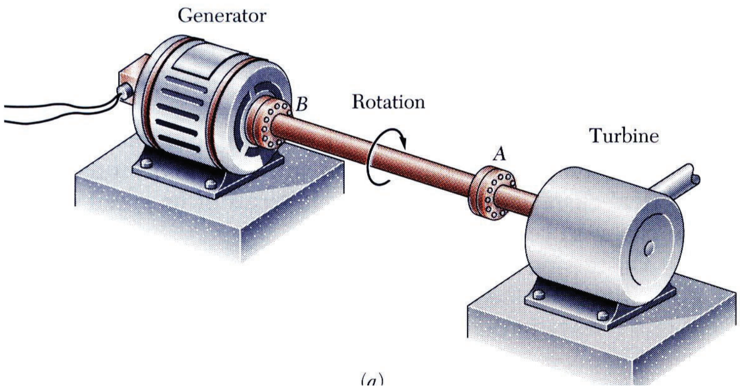


Torsion

- **Statically Indeterminate Shafts**

Example 10

What size of shaft should be used for a rotor of 5-hp motor operating at 3600 rpm if the shearing stress is not to exceed 8500 psi in the shaft?



Torsion

- **Statically Indeterminate Shafts**

Example 10

Torsion

❑ Stress Concentrations

- The derivation of the torsion formula,

$$\tau_{\max} = \frac{Tc}{J}$$

assumed a circular shaft with uniform cross-section loaded through rigid end plates.

- The use of flange couplings, gears and pulleys attached to shafts by keys in keyways, and cross-section discontinuities can cause stress concentrations

- Experimental or numerically determined concentration factors are applied as

$$\tau_{\max} = K\frac{Tc}{J}$$

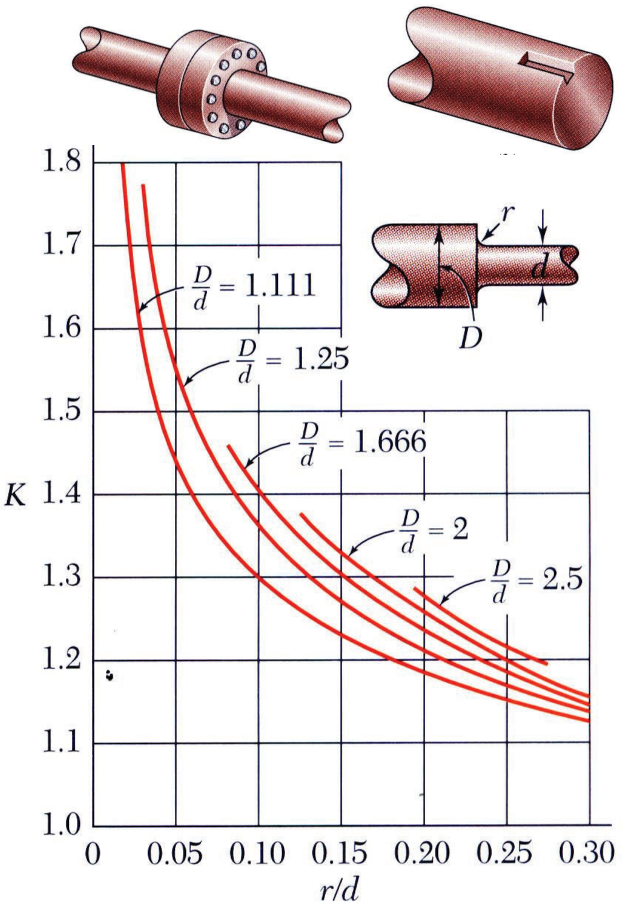


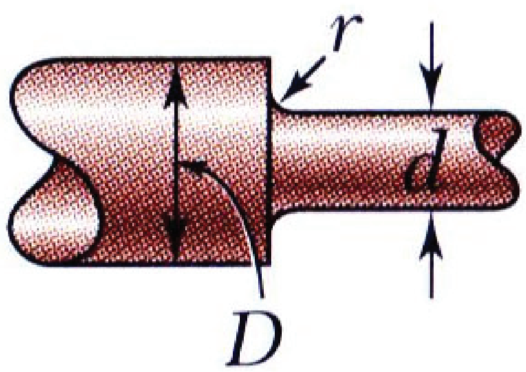
Fig. 3.32 Stress-concentration factors for fillets in circular shafts.†

Torsion

❑ Statically Indeterminate Shafts

Example 11

The stepped shaft shown is to rotate at 900 rpm as it transmits power from a turbine to a generator. The grade of steel specified in the design has an allowable shearing stress of 8 ksi. (a) For preliminary design shown, determine the maximum power that can be transmitted. (b) If in the final design the radius of the fillet is increased so that *r*= 15/16 in., what will be the percent change , relative to the preliminary design in the power?



$$D = 7.5\ in$$

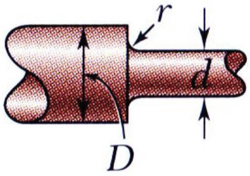
$$d = 3.75\ in$$

$$r = \frac{9}{16}\ in$$

Torsion

- **Statically Indeterminate Shafts**

Example 11



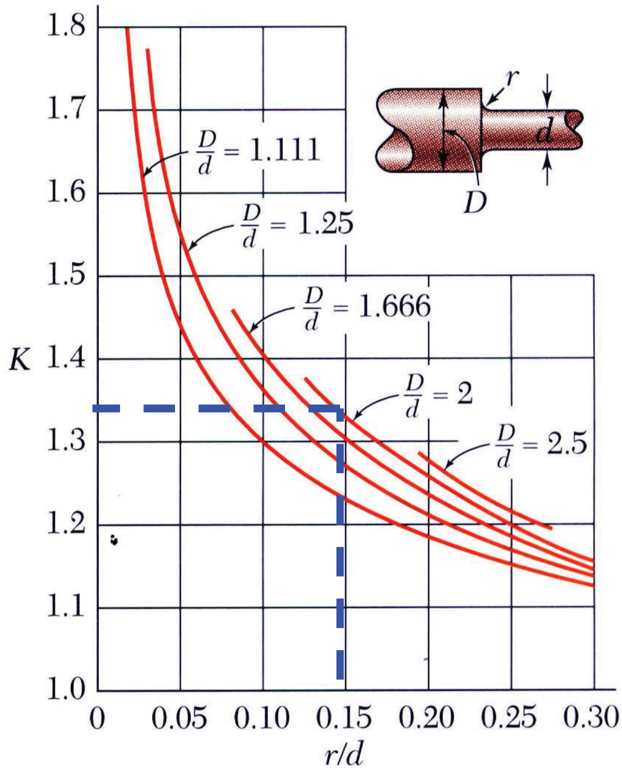
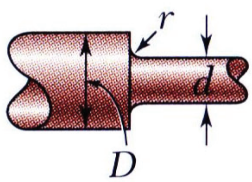


Fig. 3.32 Stress-concentration factors for fillets in circular shafts.†

Torsion

- **Statically Indeterminate Shafts**

Example 11



$D = 7.5\ in$

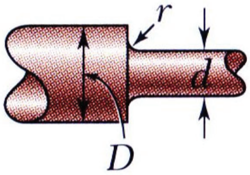
$d = 3.75\ in$

$r = \frac{9}{16} in$

Torsion

❑ Statically Indeterminate Shafts

Example 11



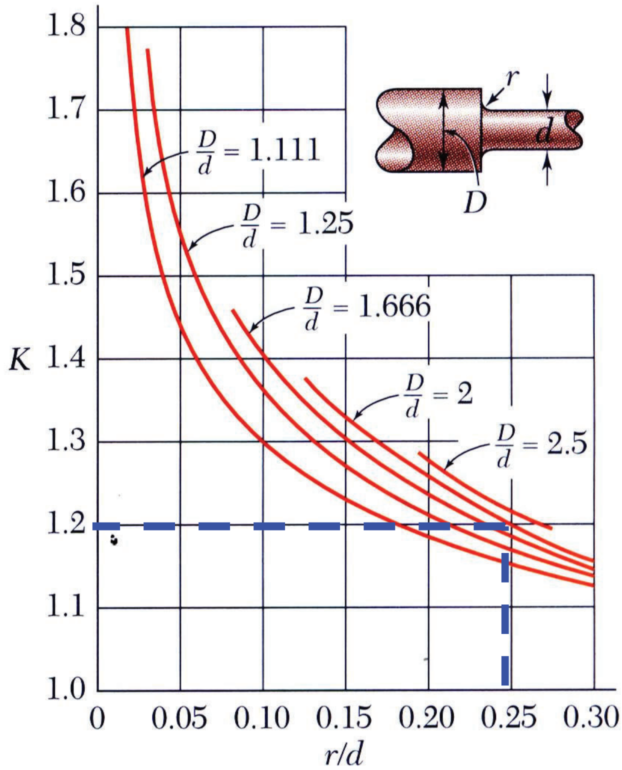
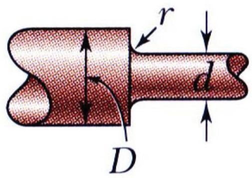


Fig. 3.32 Stress-concentration factors for fillets in circular shafts.†

Torsion

❑ Statically Indeterminate Shafts

Example 11



$D = 7.5\ in$

$d = 3.75\ in$

$r = \frac{15}{16} in$