Mechanics of Materials

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Ferdinand P.Beer, E.Russel Johnston, Jr., John T.Dewolf

Other Reference:
J.Wat Oler "Lectures notes on Mechanics od Materials"
Ibrahim A.Assakkaf "Lectures notes on Mechanics od Materials"

Pure Bending

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Pure Bending

- **Introduction**

- The most common type of structural member is a beam.

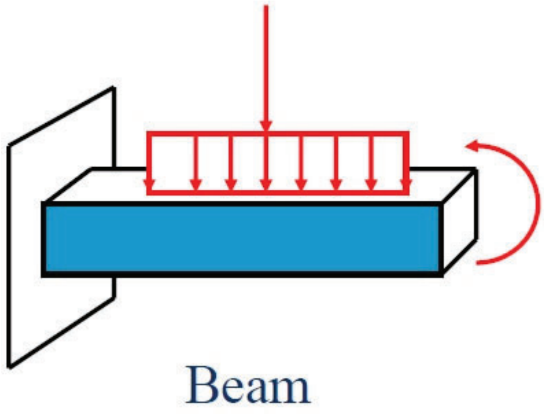
- In actual structures beams can be found in an infinite variety of
  - Sizes
  - Shapes, and
  - Orientations

Pure Bending

❑ Introduction

Beam Definition

A beam may be defined as a member whose length is relatively large in comparison with its thickness and depth, and which is loaded with transverse loads that produce significant bending effects.



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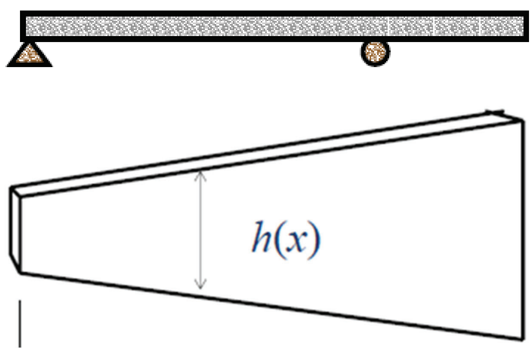
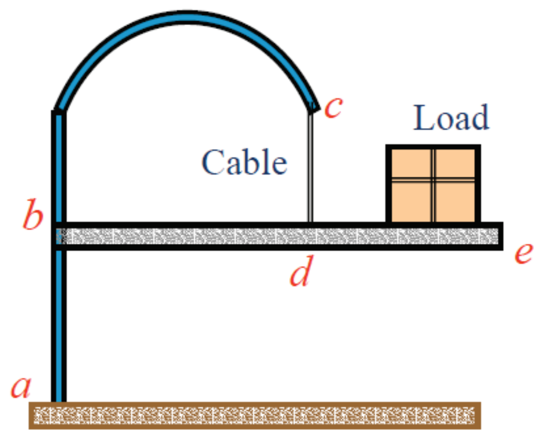
❑ Introduction

Geometrical classification

includes such features as the shape of the cross section, whether the beam is:

- *straight* or
- *curved*
- *Tapered*
- Has a *constant cross section.*





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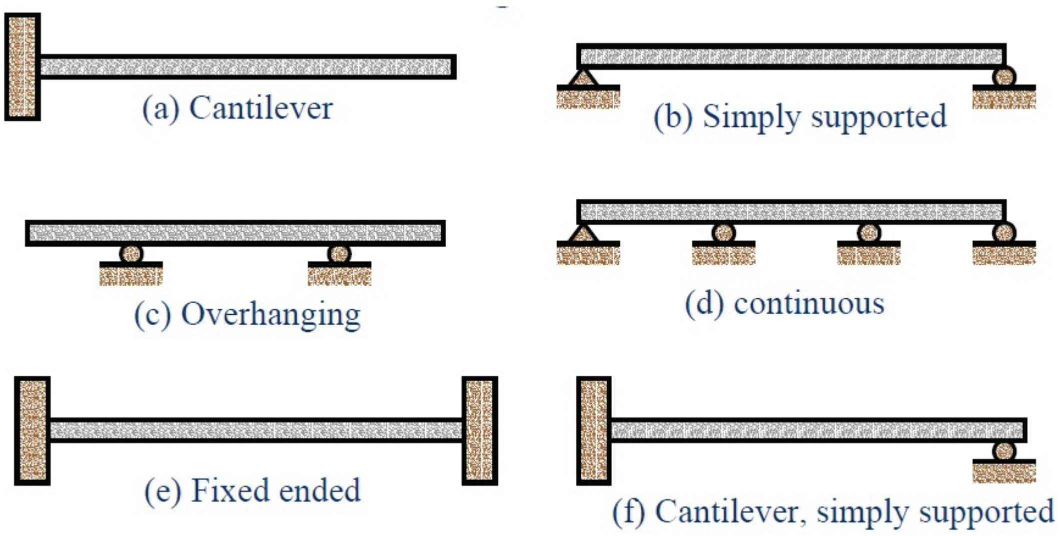
❑ Introduction

Classified based on supports

Beams are generally classified according to their geometry and the manner in which they are supported..



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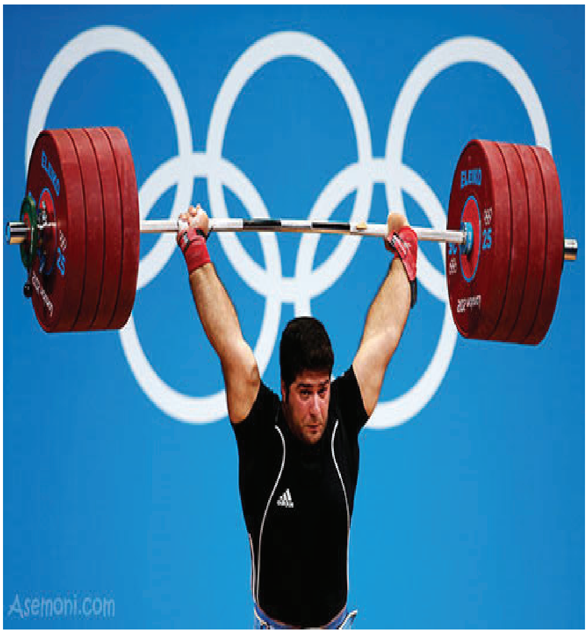
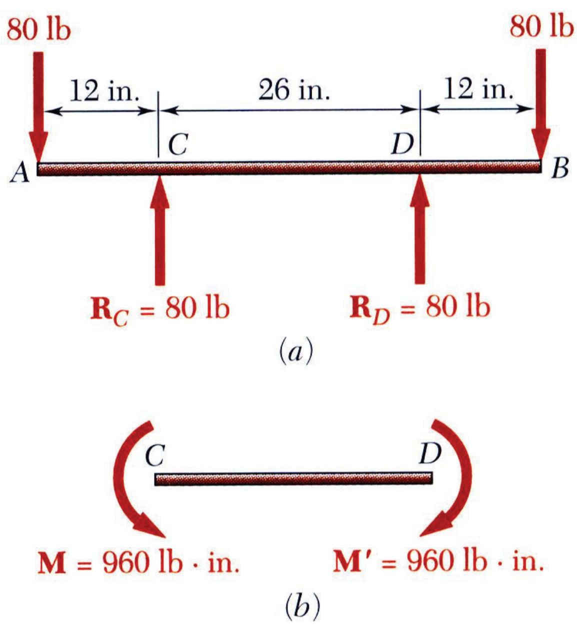
❑ Introduction



Pure Bending: Prismatic members subjected to equal and opposite couples acting in the same longitudinal plane



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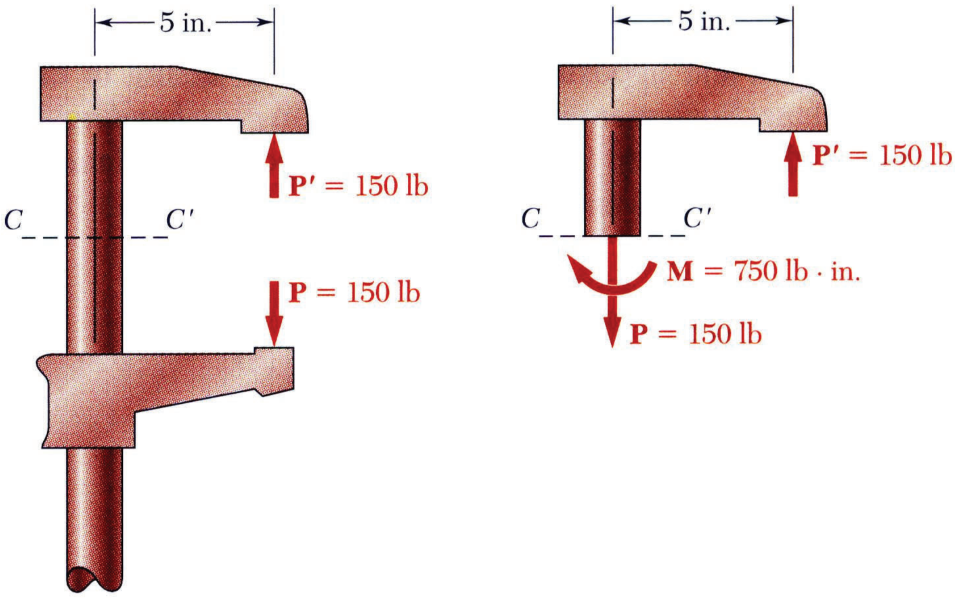
❑ ***Loading Types***

- ***Eccentric Loading:*** Axial loading which does not pass through section centroid produces internal forces equivalent to an axial force and a couple



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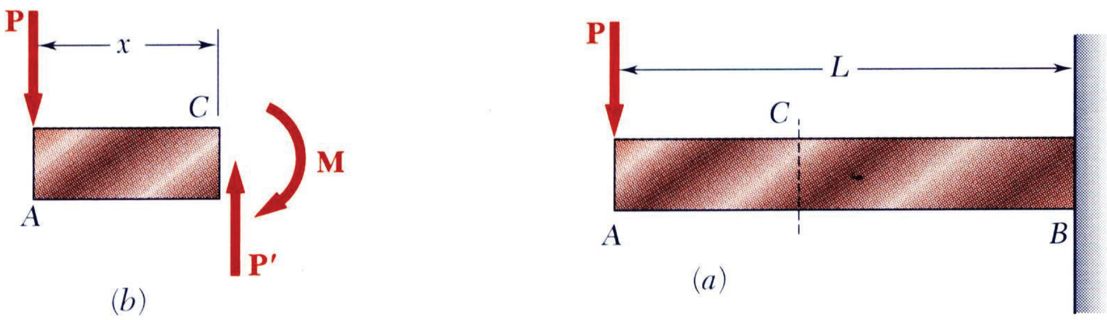
❑ ***Loading Types***

- ***Transverse Loading:*** Concentrated or distributed transverse load produces internal forces equivalent to a shear force and a couple



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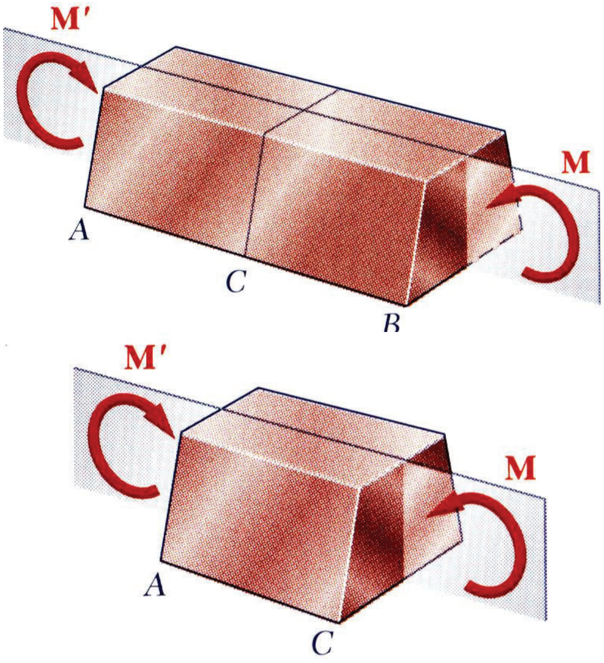
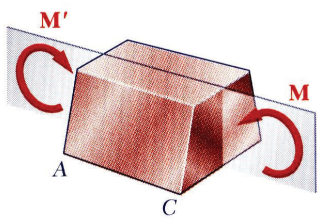
❑ ***Symmetric Member in Pure Bending***

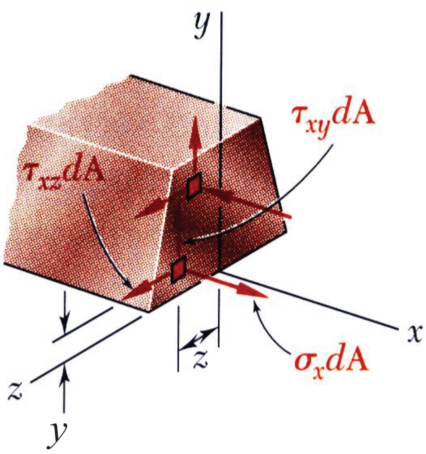


- Internal forces in any cross section are equivalent to a couple. The moment of the couple is the ***section bending moment***.

Pure Bending

❑ ***Symmetric Member in Pure Bending***





A couple M consists of two equal and opposite forces.

- The sum of the components of the forces in any direction is zero.

$$F_x = \int \sigma_x \, dA = 0$$

- The moment is **M** about any axis perpendicular to the plane of the couple

$$M_z = \int -y\sigma_x \, dA = M$$

- The moment is zero about any axis contained in the plane.

$$M_y = \int z\sigma_x \, dA = 0$$

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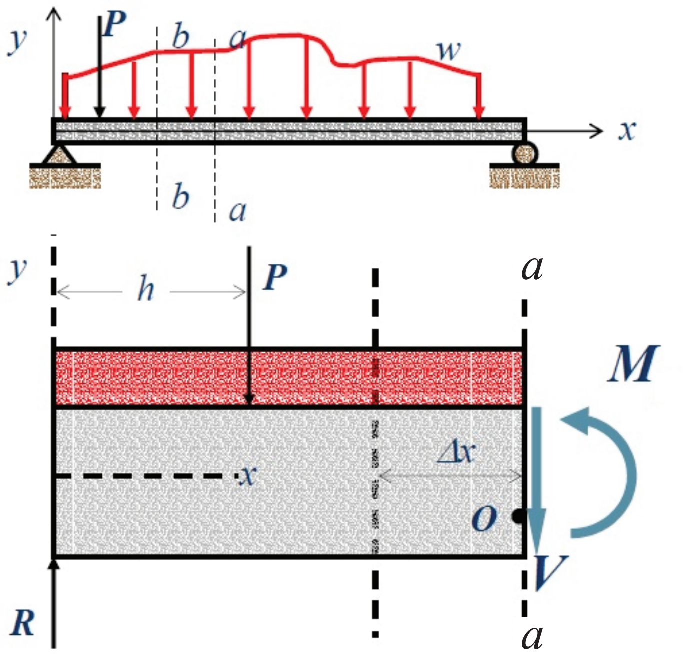
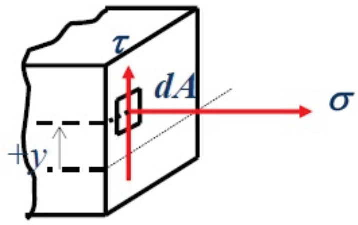
❑ ***Normal and Shearing Stress***

The normal stress on plane a-a is related to the resisting moment **M** and also the shearing stress on plane a-a is related to the resisting shear **V** .



$$M = -\int_{area} y\sigma \, dA$$

$$V = -\int_{area} \tau \, dA$$



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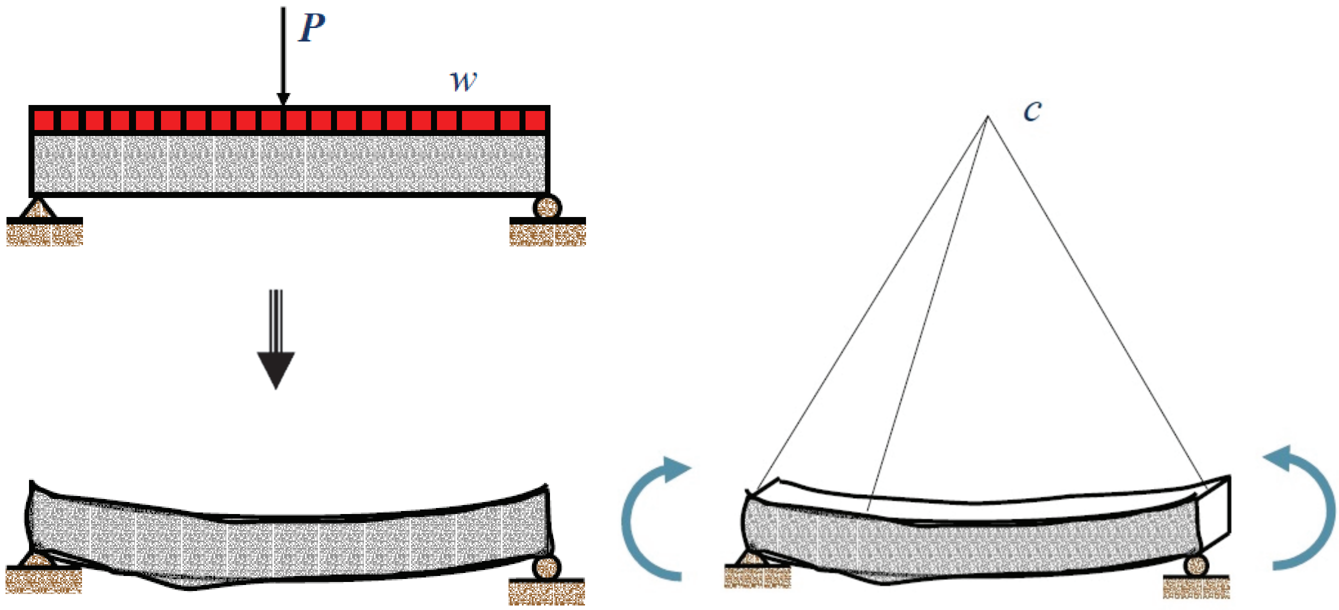
❑ ***Bending Deformations***

Deformation of Beam due to Lateral Loading



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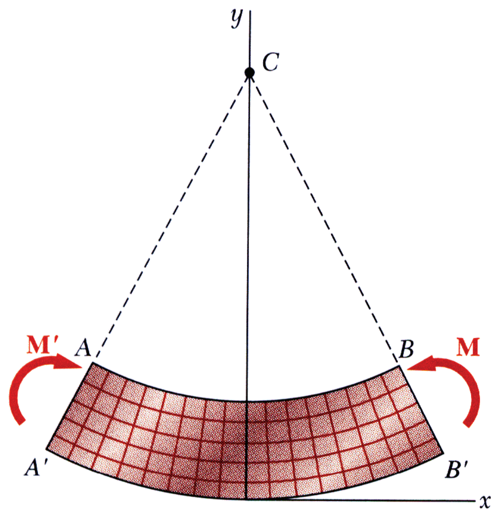
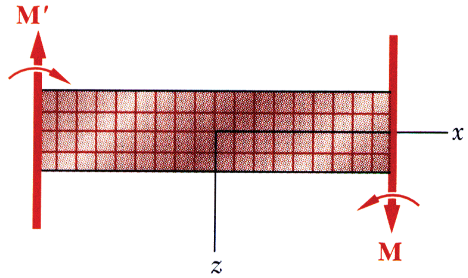
❑ Bending Deformations

Beam with a plane of symmetry in pure bending:

- member remains symmetric.
- bends uniformly to form a circular arc.
- cross-sectional plane passes through arc center and remains planar.
- length of top decreases and length of bottom increases.
- a *neutral surface* must exist that is parallel to the upper and lower surfaces and for which the length does not change.
- stresses and strains are negative (compressive) above the neutral plane and positive (tension) below it.



(a) Longitudinal, vertical section (plane of symmetry)



(b) Longitudinal, horizontal section

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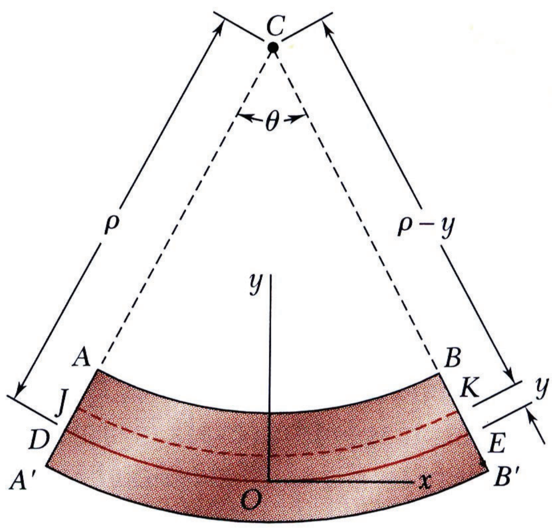
❑ Bending Deformations

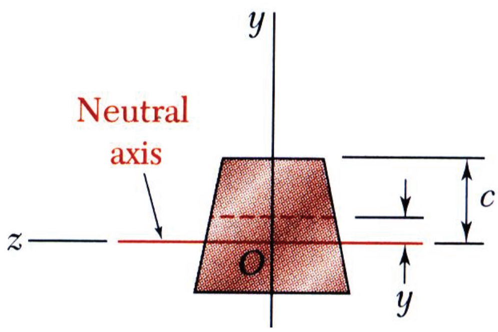
Consider a beam segment of length L. After deformation, the length of the neutral surface remains L. At other sections,

$$L_1 = \rho\theta$$
$$L_2 = (\rho - y)\theta$$

$$\delta = L_2 - L_1 = (\rho - y)\theta - \rho\theta$$
$$\Rightarrow \boxed{\delta = -y\theta}$$

$$\varepsilon_x = \frac{\delta}{L_1} = -\frac{y\theta}{\rho\theta}$$
$$\Rightarrow \boxed{\varepsilon_x = -\frac{y}{\rho}} \quad \text{(strain varies linearly)}$$





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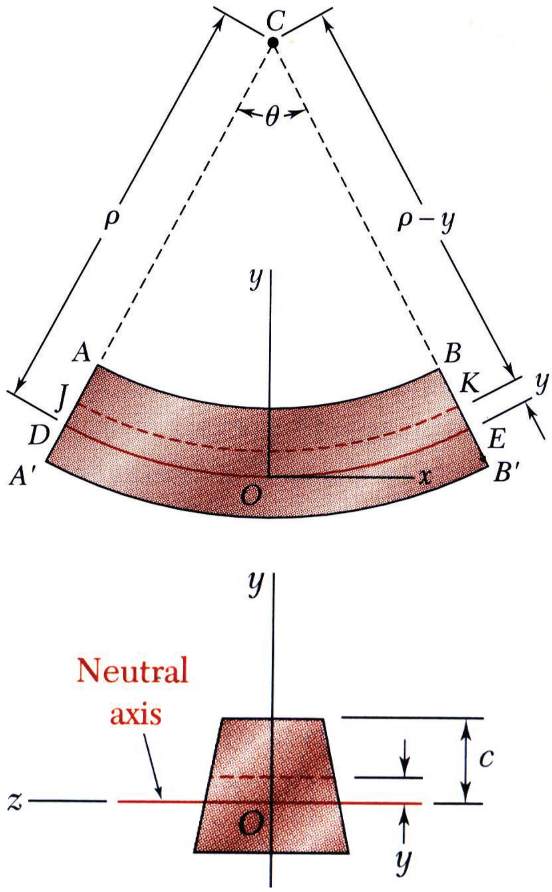
❑ ***Bending Deformations***

$$\varepsilon_x = -\frac{y}{\rho}$$

$$if \quad y = c \quad \Rightarrow \quad \varepsilon_m = -\frac{c}{\rho} \quad \text{or} \quad \rho = -\frac{c}{\varepsilon_m}$$

$$\varepsilon_x = \frac{y}{c}\varepsilon_m$$



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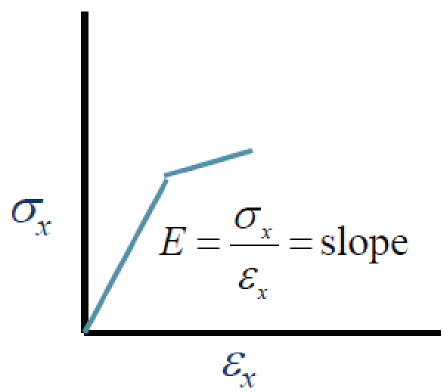
❑ ***Flexural Stress***

For special case of linearly elastic deformation, the relationship between the normal stress σ_x and the normal strain ε_x is given by Hooke's law as

$$E = \frac{\sigma_x}{\varepsilon_x} = slope$$

$$\sigma_x = E\varepsilon_x = -\frac{y}{\rho}E$$



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❑ *Flexural Stress*

The maximum normal stress on the cross section is given by

$$\varepsilon_m = \frac{c}{\rho} \quad \Rightarrow \qquad \sigma_m = -\frac{E}{\rho}c$$

$$\varepsilon_x = \frac{y}{c}\varepsilon_m \quad \Rightarrow \qquad \sigma_x = \frac{y}{c}\sigma_m$$

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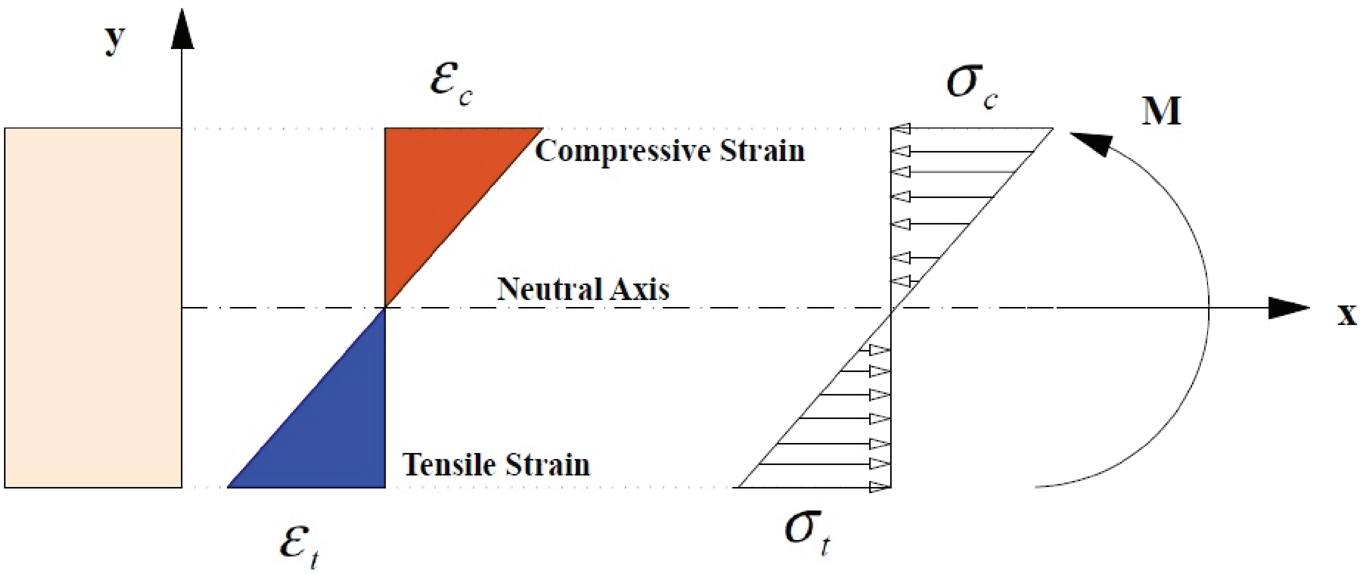
❑ *Flexural Normal Stress*

The resisting moment M that can be develop by the normal stress in a typical beam with loading in a plane of symmetry.

$$M = -\int_{area} y\sigma \, dA$$



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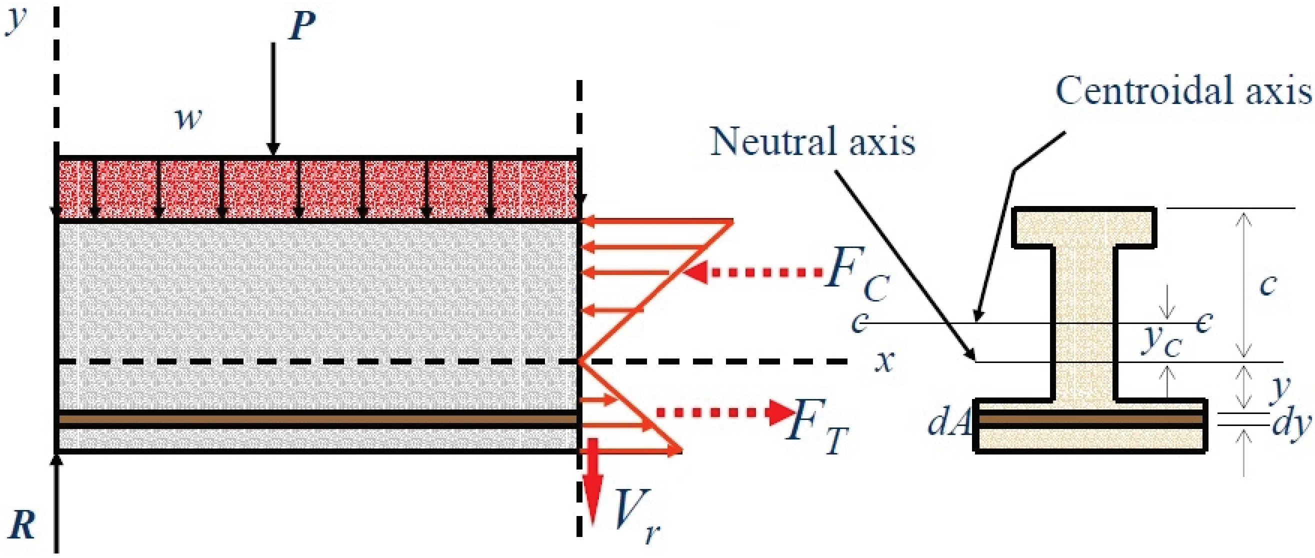
❑ ***Stress Due to Bending***

Since y is measured from the neutral axis (surface), it is necessary to locate this axis by means of the equilibrium equation as follows:

$$\sum F_x = 0 \quad \Rightarrow \quad \int_A \sigma_x \cdot dA = 0$$



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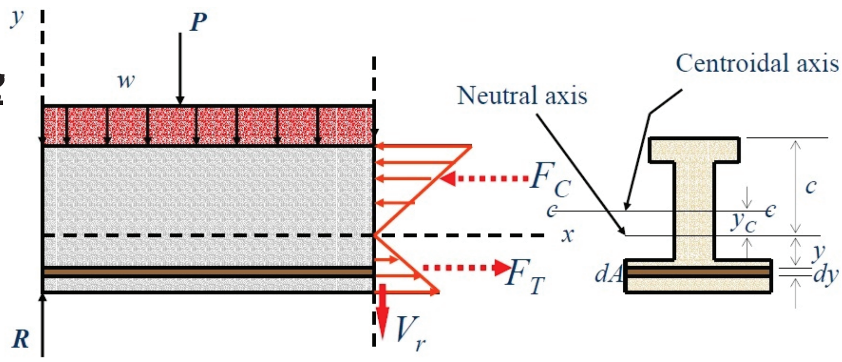
❑ ***Stress Due to Bending***



- For static equilibrium,

$$\int_A \sigma_x \cdot dA = 0$$

$$\sigma_x = -\frac{y}{\rho} E \quad \Rightarrow \quad -\frac{E}{\rho}\int_A y \cdot dA = 0$$

$$y_c = \frac{\int_A y \cdot dA}{A}$$ = distance from neutral axis to centroid axis (c-c)

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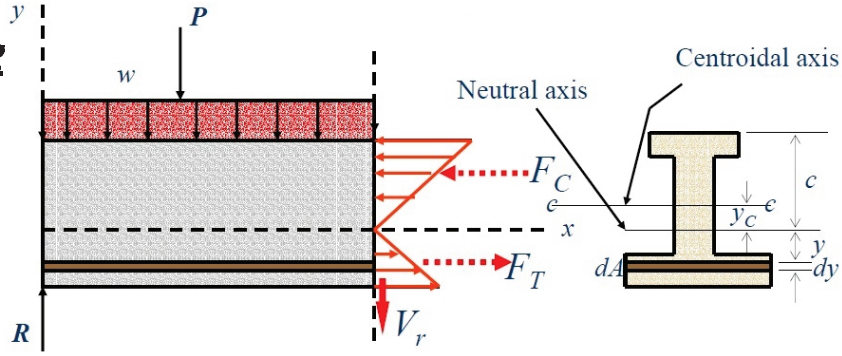
❑ ***Stress Due to Bending***



$$-\frac{E}{\rho}\int_A y \cdot dA = 0$$

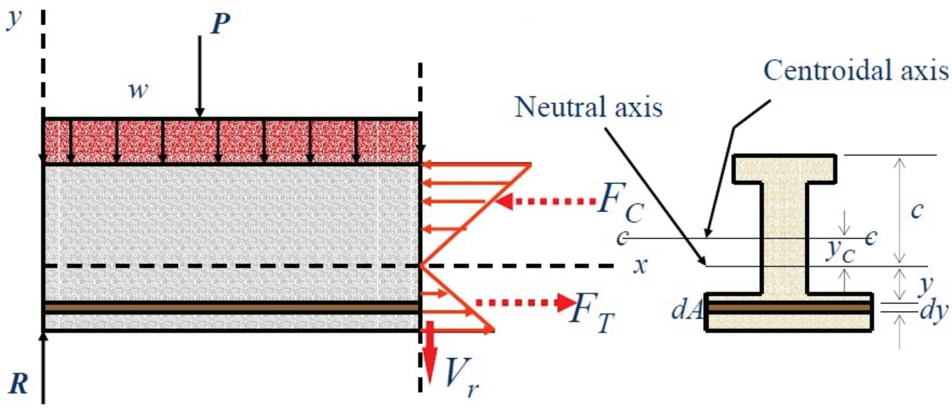
$$\Rightarrow \quad -\frac{EA}{\rho} y_c = 0 \quad \Rightarrow \quad y_c = 0$$

$$y_c \cdot A = \int_A y \cdot dA$$

For flexural loading and linearly elastic action, the neutral axis passes through the centroid of the cross section of the beam

Pure Bending

❑ ***Stress Due to Bending***



$$M = -\int_{area} y\sigma_x \, dA$$

$$\Rightarrow \quad M = -\int_{area} y(\frac{y}{c}\sigma_m)\, dA = -\frac{\sigma_m}{c}\int_{area} y^2 dA$$

$$\sigma_x = \frac{y}{c}\sigma_m$$

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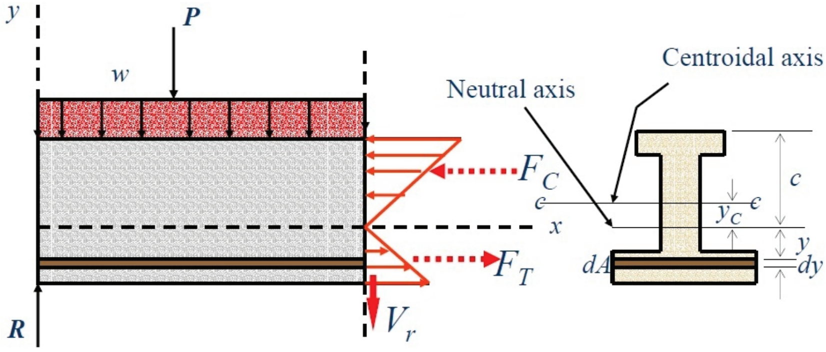
❑ ***Stress Due to Bending***



$$M = -\frac{\sigma_m}{c} \int_{area} y^2 dA$$

$$\Rightarrow \quad M = -\frac{\sigma_m}{c} I \quad \Rightarrow \quad \sigma_m = -\frac{M}{I} c$$

$$I = \int_{area} y^2 dA$$

I = Second moment of area

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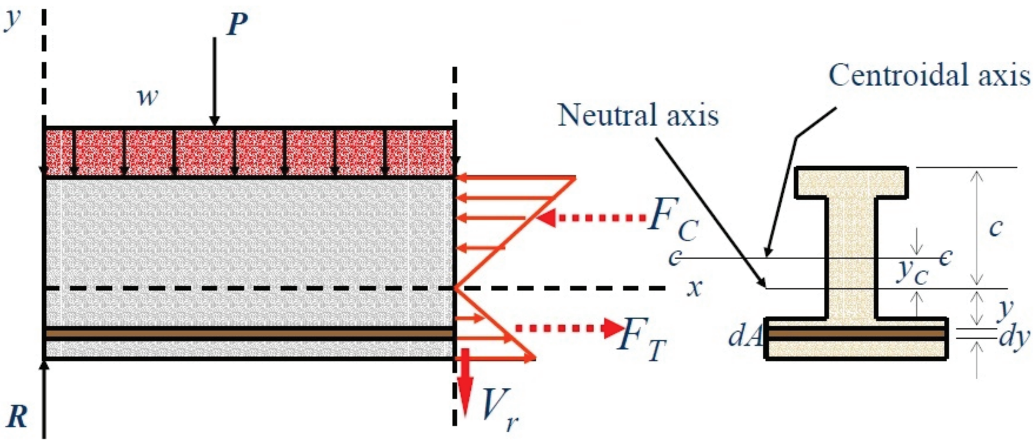
❑ ***Stress Due to Bending***



$$\sigma_m = -\frac{M}{I} c \qquad \sigma_x = -\frac{M}{I} y$$

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❑ ***Beam Section Properties***

The maximum normal stress due to bending

$$\sigma_m = \frac{Mc}{I} = \frac{M}{S}$$

$I =$ section moment of inertia

$S = \dfrac{I}{c} =$ section modulus

A beam section with a larger section modulus will have a lower maximum stress

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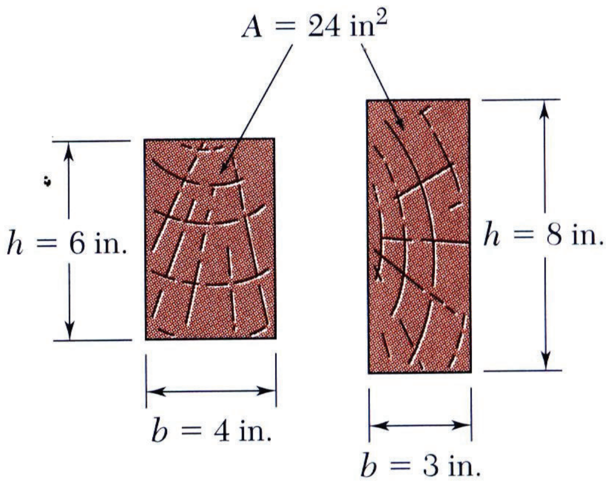
❑ ***Beam Section Properties***

- Consider a rectangular beam cross section,



$$S = \frac{I}{c} = \frac{\frac{1}{12}bh^3}{h/2} = \frac{1}{6}bh^2 = \frac{1}{6}Ah$$

Between two beams with the same cross sectional area, the beam with the greater depth will be more effective in resisting bending.

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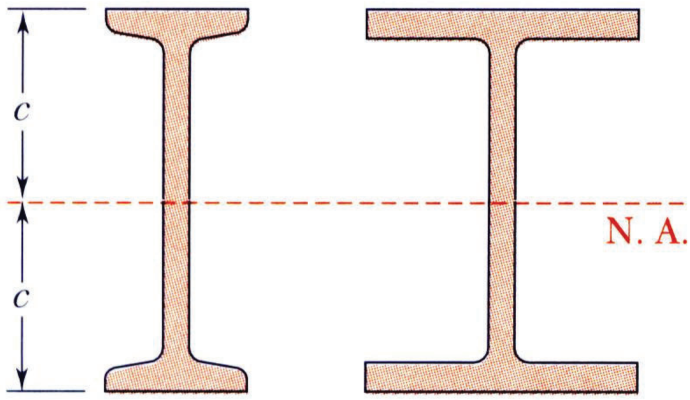
❑ ***Beam Section Properties***

- Structural steel beams are designed to have a large section modulus.

$$S \propto A \cdot h$$



(*a*) S-beam (*b*) W-beam

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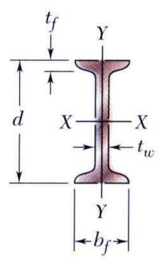
❑ ***Properties of American Standard Shapes***

Appendix C. **Properties of Rolled-Steel Shapes** (SI Units)

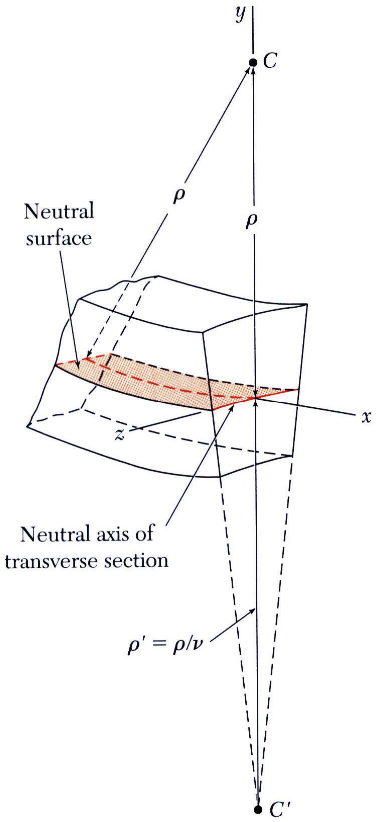
S Shapes
(American Standard Shapes)



Designation†	Area A, mm²	Depth d, mm	Flange Width b_f, mm	Flange Thickness t_f, mm	Web Thickness t_w, mm	Axis X-X I_x 10⁶ mm⁴	Axis X-X S_x 10³ mm³	Axis X-X r_x mm	Axis Y-Y I_y 10⁶ mm⁴	Axis Y-Y S_y 10³ mm³	Axis Y-Y r_y mm
S610 × 180	22900	622	204	27.7	20.3	1320	4240	240	34.9	341	39.0
158	20100	622	200	27.7	15.7	1230	3950	247	32.5	321	39.9
149	19000	610	184	22.1	18.9	995	3260	229	20.2	215	32.3
134	17100	610	181	22.1	15.9	938	3080	234	19.0	206	33.0
119	15200	610	178	22.1	12.7	878	2880	240	17.9	198	34.0
S510 × 143	18200	516	183	23.4	20.3	700	2710	196	21.3	228	33.9
128	16400	516	179	23.4	16.8	658	2550	200	19.7	216	34.4
112	14200	508	162	20.2	16.1	530	2090	193	12.6	152	29.5
98.3	12500	508	159	20.2	12.8	495	1950	199	11.8	145	30.4
S460 × 104	13300	457	159	17.6	18.1	385	1685	170	10.4	127	27.5
81.4	10400	457	152	17.6	11.7	333	1460	179	8.83	113	28.8
S380 × 74	9500	381	143	15.6	14.0	201	1060	145	6.65	90.8	26.1
64	8150	381	140	15.8	10.4	185	971	151	6.15	85.7	27.1

Pure Bending

❑ ***Deformations in a Transverse Cross Section***



- Deformation due to bending moment *M* is quantified by the curvature of the neutral surface

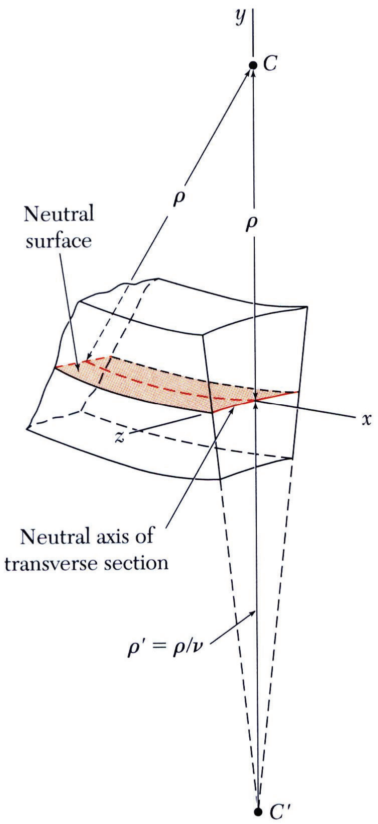
$$\frac{1}{\rho}=\frac{\varepsilon_m}{c}=\frac{\sigma_m}{Ec}=\frac{1}{Ec}\frac{Mc}{I}\Rightarrow\boxed{\frac{1}{\rho}=\frac{M}{EI}}$$

- Although cross sectional planes remain planar when subjected to bending moments, in-plane deformations are nonzero,

$$\varepsilon_y=-\nu\varepsilon_x=\frac{\nu y}{\rho}\qquad\varepsilon_z=-\nu\varepsilon_x=\frac{\nu y}{\rho}$$

Pure Bending

❑ ***Deformations in a Transverse Cross Section***



- Expansion above the neutral surface and contraction below it cause an in-plane curvature,

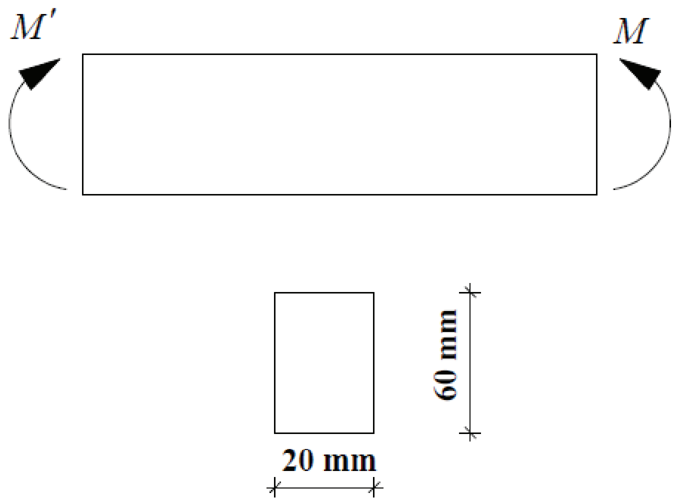
$$\frac{1}{\rho'}=\frac{\nu}{\rho}=\text{anticlastic curvature}$$

Pure Bending

Example 1

A stee1 bar with rectangular cross section is subjected to two equal and opposite couples acting in the vertical plane of symmetry of the bar. Determine the value of the bending moment M that causes the bar to yield. Assume

$$\sigma_Y = 250\ Mpa$$



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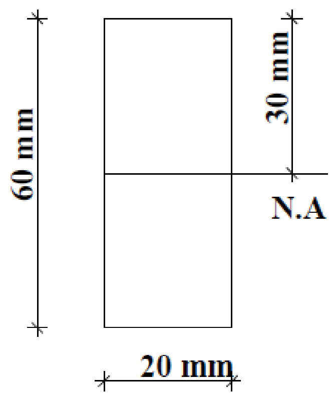
Example 1

Since the neutral axis must pass through the centroid C of the cross section, we have c = 30 mm.



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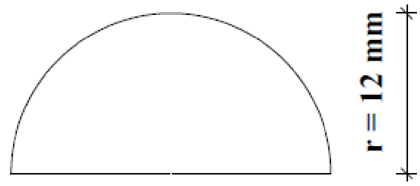
Example 2

An aluminum rod with a semicircular cross section of radius r=12 mm is bent into the shape of a circular arc of mean radius $\rho = 2.5\ m$. Knowing that the flat face of the rod is turned toward the center of curvature of the arc, determine the maximum tensile and compressive stress in the rod. Use E = 70 GPa.



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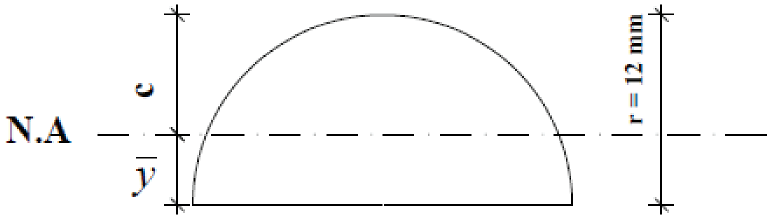
Example 2



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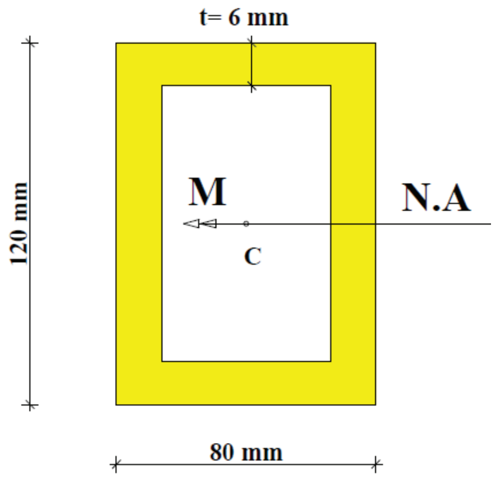
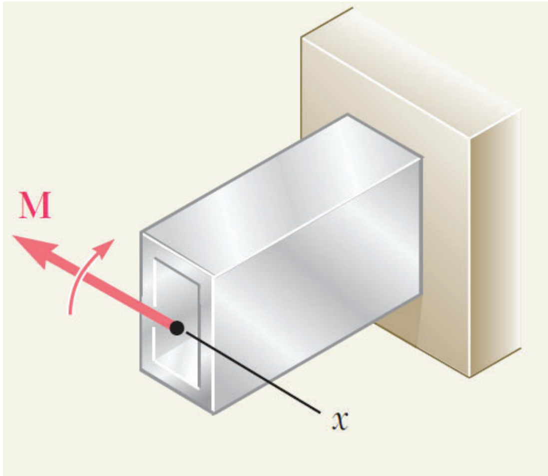
Example 3





The rectangular tube shown is extruded from an aluminum alloy for which $\sigma_y = 275 Mpa$, $\sigma_U = 415 Mpa$ and E = 73 GPa.

Neglecting the effect of fillets, determine (a) the bending moment M for which the factor of safety will be 3.00, (b) the corresponding radius of curvature of the tube.

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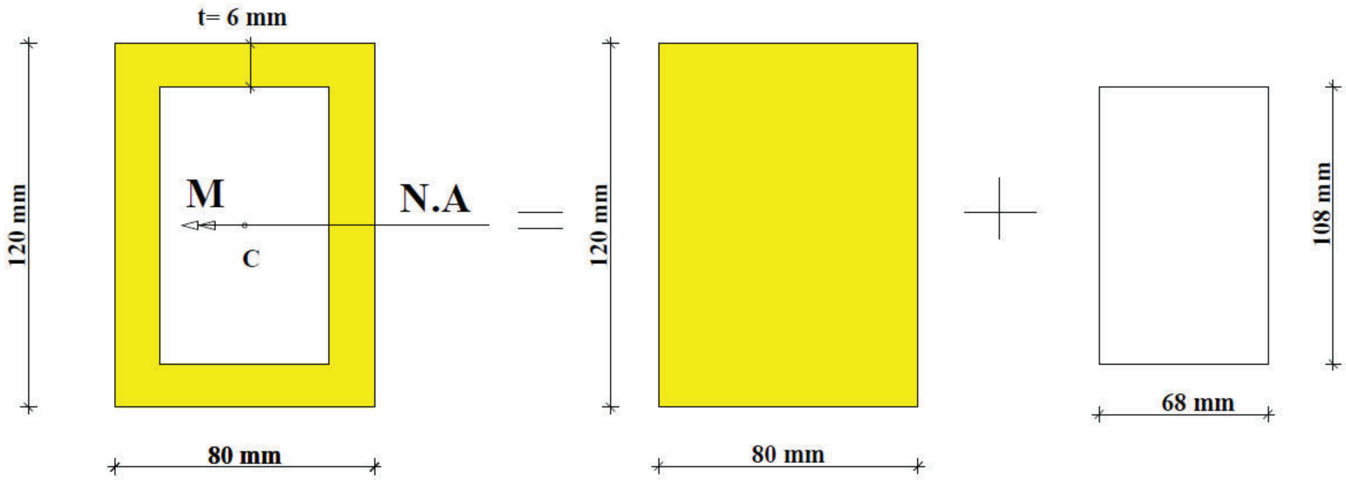
Example 3



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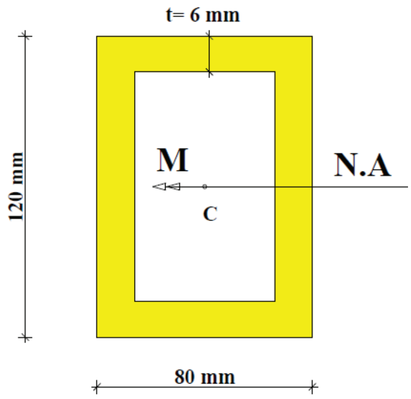
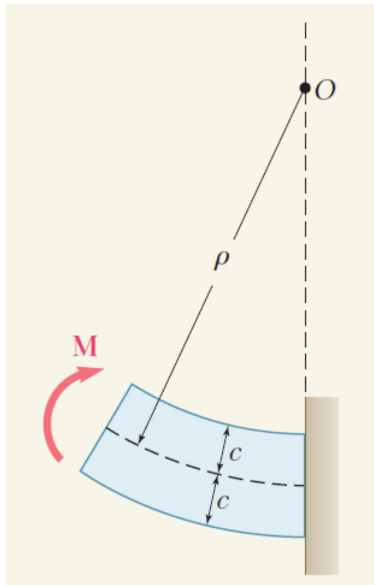
Example 3





Alternative solution

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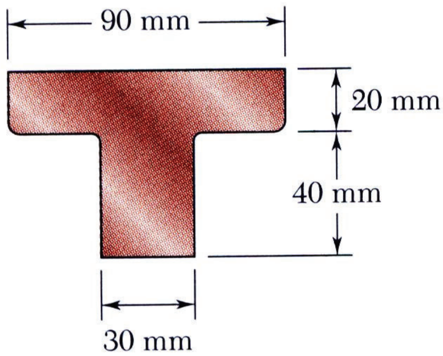
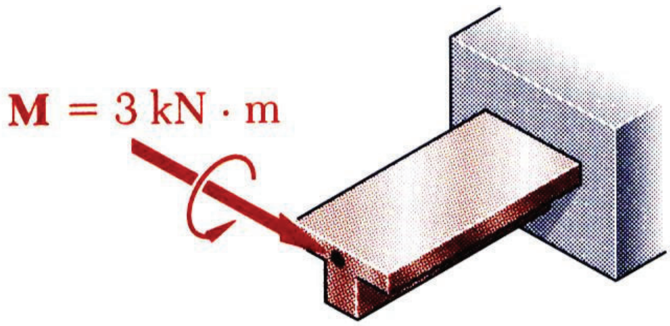
Example 4

A cast-iron machine part is acted upon by a 3 kN-m couple. Knowing $E = 165$ GPa and neglecting the effects of fillets, determine (a) the maximum tensile and compressive stresses, (b) the radius of curvature.





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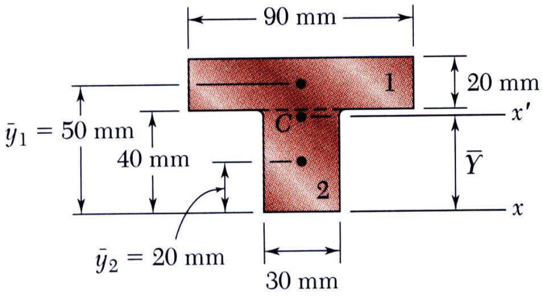
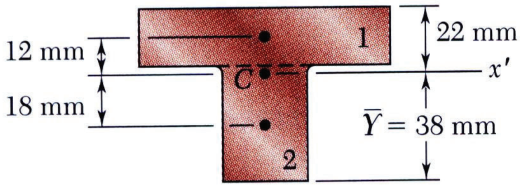
Example 4





SOLUTION:

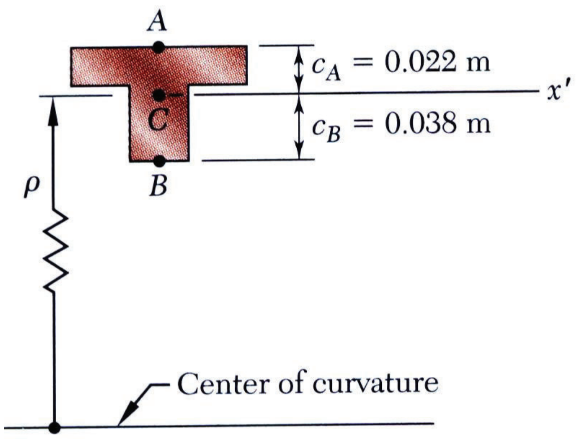
Based on the cross section geometry, calculate the location of the section centroid and moment of inertia.

	Area, mm^2	$\bar{y}$, mm	$\bar{y}A$, mm^3
1			
2			

Pure Bending

Example 4

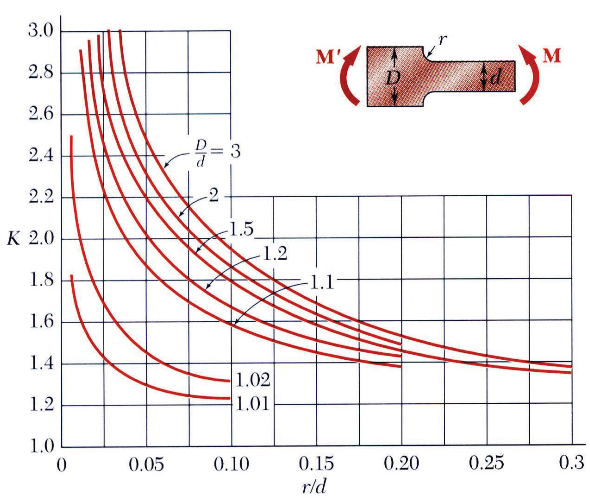
- Apply the elastic flexural formula to find the maximum tensile and compressive stresses.

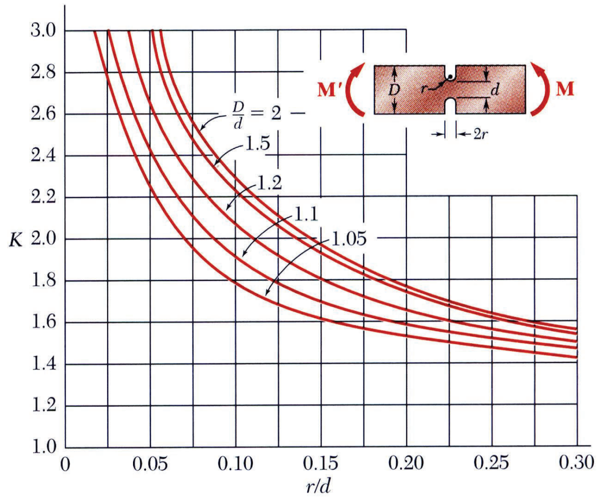


- Calculate the curvature

Pure Bending

❑ ***Stress Concentrations***





Stress concentrations may occur:

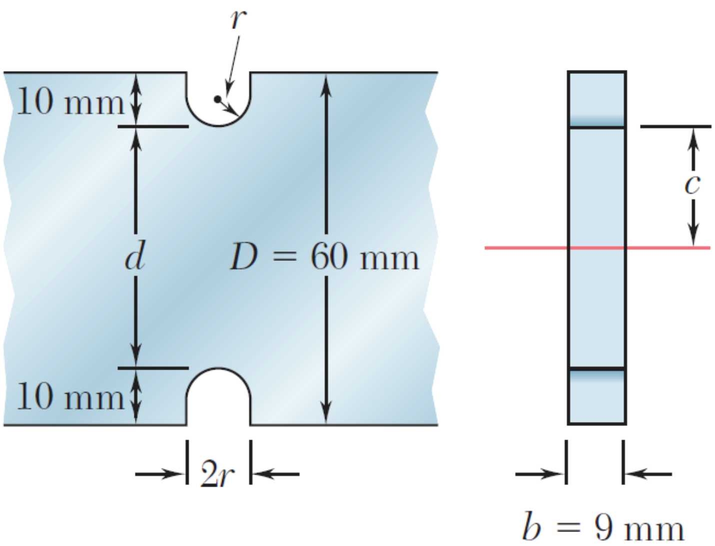
- in the vicinity of points where the loads are applied
- in the vicinity of abrupt changes in cross section

$$\sigma_m = K\frac{Mc}{I}$$

Pure Bending

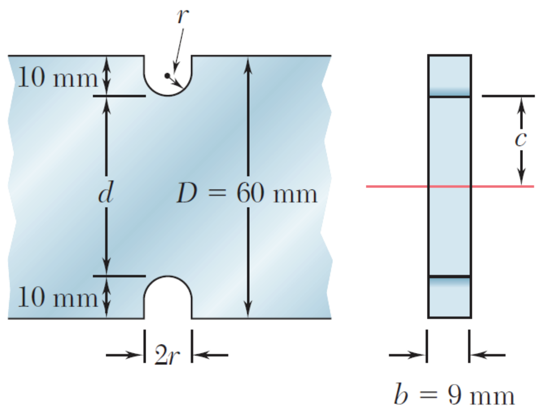
Example 5

Grooves 10 mm deep are to be cut in a steel bar which is 60 mm wide and 9 mm thick . Determine the smallest allowable width of the grooves if the stress in the bar is not to exceed 150 MPa when the bending moment is equal to l80 N·m.



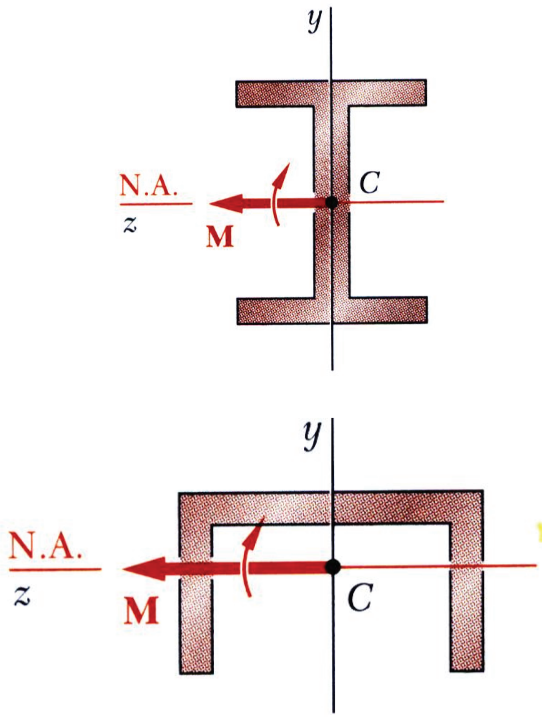
Pure Bending

Example 5



Pure Bending

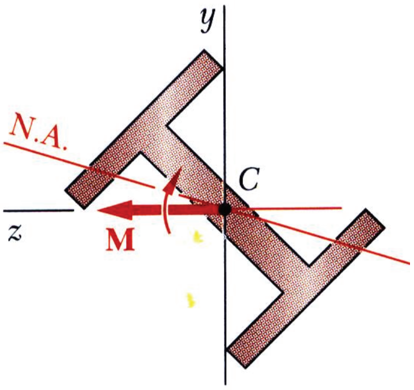
❑ ***Symmetric Bending***



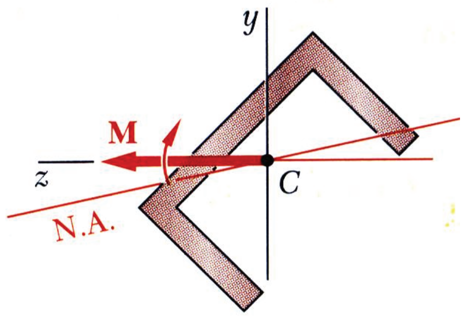
- Analysis of pure bending has been limited to members subjected to bending couples acting in a plane of symmetry.
- Members remain symmetric and bend in the plane of symmetry.
- The neutral axis of the cross section coincides with the axis of the couple

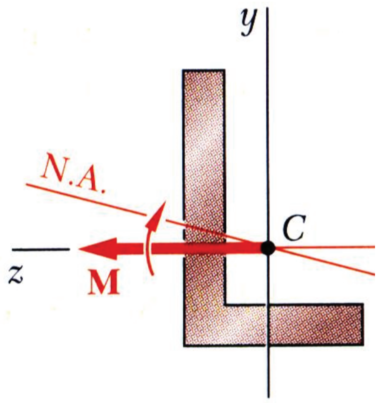
Pure Bending

❑ ***Unsymmetric Bending***



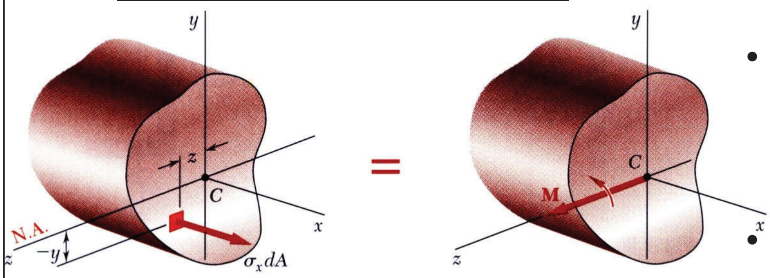
- Will now consider situations in which the bending couples do not act in a plane of symmetry.
- Cannot assume that the member will bend in the plane of the couples.
- In general, the neutral axis of the section will not coincide with the axis of the couple.





Pure Bending

❑ ***Unsymmetric Bending***



Wish to determine the conditions under which the neutral axis of a cross section of arbitrary shape coincides with the axis of the couple as shown.

- The resultant force and moment from the distribution of elementary forces in the section must satisfy

$$F_x = M_y = 0 \quad M_z = M = applied\ couple$$

- $F_x = \int \sigma_x dA = 0 \quad \Rightarrow \quad \boxed{\int y\, dA = 0}$

neutral axis passes through centroid

- $M_z = -\int y \sigma_x dA = M \quad \Rightarrow \quad \boxed{\sigma_x = -\frac{My}{I}}$

defines stress distribution

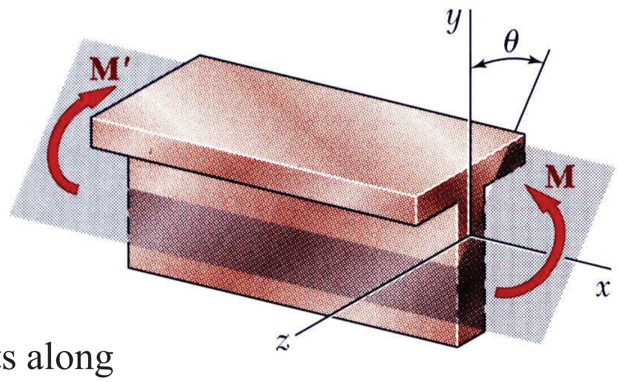
- $M_y = \int z \sigma_x dA = 0 \quad \Rightarrow \quad \int z\left(-\frac{y}{c}\sigma_m\right) dA$

$\Rightarrow \quad \boxed{\int yz\, dA = I_{yz} = product\ of\ inertia = 0}$

If one of y or z axis is principle then I_{yz}=0, so couple vector must be directed along a principal centroidal axis

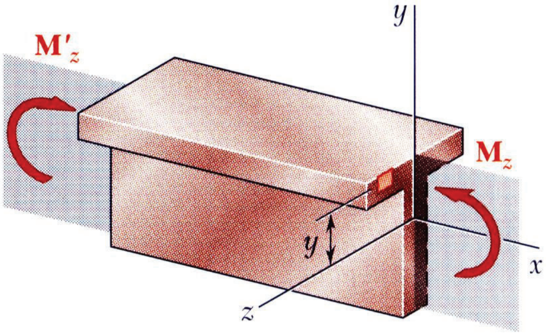
Pure Bending

❑ *Unsymmetric Bending*

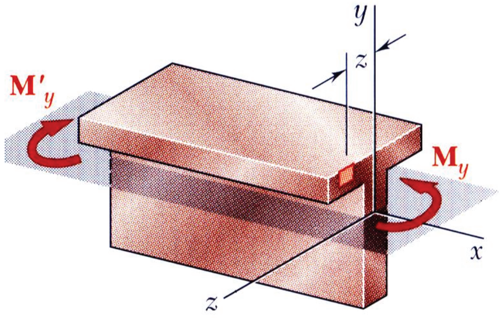


Superposition is applied to determine stresses in the most general case of unsymmetric bending.

- Resolve the couple vector into components along the principle centroidal axes.



$$M_z = M\cos\theta$$



$$M_y = M\sin\theta$$

Pure Bending

❑ *Unsymmetric Bending*

- **Superpose the component stress distributions**

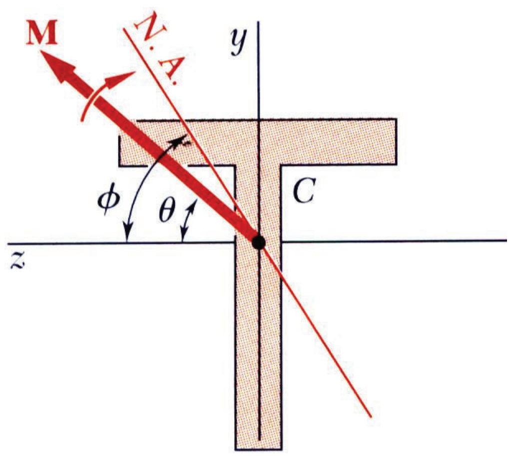
$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

- **Along the neutral axis,**

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y} = 0 \quad \Rightarrow$$

$$-\frac{(M\cos\theta)y}{I_z} + \frac{(M\sin\theta)z}{I_y} = 0$$

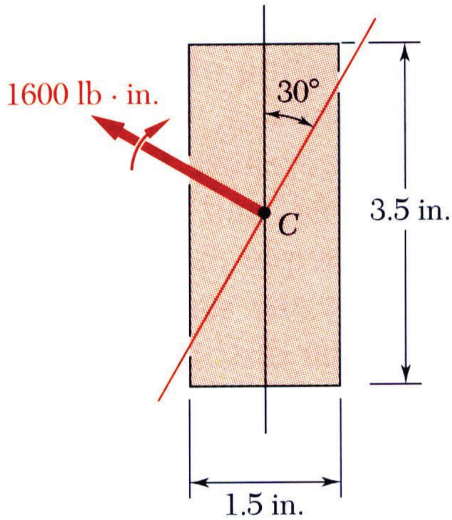
$$\Rightarrow \tan\phi = \frac{y}{z} = \frac{I_z}{I_y}\tan\theta$$



Pure Bending

Example 6

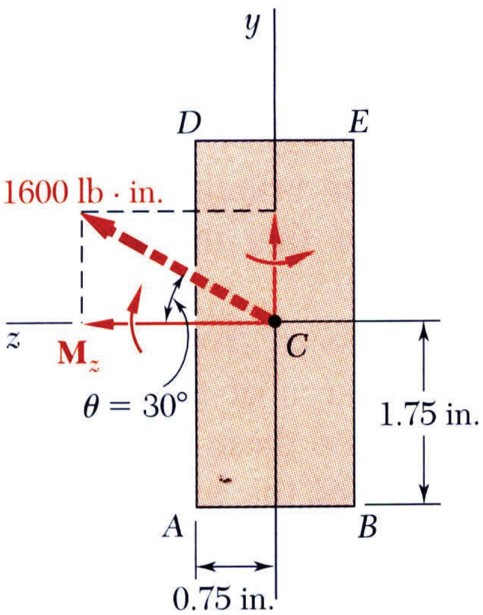
A 1600 lb-in couple is applied to a rectangular wooden beam in a plane forming an angle of 30 deg. with the vertical. Determine (a) the maximum stress in the beam, (b) the angle that the neutral axis forms with the horizontal plane.



Pure Bending

Example 6

- Resolve the couple vector into components and calculate the corresponding maximum stresses.

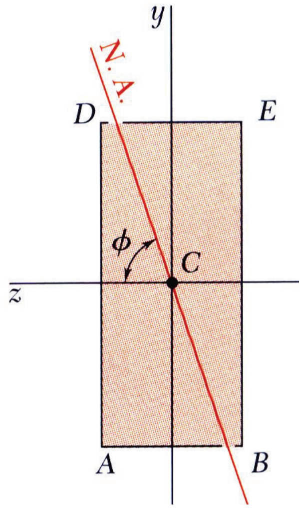


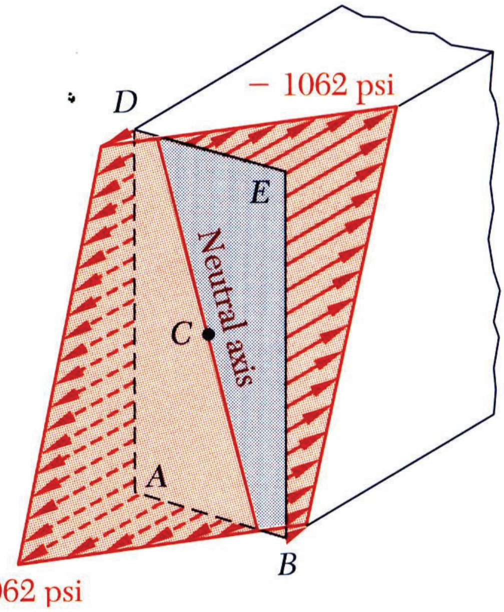
Pure Bending

Example 6

- The largest tensile stress due to the combined loading occurs at *A*.

- Determine the angle of the neutral axis.

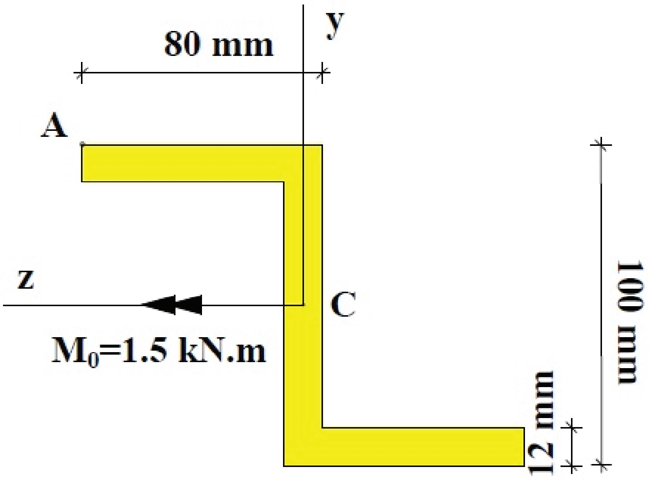




Pure Bending

Example 7

A couple of magnitude $M_0 = 1.5\ kN.m$ acting in a vertical plane is applied to a beam having the Z-shaped cross section shown. Determine (a) the stress at point A, (b) the angle that the neutral axis forms with the horizontal plane. The moments and product of inertia of the section with respect to the y and z axes have been computed and are as follows:



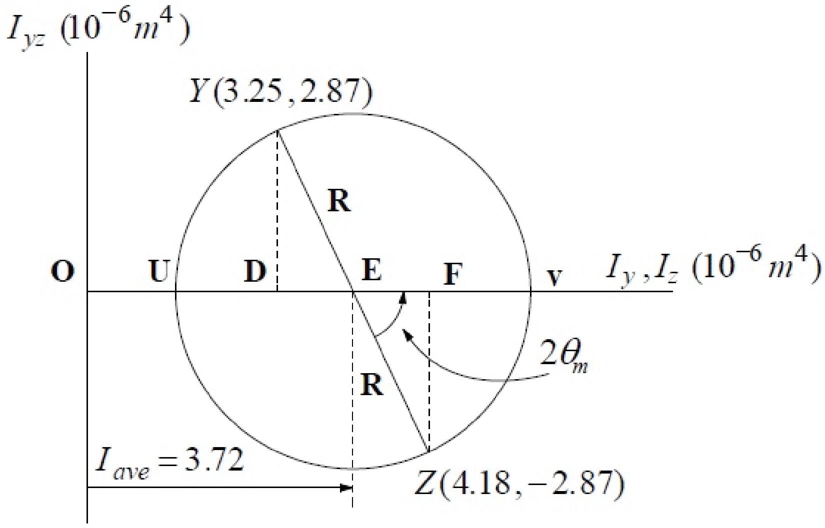
$$I_y = 3.25\times10^{-6} m^4$$

$$I_z = 4.18\times10^{-6} m^4$$

$$I_{yz} = 2.87\times10^{-6} m^4$$

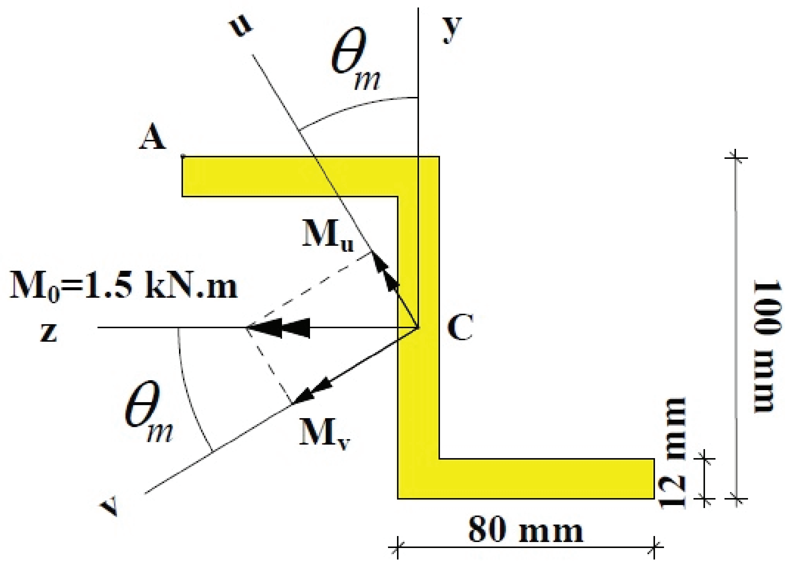
Pure Bending

Example 7



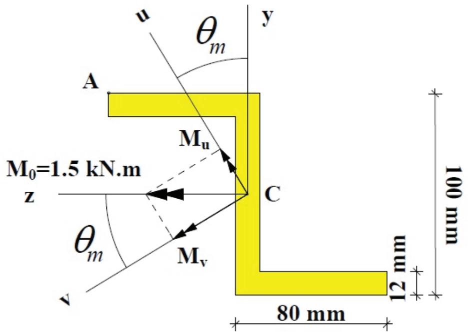
Pure Bending

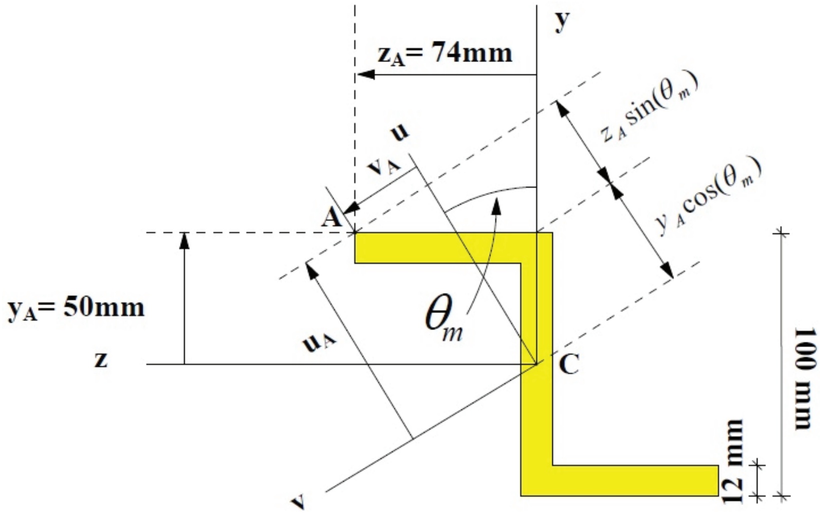
Example 7



Pure Bending

Example 2





Pure Bending

Example 7