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Complex Networks

Network Models

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Network Model

- A **network model**:
  an algorithm which **generates artificial networks**
- It generates artificial graphs which are **similar to real-world networks**
- How a graph becomes similar to real networks?
  - **Small-worlds**, **transitivity**, **long-tail degree distribution**, **community structure**, …
- How to generate a network that conforms to such properties?
  - Network models try to answer that question

Network Models

- Terminology:
  - Network model
  - Network generation method
  - Generative model
  - Random graph generation model
- Examples:
  - Erdős–Rényi (ER) model: random networks
  - Watts–Strogatz (WS) model: small-world networks
  - Barabási–Albert model: scale-free neworks
  - Many other models (a research topic)
    - How efficient? How similar to real networks? How tunable/adaptive?

Why Network Models?

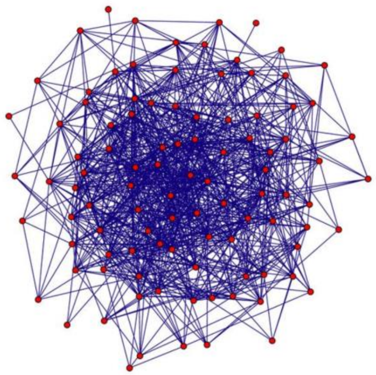
- Uncover/explain the generative mechanisms underlying networks
  - Models can uncover the hidden reality of networks
  - Reveal the processes which results in real-world networks
- Predict the future
- They may simulate real networks:
  - When we want to study the properties/dynamics of networks
  - When we have no access to real-world networks
  - When it is not safe to publish a network dataset
  - And many other applications

Why Network Models? (cont'd)

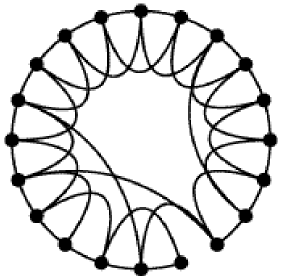
- Network structure
  - The parameters give us insight into the global structure of the network itself.
- Simulations
  - Given an algorithm working on a graph we would like to evaluate how its performance depends on various properties of the network.
- Extrapolations & Sampling
  - We can use the model to generate a larger/smaller graph.
- Graph similarity
  - To compare the similarity of the structure of different networks (even of different sizes) one can use the differences in estimated parameters as a similarity measure.
- Graph compression
  - We can compress the graph, by storing just the model parameters.

Basic Network Models

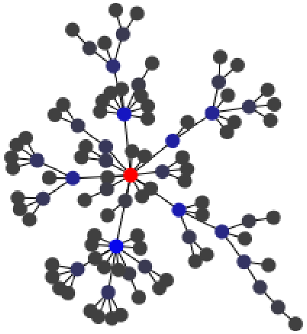
Random graph model
(Erdős and Rényi, 1959)



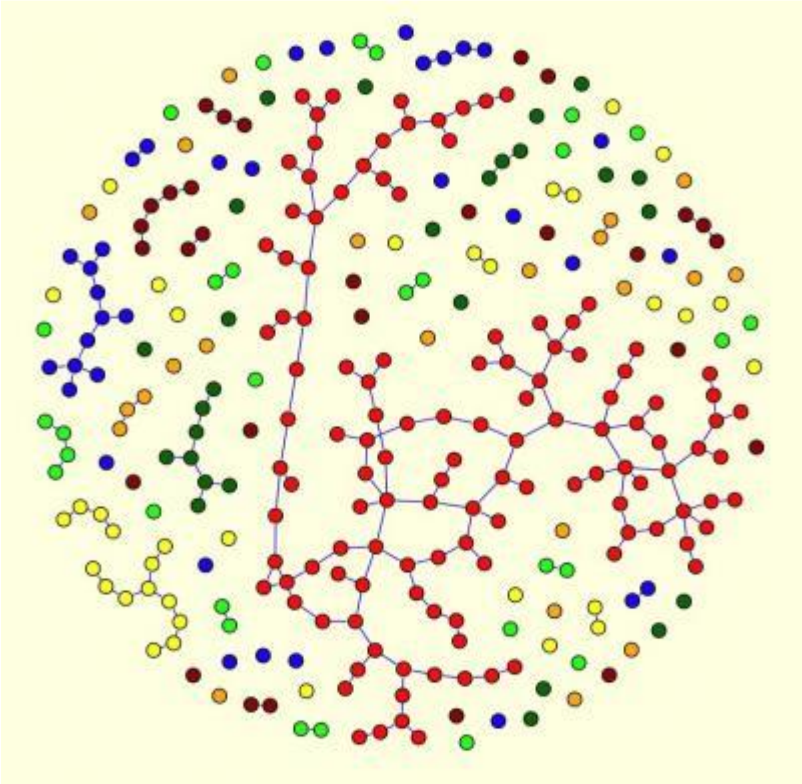
"Small world" model
(Watts & Strogatz, 1998)

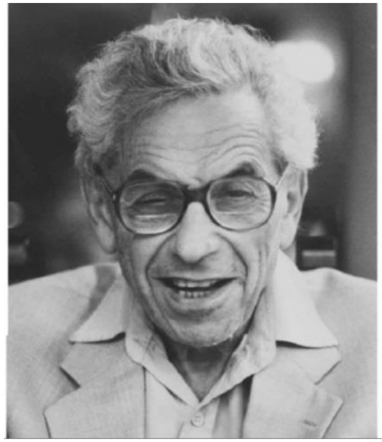


Preferential attachement model
(Barabasi & Albert, 1999)



Erdos- Renyi Random graph model





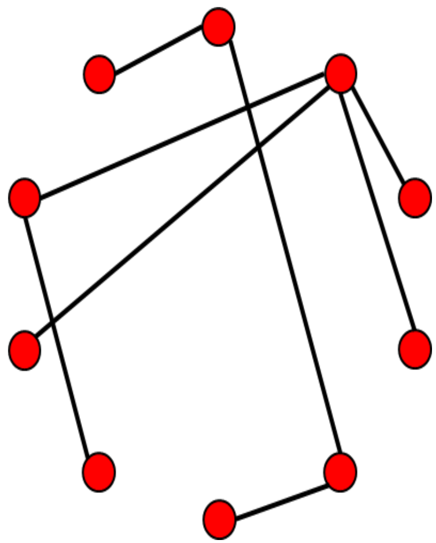
Pál Erdös
(1913-1996)

Alfréd Rényi
(1921-1970)

G_{np}

Random Network Model

- Definition: A random graph is a graph of ***N*** nodes where each pair of nodes is connected by probability ***p***.   G(N,p)



Erdös-Rényi model (1959)

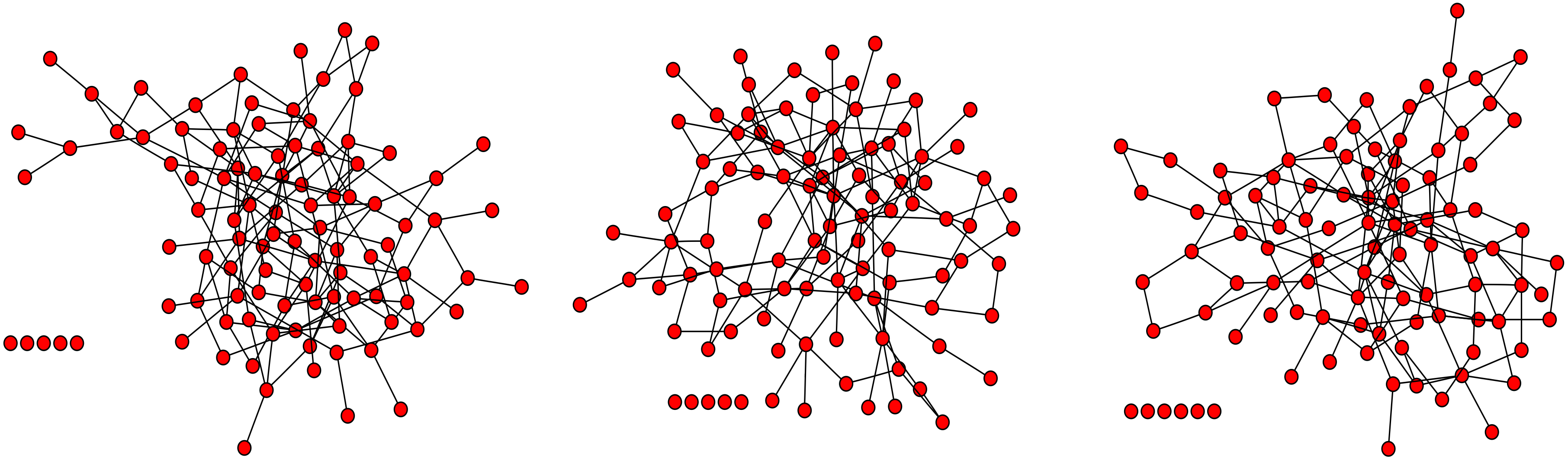
Connect with probability p

p=1/6
N=10

<k> ~ 1.5

Erdős–Rényi (ER) Model, Example:

p=0.03
N=100



Clustering coefficient

- Clustering coefficient is defined as the probability that two vertices with a common neighbor are connected themselves

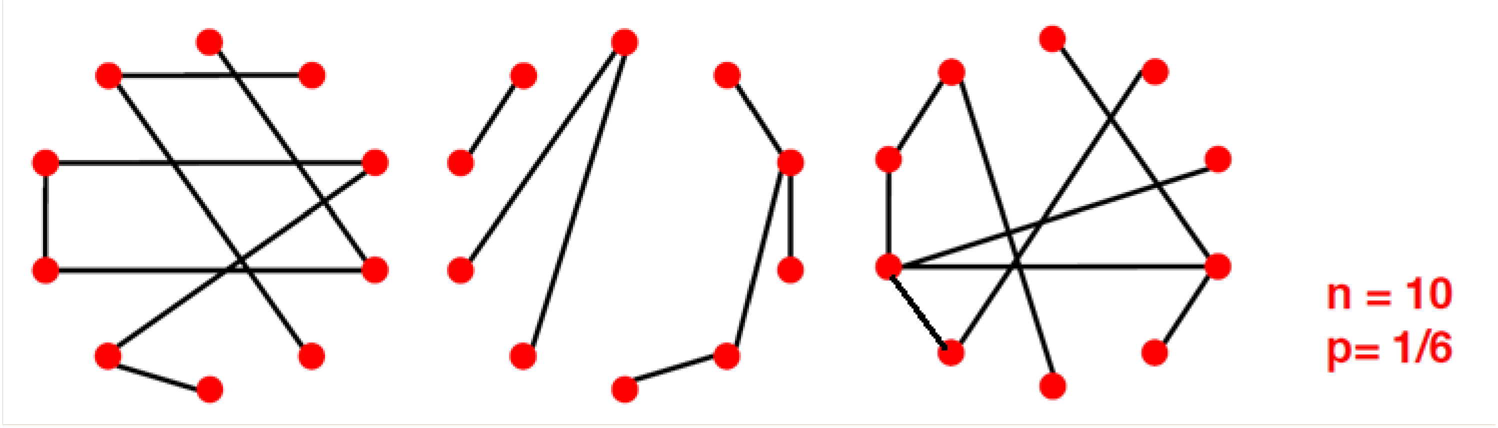
- In a random graph the probability that *any* two vertices are connected is equal to $p=\langle k \rangle/(n-1)$
  - Hence the clustering coefficient is also: $C=p=\langle k \rangle/(n-1)$

- Given that for large n, average degree is constant, it follows that the clustering coefficient goes to 0
  - This is a sharp difference between the G(n,p) model and real networks

The Number of Links is Variable

- ***n* and *p* do not uniquely determine the graph!**
  (The graph is a result of a random process)

- We can have many different realizations given the same *n* and *p*



number of edges=8 number of edges=7 number of edges=9

Number of Links in ER Networks

P(*L*): the probability to have exactly ***L*** links in a network of ***N*** nodes and probability ***p***:

The maximum number of links in a network of N nodes.

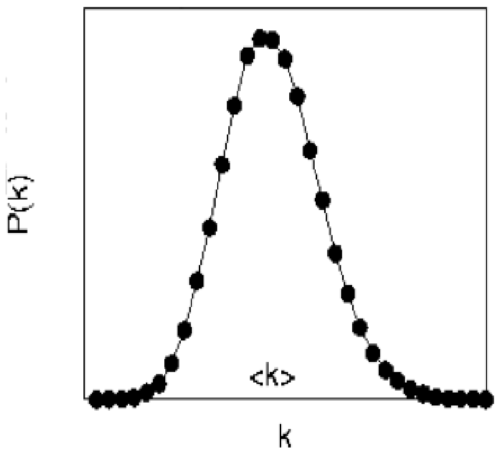
$$P(L) = \binom{\binom{N}{2}}{L} p^L (1-p)^{\frac{N(N-1)}{2}-L}$$

Number of different ways we can choose L links among all potential links.

$$P(x) = \binom{N}{x} p^x (1-p)^{N-x}$$

Binomial distribution...

Degree Distribution of Random Networks



The probability of having k links for a node? (Degree Probability Distribution)

$$P(k) = \binom{N-1}{k} p^k (1-p)^{(N-1)-k}$$

$$<k> = p(N-1)$$

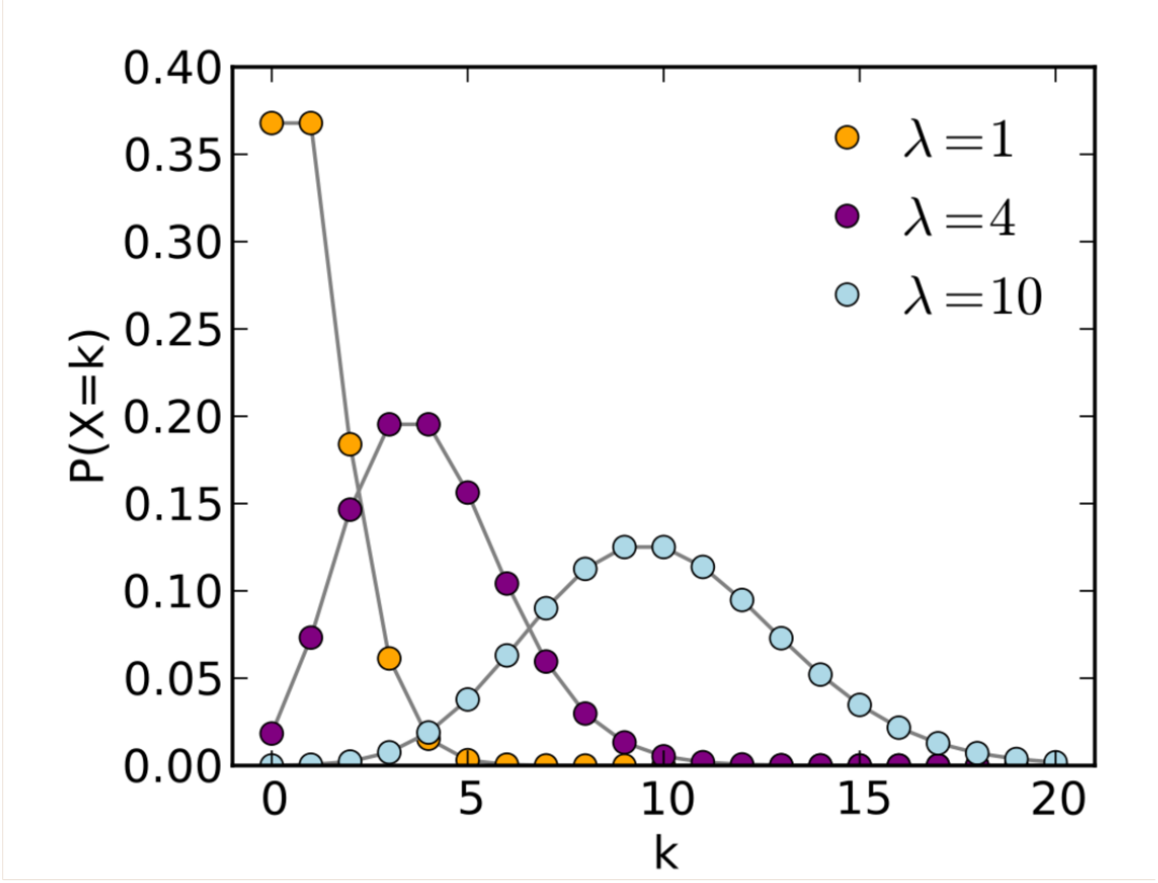
Makes sense

As the network size increases, the distribution becomes increasingly narrow—we are increasingly confident that the degree of a node is in the vicinity of <k>.

Degree Distribution of Random Networks

For large values of n, the degree distribution follows a Poisson distribution

$$p_k = e^{-\lambda} \frac{\lambda^k}{k!}$$



ER properties

- Binomial degree distribution: (biased coin experiment)

$$P(k) = \binom{N-1}{k} p^k (1-p)^{(N-1)-k}$$

- *P(L)*: the probability to have a network of exactly *L* links

$$P(L) = \binom{\binom{N}{2}}{L} p^L (1-p)^{\frac{N(N-1)}{2}-L}$$

- The average number of links <*L*> in a random graph

$$<L> = p\frac{N(N-1)}{2}$$

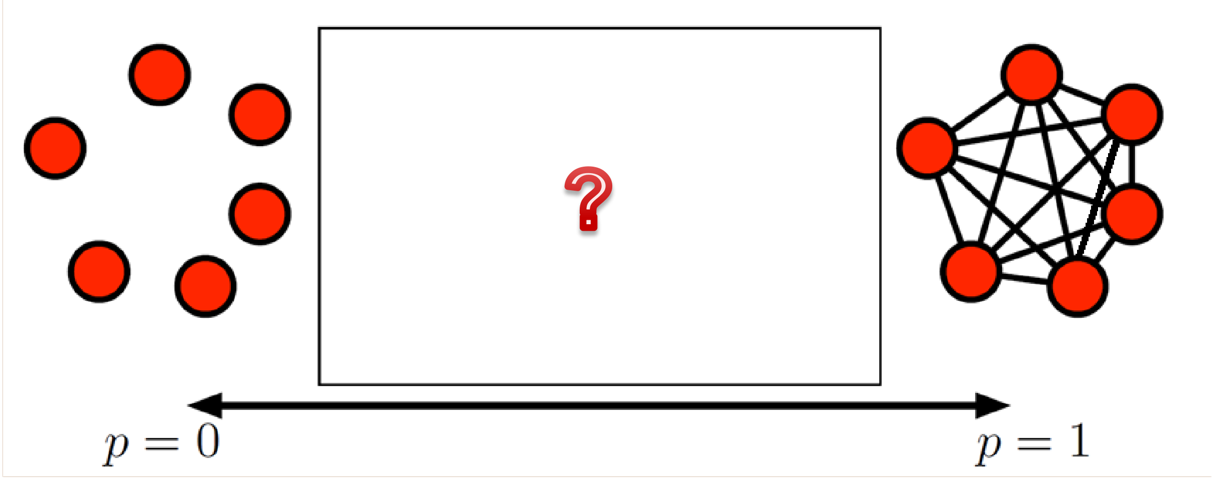
- The average degree:

$$<k> = 2L/N = p(N-1)$$

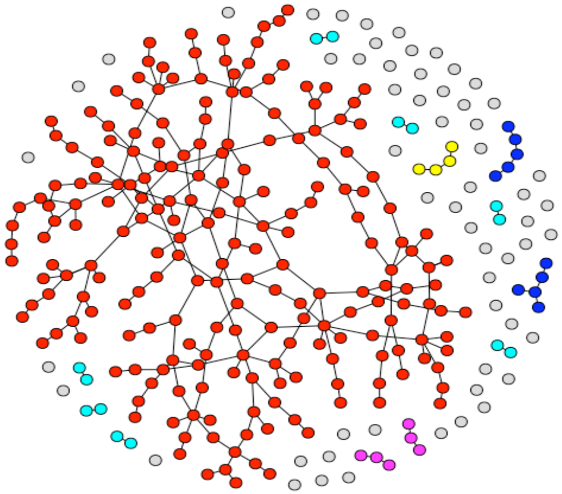
Giant component and Phase transition

- How many components exist in G(n,p) model
  - p=0 → Every node is isolated → Component size = 1 (independent of n)
  - p=1 → All nodes connected with each other → Component size = n (proportional to n)

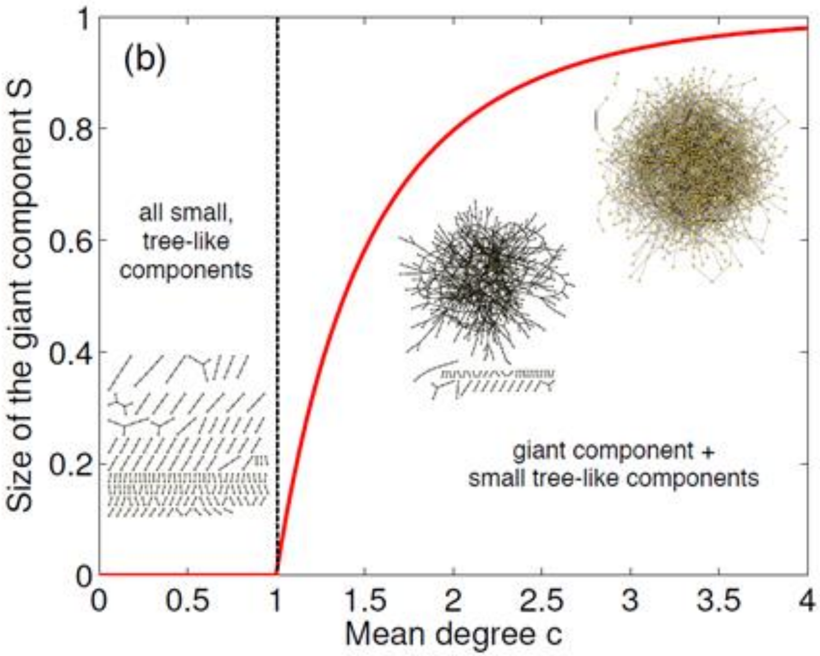
- It is interesting to examine <u>what happens for values of p in-between</u>
  - In particular, what happens to the largest component in the network as p increases?



Giant component and Phase transition



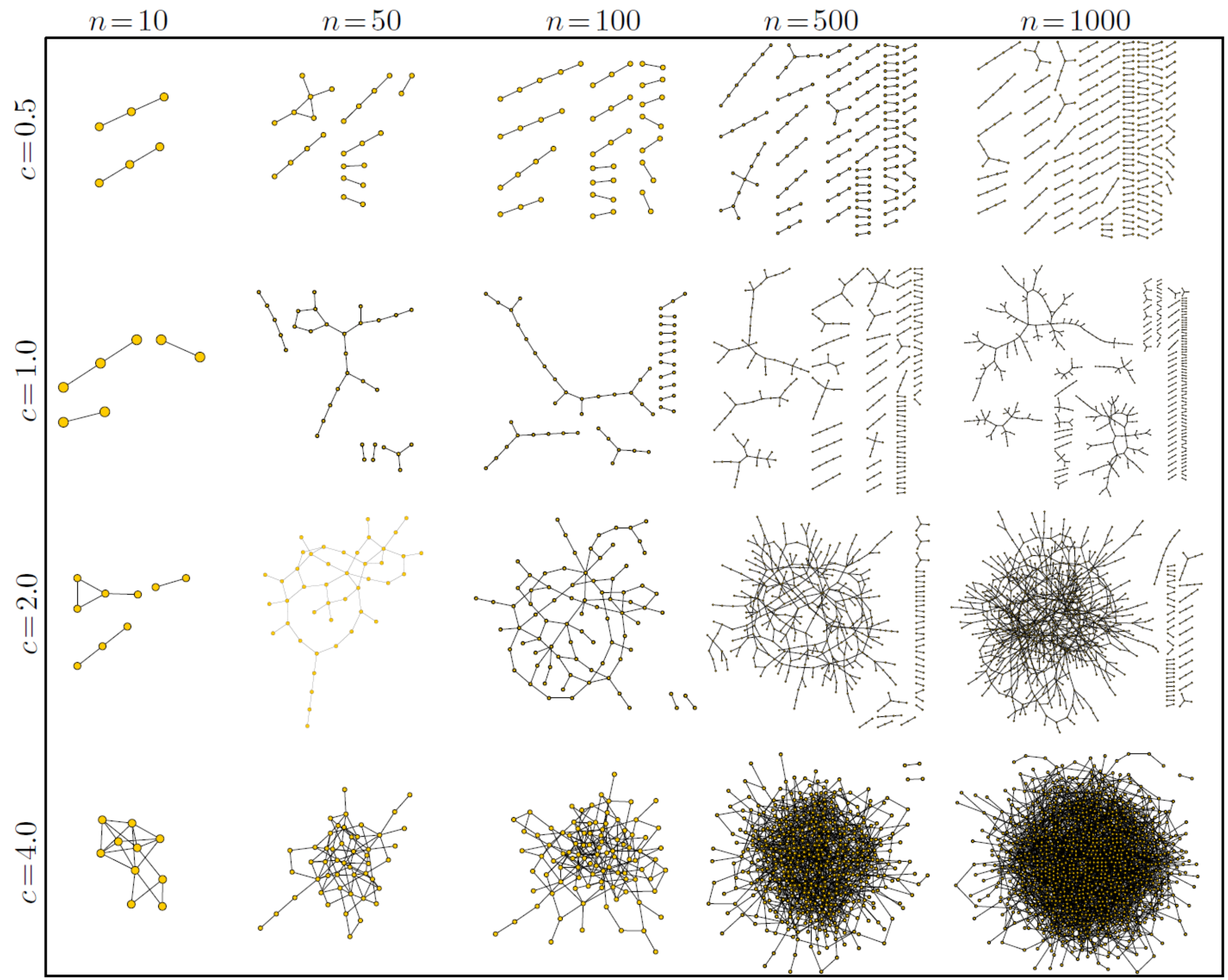
Fraction of nodes in the largest component



The size of the largest component undergoes a **sudden change**, or phase transition, from constant size to extensive size at one particular special value of p ($p_c = 1/n$)

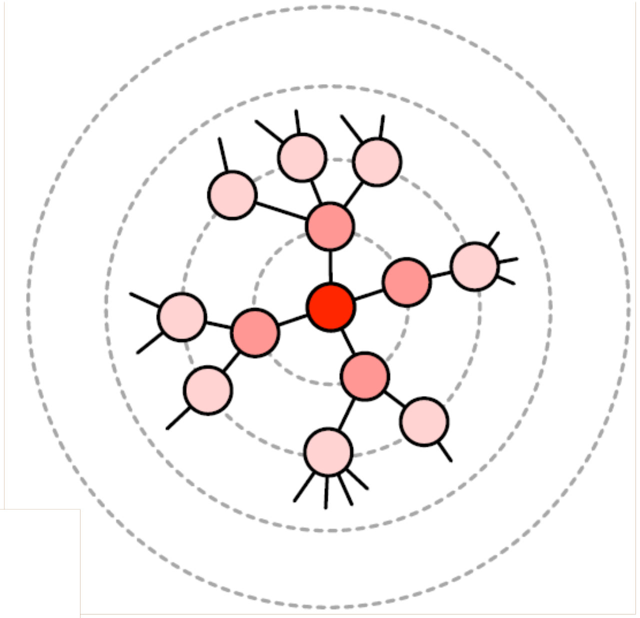
Phase transition in random graphs

What G(n, p) graphs look like?



Diameter of G(n, p) random graphs

Simple random graphs are locally tree-like (no loops; low clustering coefficient)



On average, the number of nodes ***D*** steps away from a node:

$$n = 1 + \langle k \rangle + \langle k \rangle^2 + ...\langle k \rangle^D = \frac{\langle k \rangle^{D+1} - 1}{\langle k \rangle - 1} \approx \langle k \rangle^D$$

in GCC, around p_c , $\langle k \rangle^D \sim n$,

$$D \sim \frac{\ln n}{\ln \langle k \rangle}$$

Random graph properties

- Poisson degree distribution
- Locally tree-like structure (very few triangles)
- Small diameters (small-world property)
- Sudden appearance of a giant component (Phase transition)

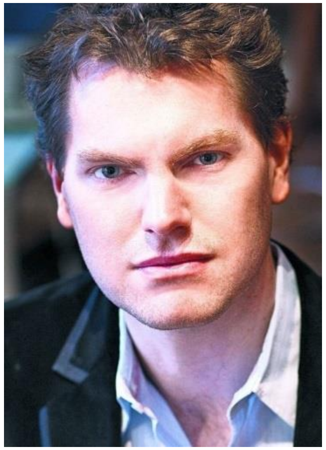
Network Properties of G(n, p)

- **Degree distribution:** $P(k) = \binom{n-1}{k} p^k (1-p)^{n-1-k}$
- **Path length:** O(log n)
- **Clustering coefficient:** C=p=<k>/(n-1)

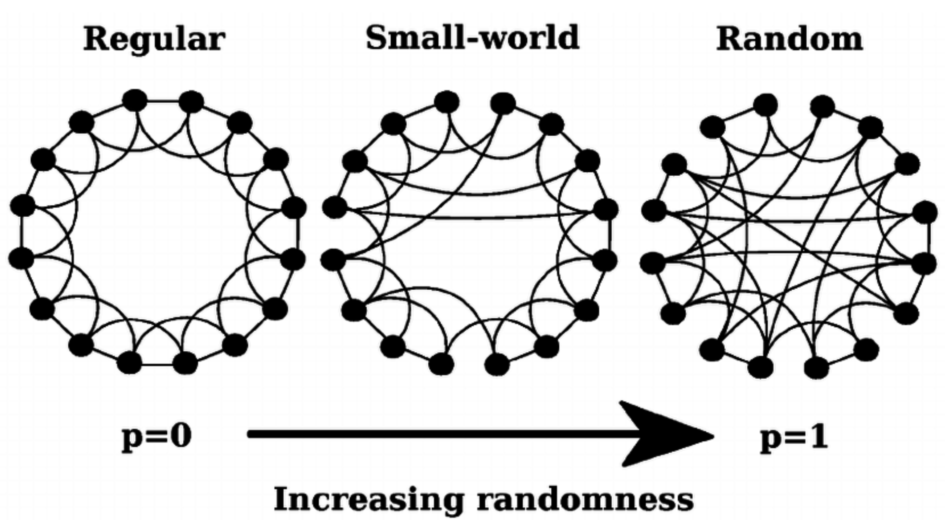
Does ER Represent Real Networks?

- It is a simple and old model
- Not compatible to many characteristics of real networks
  - No Transitivity
  - Degree distribution differs from real networks (Poisson vs. Long-tail)
  - No community structure
  - No Assortativity (No correlation between the degrees of adjacent vertices)
- However, random networks show **small-world-ness**

Small World Model

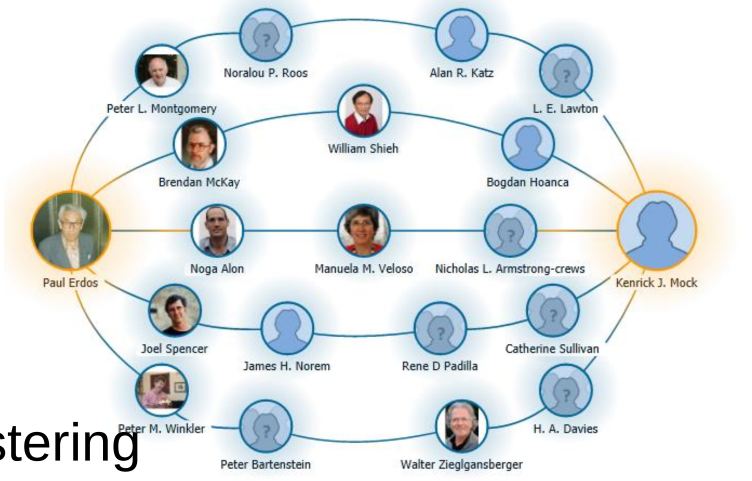
Duncan J. Watts 

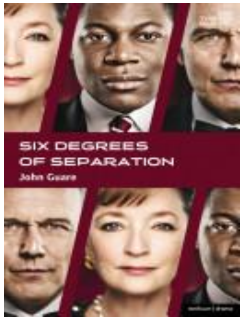
 Steven Strogatz



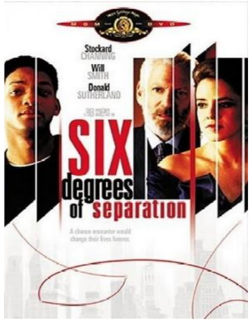
Small World Networks

- The World is Small. many evidences:
  - Milgram experiment
  - Six degrees of Kevin Bacon
  - Erdos number
  - Six degrees of separation
- The real networks also show high local clustering
  - A friend of my friend, is probably my friend

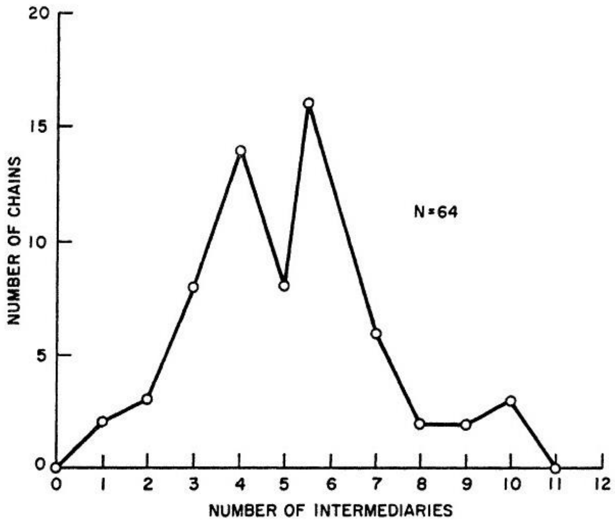




John Guare, 1990

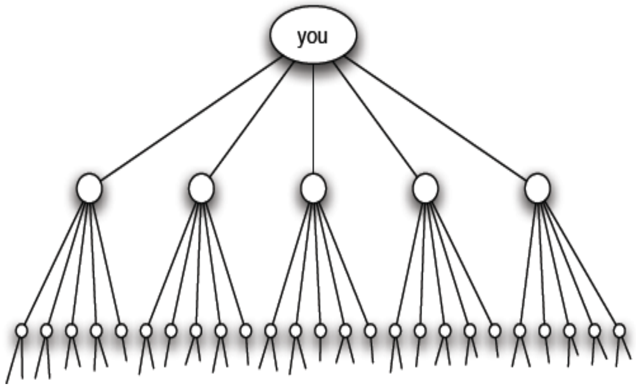


1993

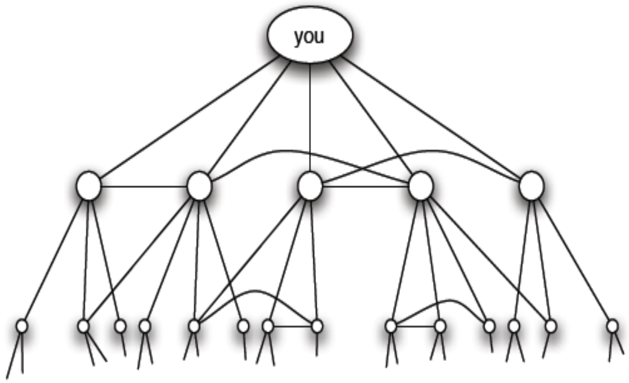


A Small-World

- **Consequence of expansion:**
  - Short paths: *O(log n)*
    This is the "best" we can do if the graph has constant degree and *n* nodes
  - Random graphs also result in short paths
- But networks have **local structure:**
  - Triadic closure:
    Friend of a friend is my friend
- **How can we have both?**



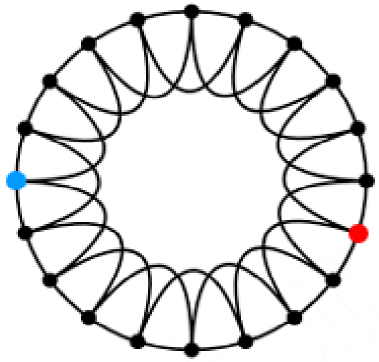
Pure exponential growth



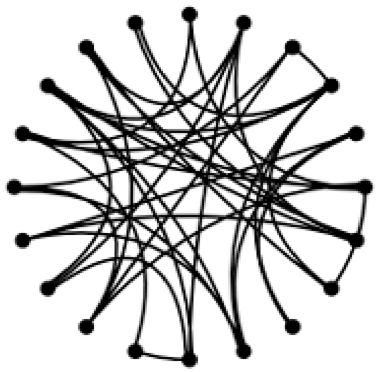
Triadic closure reduces growth rate

Small-World vs. Clustering

- Could a network with high clustering be at the same time a small world?
  - How can we at the same time have **high clustering** and **small diameter?**
  - Clustering implies edge "locality"
  - Randomness enables "shortcuts"



High clustering
High diameter



Low clustering
Low diameter

Clustering Implies Edge Locality

Data set	Avg. shortest path length (measured)	Avg. Shortest path length (random)	Clustering coefficient (measured)	Clustering coefficient (random)
Film actors (225,226 nodes, avg. degree k=61)	3.65	2.99	0.79	0.00027
Electrical power grid (4,941 nodes, k=2.67)	18.7	12.4	0.080	0.005
Network of neurons (282 nodes, k=14)	2.65	2.25	0.28	0.05
MSN (180 million edges, k=7)	6.6	...	0.114	0.00000008
Facebook (721 million, k=99)	4.7	...	0.14	...

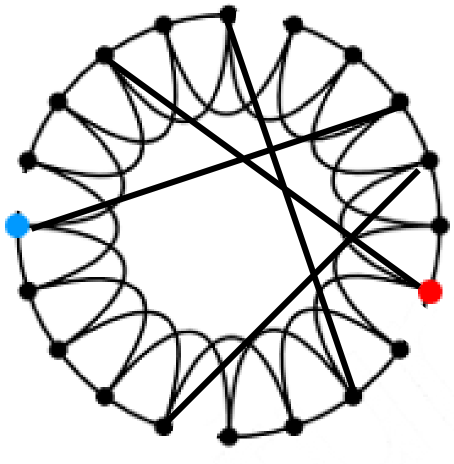
Real-world networks have high clustering and small diameter

Solution: The Small-World Model

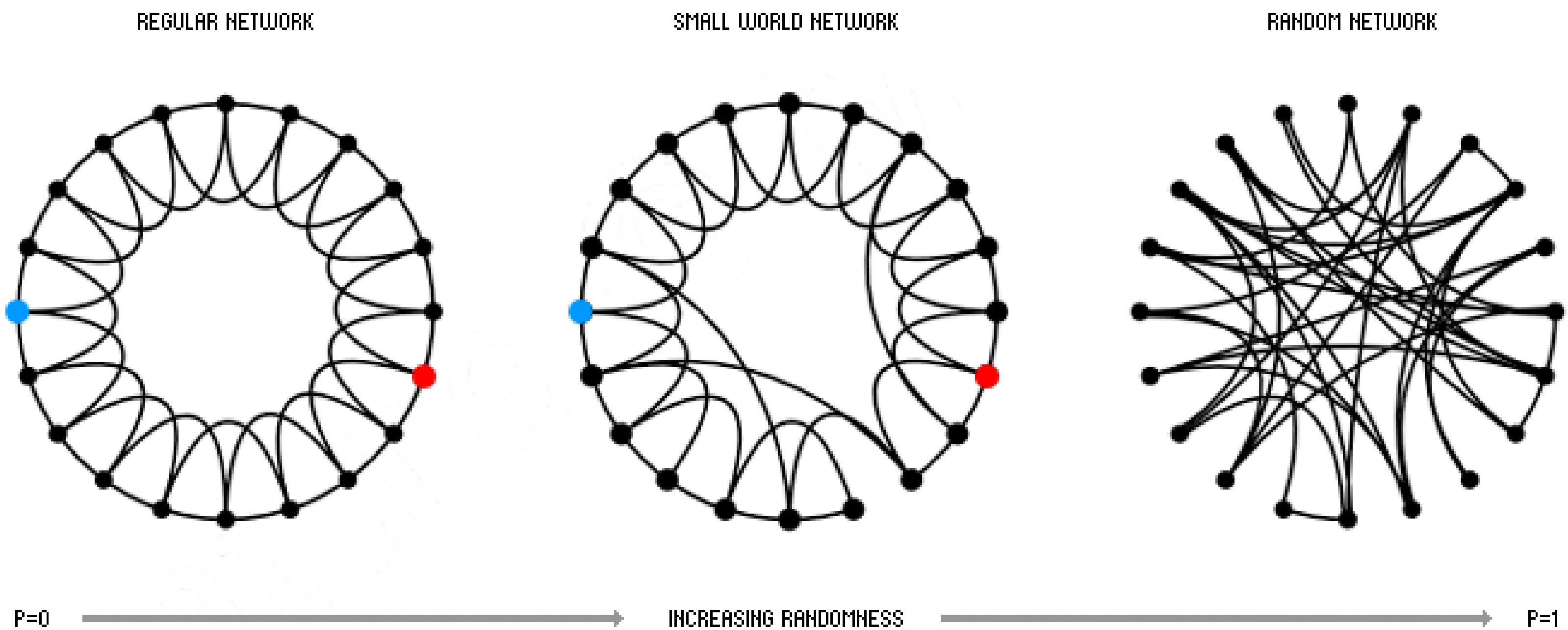
Small-world Model [Watts-Strogatz '98]:

2 components to the model:

- **(1) Start with a low-dimensional regular lattice**
  - Has high clustering coefficient
- **(2) Now introduce randomness ("shortcuts"): Rewire:**
  - Add/remove edges to create shortcuts to join remote parts of the lattice
  - For each edge with prob. p move the other end to a random node



The Small-World Model



High clustering High diameter	High clustering Low diameter	Low clustering Low diameter
$h = \frac{N}{2\bar{k}}$ $C = \frac{3}{4}$		$h = \frac{\log N}{\log \alpha}$ $C = \frac{\bar{k}}{N}$

Rewiring allows us to interpolate between regular lattice and a random graph

Watts-Strogatz (WS) Model

- Watts-Strogatz networks: $l_{network} \approx \ln(N)$

$$C_{network} >> C_{randomgraph}$$

- Random networks:

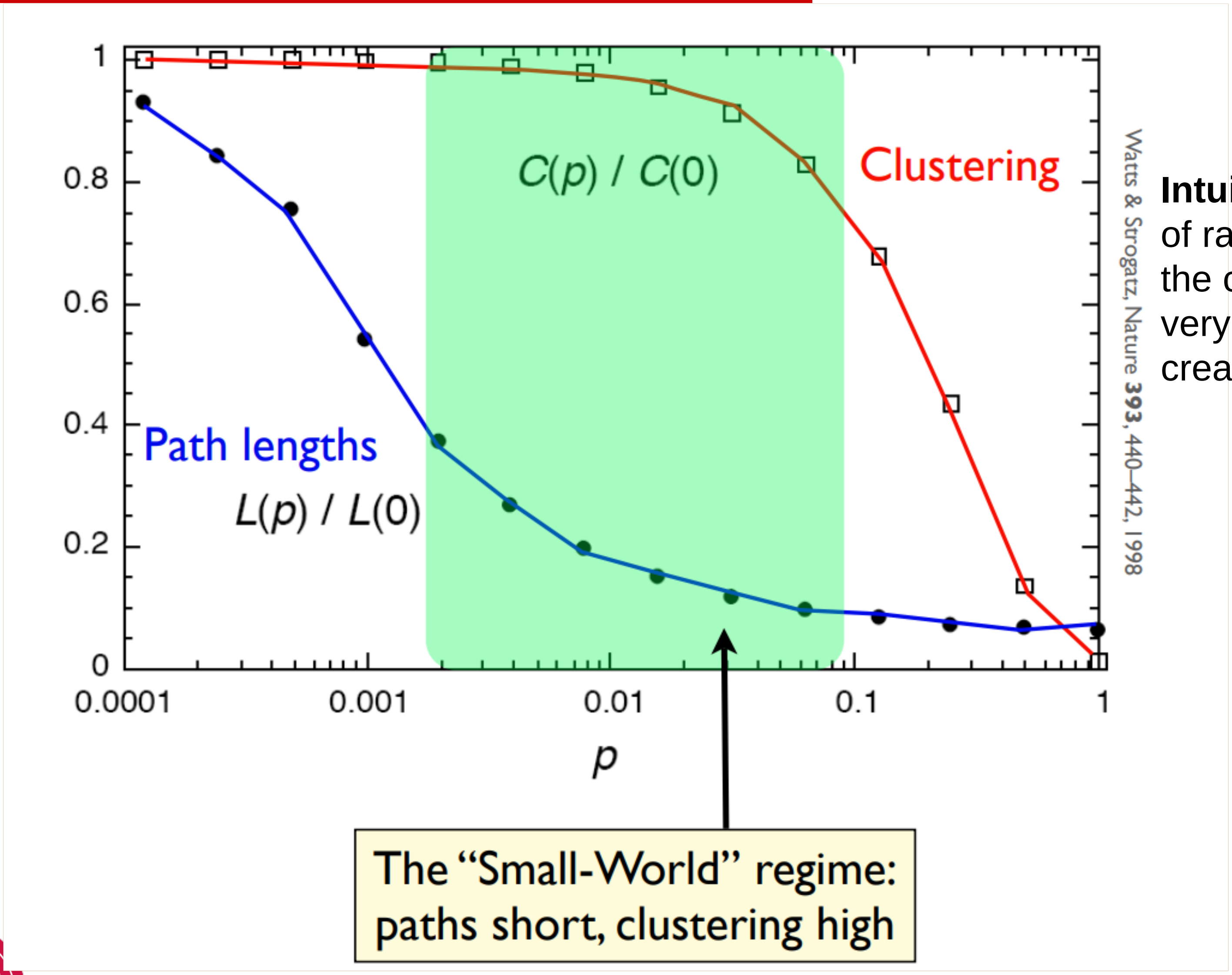
$$l \approx \frac{\ln N}{\ln K} \quad \text{small}$$

$$C \approx \frac{K}{N} \quad \text{small}$$

What happens in between?

- Small shortest path means small clustering?
- Large shortest path means large clustering?
- Through numerical simulation
  - As we increase p from 0 to 1
    - Fast decrease of mean distance
    - Slow decrease in clustering

What happens in between?

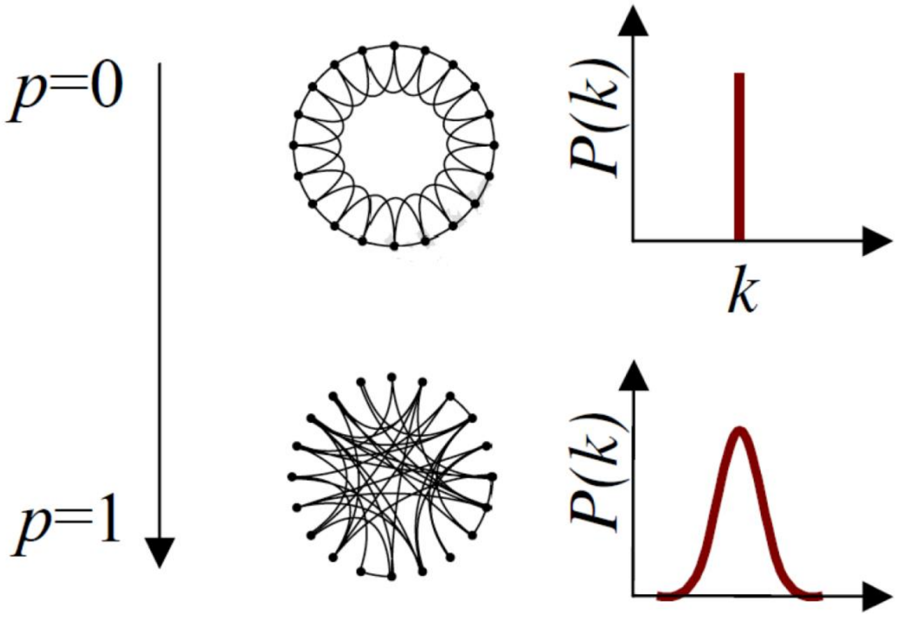


Intuition: It takes a lot of randomness to ruin the clustering, but a very small amount to create shortcuts.

Degree distribution

- p=0 delta-function
- p>0 broadens the distribution
- p=1 ➔ random networks ➔ Binomial distribution

- The shape of the degree distribution is similar to that of a random graph and has a pronounced peak at k=K and decays exponentially for large |k-K|



Small World Model: Summary

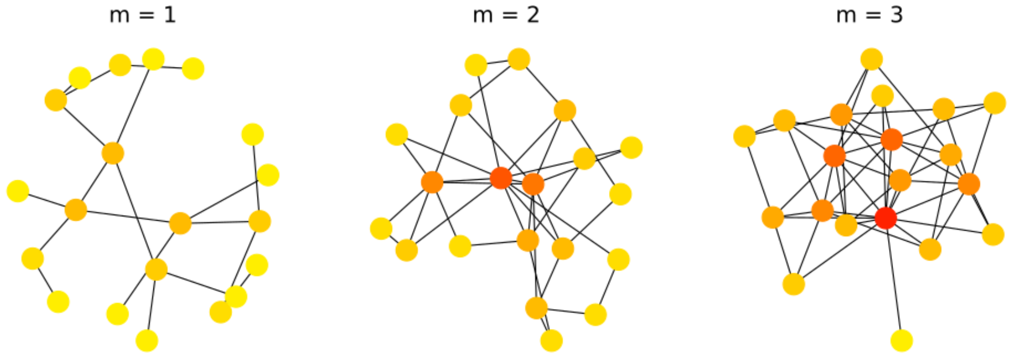
- Can a network with high clustering also be a small world?
  - **Yes!** Only need a few random links.

- **The Watts-Strogatz Model:**
  - A random graph generation model
  - Provides insight on the interplay between clustering and the small-world
  - Captures the structure of many realistic networks
  - Accounts for the **high clustering** of real networks

Preferential Attachment Model

Albert-László Barabási

Réka Albert



Preferential Attachment Model

Hubs represent the most striking difference between a random and a scale-free network. Their emergence in many real systems raises several fundamental questions.

- Why does the random network model of Erdős and Rényi fail to reproduce the hubs and the power laws observed in many real networks?
- Why do so different systems as the WWW or the cell converge to a similar scale-free architecture?

Growth and Preferential Attachment

The random network model differs from real networks in <u>two</u> important characteristics:

1-Growth: While the random network model assumes that the number of nodes is fixed (time invariant), real networks are the result of a growth process that continuously increases.

2-Preferential Attachment: While nodes in random networks randomly choose their interaction partner, in real networks new nodes prefer to link to the more connected nodes.

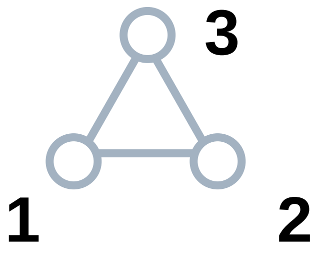
Preferential attachment (PA) model

- parameters: **m, n** (positive integers)
  - n: number of nodes
  - m: number of attachments of each new node
- at time 0, consider an arbitrary initial graph
  - E.g., a single edge or a 10-clique
- at time t+1, add m edges from a new node v_{t+1} to existing nodes forming the graph G_t
  - the edge $v_{t+1}\, x_i$ is added with probability: $$\frac{\deg(x_i)}{\sum_{1\le i\le n}\deg(x_i)} = \frac{\deg(x_i)}{2|E(G)|}$$

The larger $\deg(x_i)$, the higher the probability that new node is joined to x_i

Basic BA-model

- Very simple algorithm to implement
  - start with an initial set of m_0 fully connected nodes
    - e.g. $m_0 = 3$

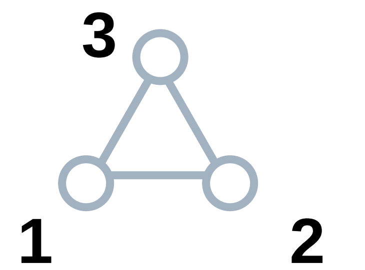


  - now add new vertices one by one, each one with exactly m edges
  - each new edge connects to an existing vertex in proportion to the number of edges that vertex already has → ***preferential attachment***
  - easiest if you keep track of edge endpoints in one large array and select an element from this array at random
    - the probability of selecting any one vertex will be proportional to the number of times it appears in the array – which corresponds to its degree

Generating BA graphs – cont'd

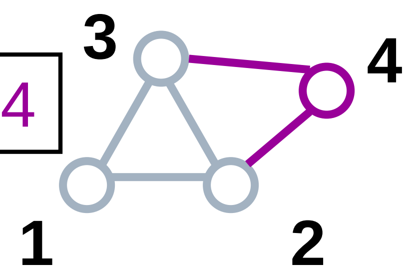
- To start, each vertex has an equal number of edges (2)
  - the probability of choosing any vertex is 1/3

1 1 2 2 3 3

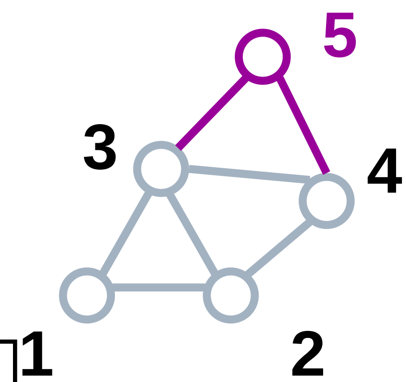


- We add a new vertex, and it will have m edges, here take m=2
  - draw 2 random elements from the array – suppose they are 2 and 3

1 1 2 2 2 3 3 3 4 4



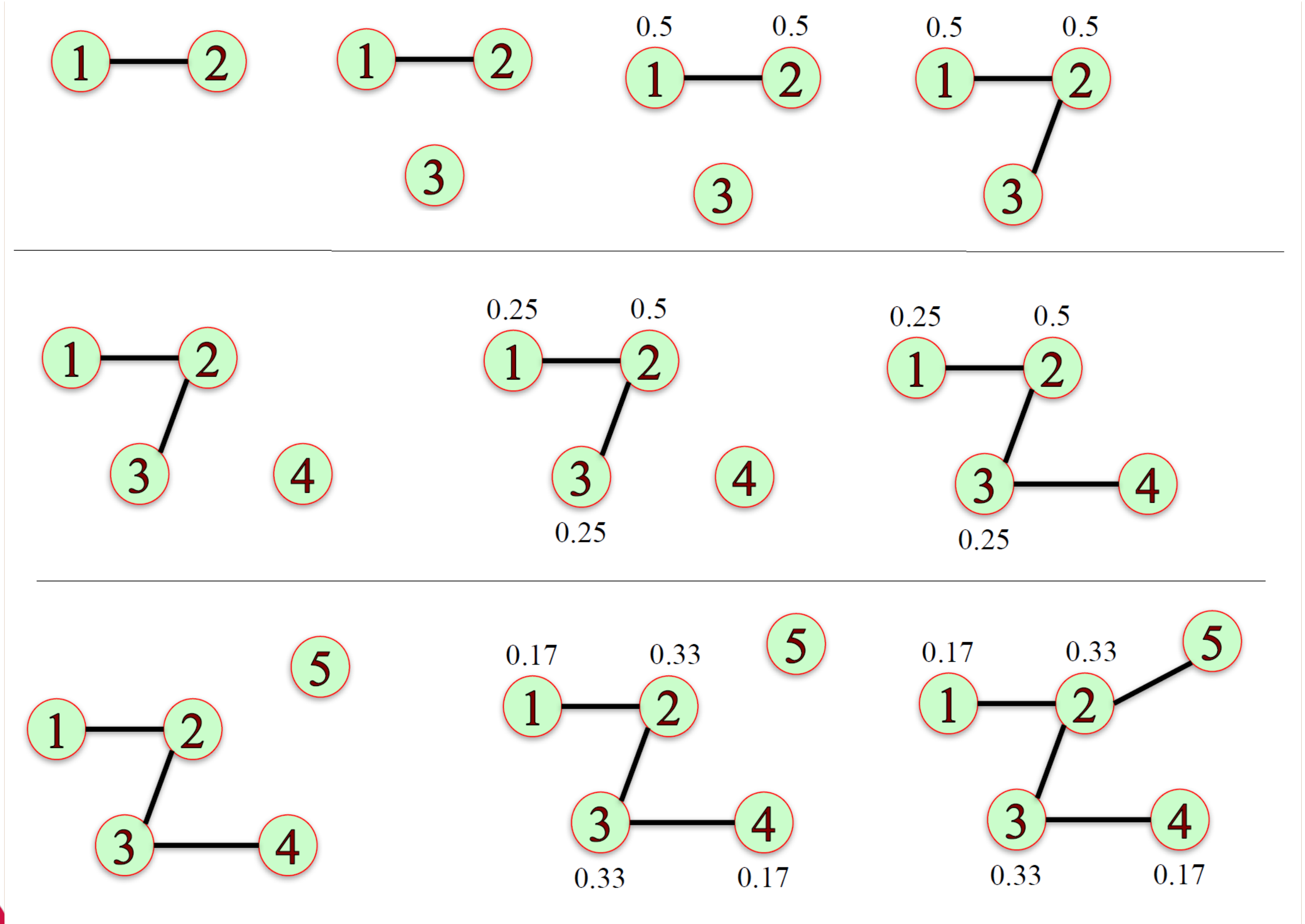
- Now the probabilities of selecting 1,2,3,or 4 are
  1/5, 3/10, 3/10, 1/5



- Add a new vertex, draw a vertex for it to connect from the array
  - etc.

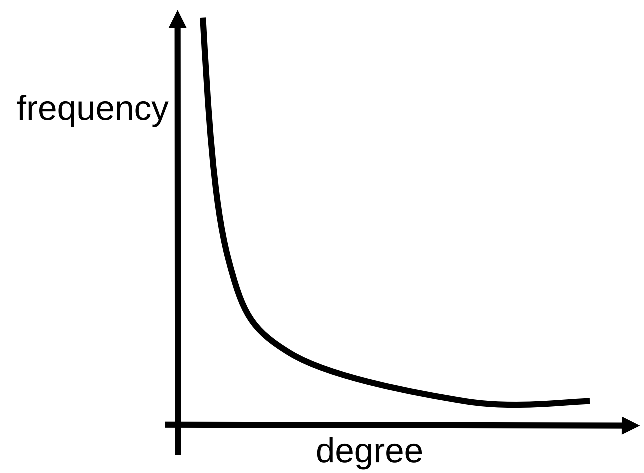
1 1 2 2 2 3 3 3 3 4 4 4 5 5

Preferential Attachment



Preferential Attachment and Scale-free Networks

- Preferential attachment (PA) results in **scale-free** networks
- Networks with **power-law** degree distribution are called scale-free
- PA ➔ **rich get richer**
  - A few nodes become important hubs with many attachments
  - Many nodes stay with little relationships



Properties of BA Networks

- The graph is connected

  - Every vertex is born with a link (m= 1) or several links (m > 1)

  - It connects to older vertices, which are part of the giant component

- The older are richer

  - Nodes accumulate links as time goes on

  - preferential attachment will prefer wealthier nodes, who tend to be older and had a head start

- BA networks are not clustered.(Can you think of a growth model of having preferential attachment and clustering at the same time?)

Properties of BA Networks

- Degree distribution

  - power law degree distribution with $P(k) \sim k^{-3}$

- Average path length $\ell \sim \frac{\ln N}{\ln \ln N}.$

  - Which is even shorter than in random networks

- Average degree

  - *2m*

- Clustering coefficient

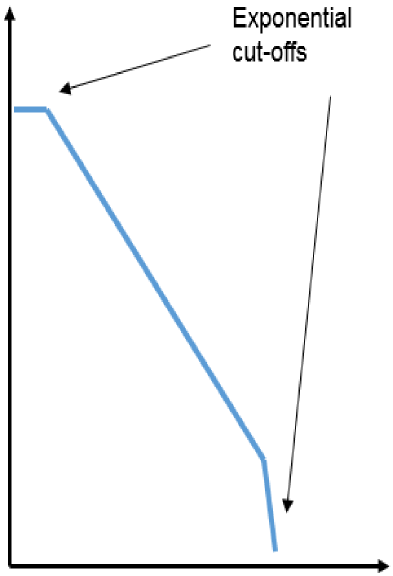
  - no analytical result
  - higher for the BA model than for random networks

Problems of the BA Model

- BA model is a nice one, but is not fully satisfactory!
- BA model does not give satisfactory answers with regard to **clustering**
  - While the small world model of Watts and Strogatz does!
- BA predicts a **fixed exponent of 3** for the powerlaw
  - However, real networks shows exponents between 2 and 3

Problems of the BA Model (cont'd)

- Real networks are **not "completely" power law**
  - After having obeyed the power-law for a large amount of k, for very large k, the distribution suddenly becomes **exponential**
  - They exhibit a so called **exponential cut-off**
- In general
  - The distribution has still a "**heavy tailed"**
  - However, such tail is not infinite
- This can be explained because
  - The number of resources (i.e., of links) that an individual c can properly handled) is often limited



Growing Networks

- In general, networks are not static entities
- They grow, with the continuous addition of new nodes
  - The Web, Internet, acquaintances, scientific literature, etc.
- Thus, edges are added in a network with time

- Preferential-Attachment, is a growing-network model

Evolving Networks

- More in general…
  - Network **grows** AND network **evolves**
- The evolution may be driven by various forces
  - Connection **age**
  - Connection **satisfaction**
- Connections can change during the life of the network
  - Not necessarily in a random way
  - But following characteristics of the network…
- Preferential-Attachment is **not** an evolving-network model

Variations on the BA Model: Evolving Networks

- The problems of the BA Model may depend on the fact that networks not only **grow** but also **evolve**
  - BA does not account for **evolutions following the growth**
- Evolution is frequent in real networks, otherwise:
  - **Google** would have never replaced Altavista
  - All new **Routers** in the Internet would be unimportant ones
  - A **Scientist** would have never the chance of becoming a highly-cited one

Variations on the BA Model: Edges Rewiring

- By coupling the model for node additions
  - Adding new nodes at new time interval
- One can consider also mechanisms for edge rewiring
  - E.g., adding some edges at each time interval
  - Some of these can be added randomly
  - Some of these can be added based on preferential attachment
- Then, it is possible to show (Albert and Barabasi, 2000)
  - That the network evolves as a power law with an exponent that can vary between 2 and infinity
  - This enables explaining the various exponents that are measured in real networks

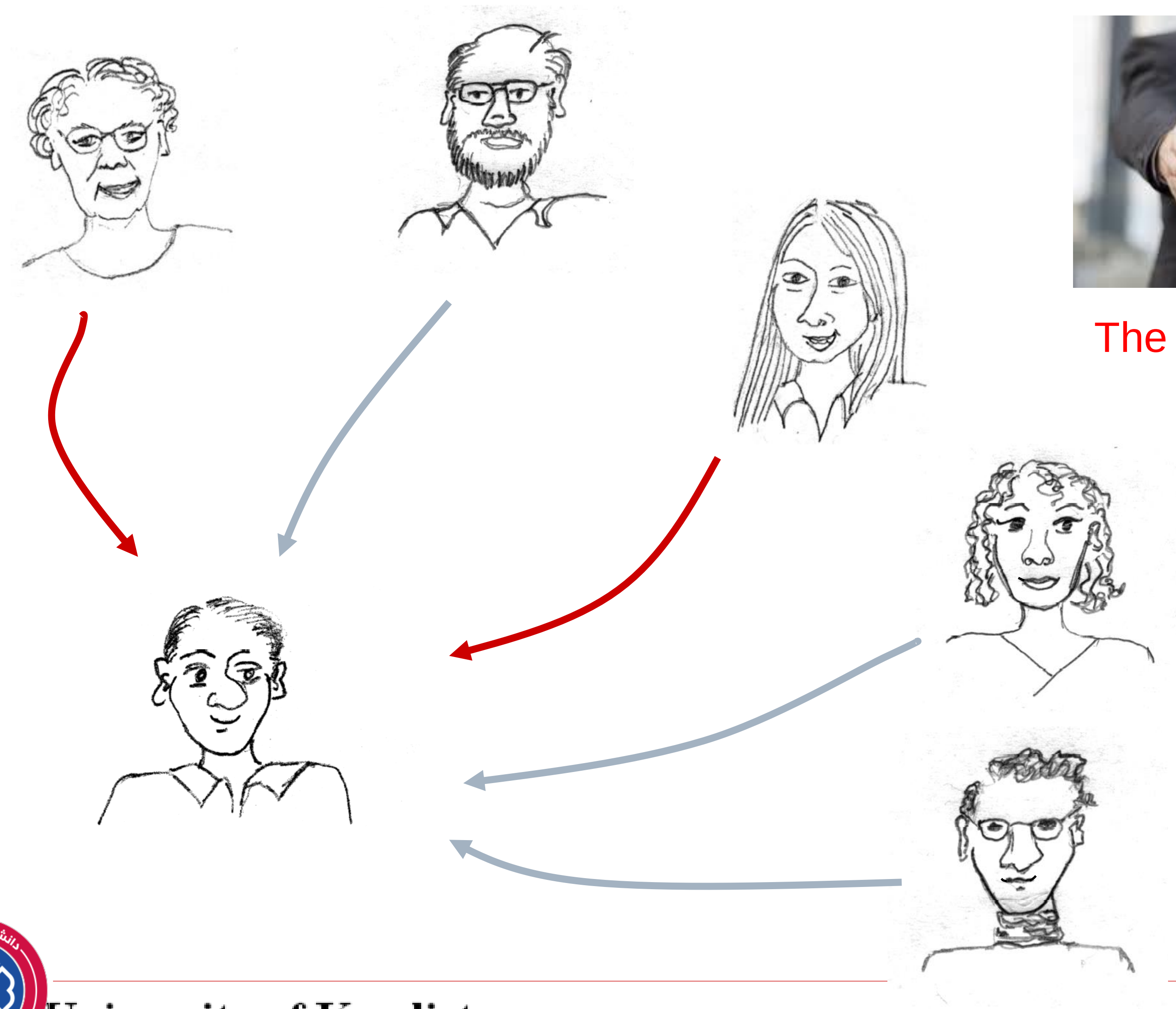
Variations on the BA Model: Aging and Cost

- **Node Aging**
  - The possibility of hosting new links decreased with the “age” of the node
  - E.g. nodes get tired or out-of-date
- **Link cost**
  - The cost of hosting new link increases with the number of links
  - E.g., for a Web site this implies adding more computational power, for a router this means buying a new powerful router

What implications does this have?

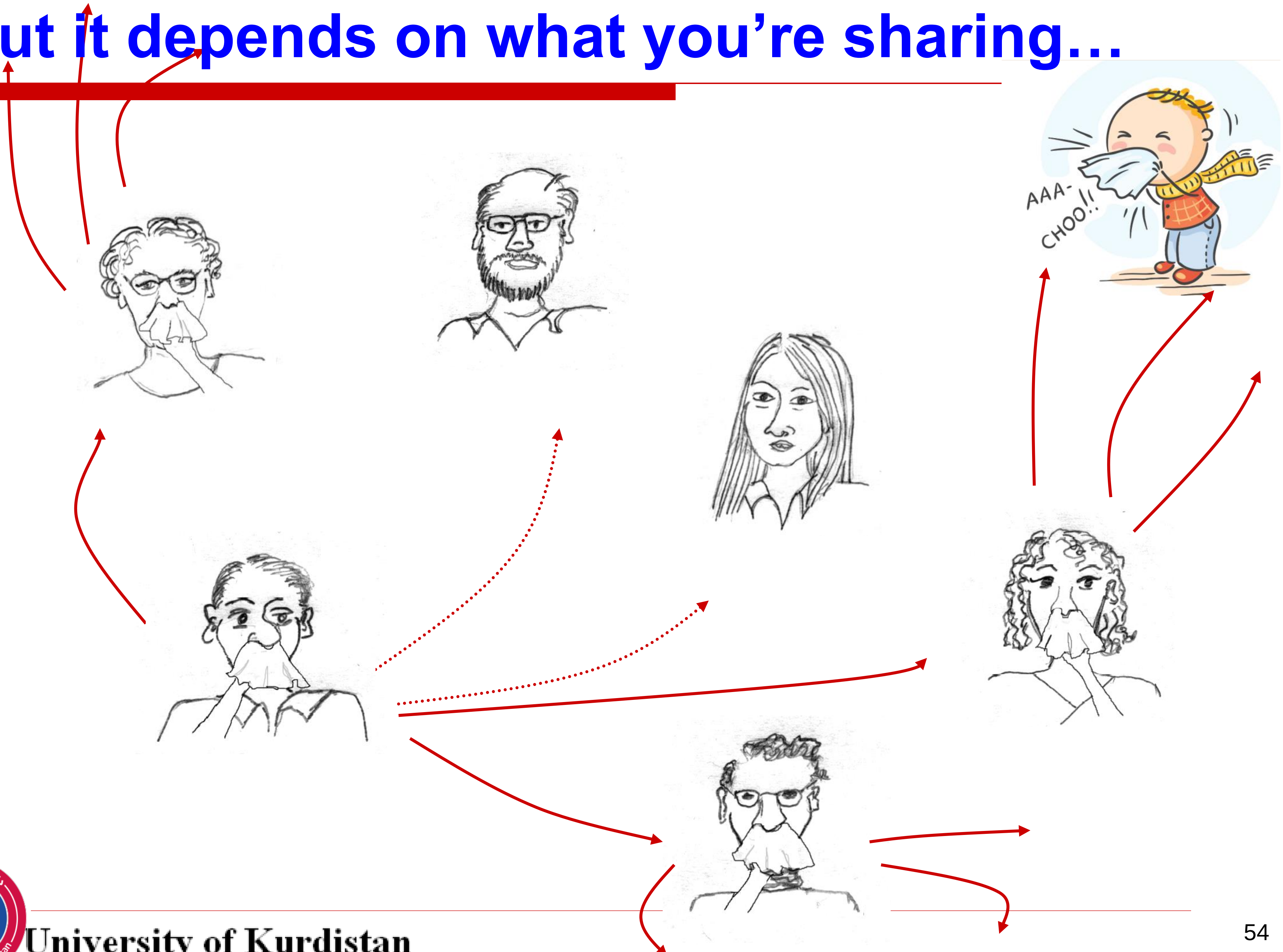
- Robustness
- Search
- Spread of disease
- Opinion formation
- Spread of computer viruses
- Gossip

In social networks, it's nice to be a hub



The concept of trust

But it depends on what you're sharing…



Failure vs. Attack

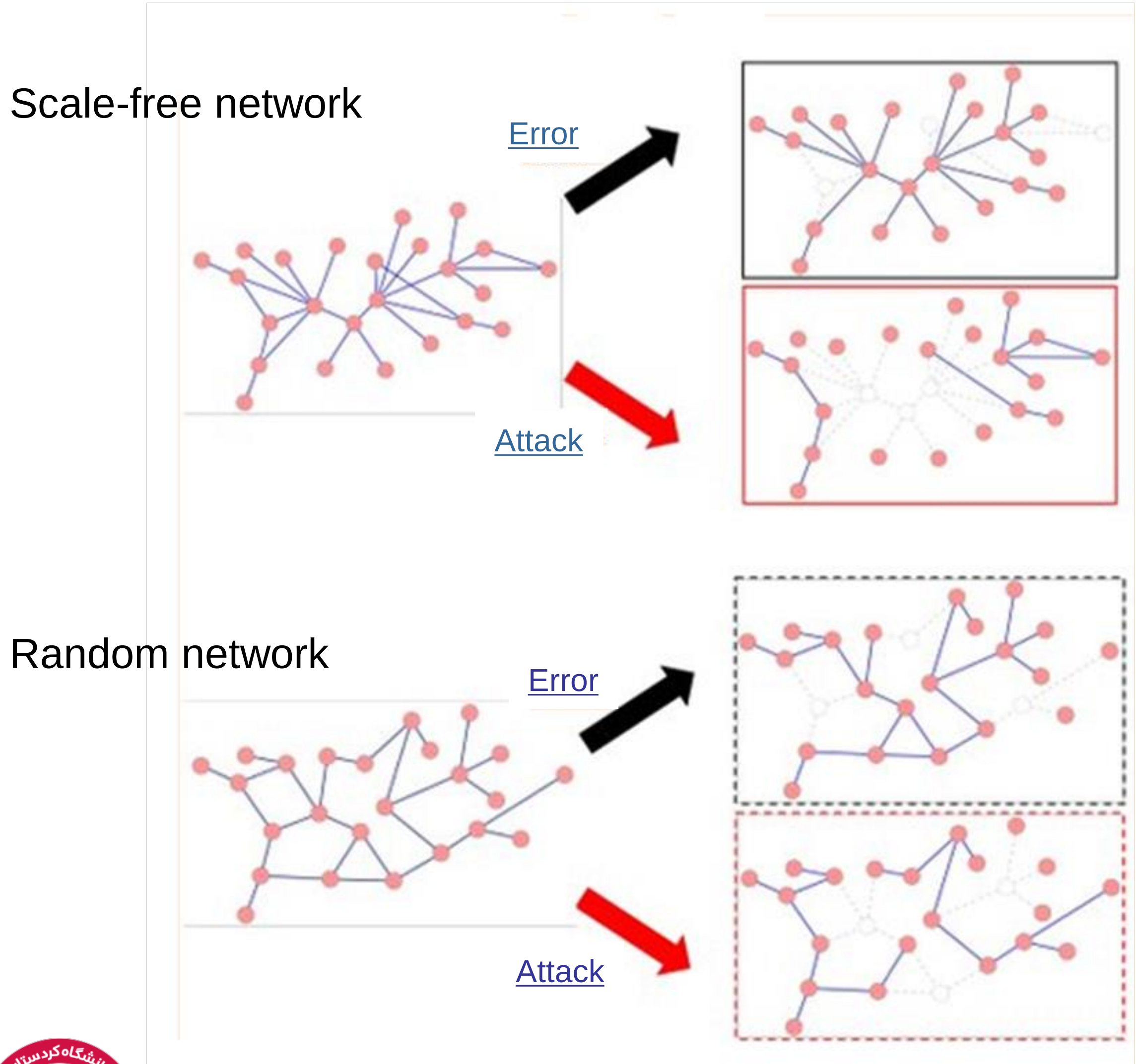
How do network connectivity change as nodes get removed?

- **Nodes can be removed:**

  - Random failure: Remove nodes uniformly at random

  - Targeted attack: Remove nodes in order of decreasing degrees

Random failure or targeted attack



In a scale-free network, the random removal (error) of even a large fraction of vertices impacts the overall connectedness of the network very little , while targeted attack destroys the connectedness very quickly, causing a rapid drop in efficiency. On the contrary, in random graphs, removal of nodes through either error or attack has the same effect on the network performance.

Network Models: Comparison

Topology	Average Path Length (L)	Clustering Coefficient (CC)	Degree Distribution ($P(k)$)
Random Graph	**Short (log N /log⟨k⟩, where N=nodes, ⟨k⟩=avg degree)**	Low (CC ≈ ⟨k⟩/N, since edges are random)	Poisson Dist.: $P(k) \approx e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}$
Small World (Watts & Strogatz, 1998)	**Short (similar to random networks)**	**High (local clustering preserved via rewiring)**	Similar to random (but depends on rewiring probability)
Scale-Free network	**Very short (~log N / log log N, "ultra-small-world")**	Low overall, but hubs can have local clustering	**Power-law Distribution: $P(k) \sim k^{-\gamma}$**


Questions